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AN EXTENSION OF THE ROOT PERTURBATION M-DIMENSIONAL POLYNOMIAL FACTORIZATION METHOD

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In this paper, an extension of an m-D (multidimensional or multivariable) polynomial factorization method is investigated. The method is the "root perturbation method" which is recently proposed by the author. According to this method, one sets to zero all complex variables, except one variable, and factorizes the 1-D polynomial. Furthermore, the values of these variables vary properly. In this way, one can "built" the m-dimensional polynomial in its factorized form. However, in the "root perturbation method", an assumption is that the 1-D polynomial must have discrete roots. In this paper, a solution is given in the case that the 1-D polynomial may have multiple roots. This is achieved by a proper transformation of the complex variables. The present method is summarized by way of algorithm. A numerical (3-D) example is presented.

1. INTRODUCTION

Multidimensional (m-D) Systems and Multidimensional Signal Processing is an important new discipline in mathematics (systems theory) and in computer science. m-D systems theory are receiving continuous attention in recent years by many mathematicians, while many computer engineers and practitioners are usually interested in the relevant applications like remote sensing, digital filtering, computerized tomography, sonar devices, artificial vision etc, [9], [23]. A serious difficulty in the development of the m-D systems is the non-factorizability of the m-D polynomials, where by the term factorizability it is meant the possibility that a given m-D polynomial can be split into factors of other multidimensional polynomials of appropriate dimensions. Polynomial theory and the relevant algorithms is in general an important modern topic in *mathematics, (algebra and system theory), electrical engineering (control systems, circuits and communication)* and *computer science (algorithms, multidimensional signal processing, information theory, codification techniques)* [8], [10].

The study of m-D polynomials and particularly their factorization into simpler factors is of fundamental importance, since such polynomials appear to be characteristic polynomials of m-D transfer functions, and the m-D stability criteria can be applied more easily to these factors. On the other hand, the transfer function factorization that is numerator and denominator factorization enable us a cascade realization of the system. The m-D polynomial factorization is also a question of a great importance in the study of distributed parameter systems (DPS) which are described by Partial Differential Equations, since each Partial Differential Equation corresponds one-to-one and onto an m-D polynomial. Finally, there is an obvious importance of the m-D factorization subject from a pure mathematical point of view.

So far, several different meaning of the term "m-D polynomial factorization" have been discussed [1]–[7], [20], [21], [24]. In [22], the factorization in factors of one variable i.e. $f(z_1, \ldots, z_m) = f_1(z_1) \cdots f_m(z_m)$, or in factors with no common variables i.e. $f(z_1, \ldots, z_m) = f_1(\bar{z}_1) \cdots f_k(\bar{z}_k)$ where $\bar{z}_1, \ldots, \bar{z}_k$ are mutually disjoint groups of independent variables has been presented. In [11], [14], the factorization is succeeded by considering the given polynomial as 1-D polynomial with respect to z_j and applying the well known formulas from 1-D algebra. In [11], [15], the factorization of the state-space model is investigated. In [11], [13], the factorization of an m-D polynomial using appropriate linear operators is considered. In [11], [16], the factorization of an m-D polynomial $f = f(z_1, \ldots, z_m) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_m=0}^{N_m} a(i_1, \ldots, i_m)\, z_1^{i_1} \ldots z_m^{i_m}$ in linear factors $f(z_1, \ldots, z_m) = \prod_{i_1=0}^{N_1}(z_1 + a_{i,2}z_2 + \ldots + a_{i,m}z_m + c_i)$ has been examined, while in [11] and [17] the more general type of factorization is studied. The method of reduction in lower-order polynomial factors is presented in [11] and [18]. In [11] and [19], an m-D polynomial factorization method based on a proper perturbation of the discrete roots of one 1-D polynomial is developed. However, in the case in which this 1-D polynomial has multiple roots the method fails. In the present paper, an attempt is made to face this difficult case of the multiple roots. In particular, a proper transformation of the variables is proposed under which the roots of this 1-D polynomial become discrete again. All the methods from [11] to [19] have been proposed by the author for first time in the international literature. See a brief description of them in [12].

The present paper is organised as follows: In Section 2, the steps of the "root perturbation method" ([11] and [19]) are briefly presented. In Section 3, the extension of the method, in the case of multiple roots, is given. The proposed algorithm is stated in Section 4, while, in Section 5, the applicability of the present technique is illustrated by an example. The author has already verified the validity, the power and the applicability of the algorithm by a computer program.

2. THE ROOT PERTURBATION METHOD

The considered problem is the factorization of the m-D polynomial $f(z_1, \ldots, z_m)$ given by (1)

$$f(z_1, \ldots, z_m) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_m=0}^{N_m} a(i_1, \ldots, i_m)\, z_1^{i_1} \ldots z_m^{i_m} \qquad (1)$$

where $\exists\, j$ such that $a(0, \ldots, 0, N_j, 0, \ldots, 0) \neq 0$ and $a(i_1, \ldots, i_{j-1}, N_j, i_{j+1}, \ldots i_m) = 0$ when $i_1 + \ldots + i_{j-1} + i_{j+1} + \ldots + i_m > 0$. Then, without loss of generality, we can set

$$a(0, \ldots, 0, N_j, 0, \ldots, 0) = 1. \tag{2}$$

We exam the possibility of the given polynomial to be written as

$$f(z_1, \ldots, z_m) = \prod_{k=1}^{N_j} (\, z_j - p_k(\tilde{z})\,) \tag{3}$$

with $p_k(\tilde{z})$ polynomial in $\tilde{z}$ ($\tilde{z} \hat{=} [z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m]^{\mathrm{T}}$). Therefore

$$p_k(\tilde{z}) = \sum_{i_1=0}^{M_1} \cdots \sum_{j-1=0}^{M_{j-1}} \sum_{i_{j+1}=0}^{M_{j+1}} \cdots \sum_{i_m=0}^{M_m} q(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_m) \cdot z_1^{i_1} \ldots z_{j-1}^{i_{j-1}} z_{j+1}^{i_{j+1}} \ldots z_m^{i_m} \tag{4}$$

with $M_k \leq N_k$, $k = 1, \ldots, j-1, j+1, \ldots, m$ and $q(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_m)$ to be real numbers. Since the exact values of M_k $(k = 1, \ldots, j-1, j+1, \ldots, m)$ are unknown, we write

$$p_k(\tilde{z}) = \sum_{i_1=0}^{N_1} \cdots \sum_{j-1=0}^{N_{j-1}} \sum_{i_{j+1}=0}^{N_{j+1}} \cdots \sum_{i_m=0}^{N_m} q(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_m) \cdot z_1^{i_1} \ldots z_{j-1}^{i_{j-1}} z_{j+1}^{i_{j+1}} \ldots z_m^{i_m} \tag{5}$$

considering $q(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_m) = 0$ for the remaining terms.

The polynomial $p_k(\tilde{z})$ can be found by knowing its values at $(N_1 + 1) \ldots (N_{j-1} + 1)\,(N_{j+1} + 1)\,(N_m + 1)$ different points and more particularly at $(z_{1_{i_1}}, \ldots, z_{j-1_{i_{j-1}}}, z_{j+1_{i_{j+1}}}, \ldots, z_{m_{i_m}})$ where

$$0 \leq i_1 \leq N_1, \ldots, 0 \leq i_{j-1} \leq N_{j-1}, \quad 0 \leq i_{j+1} \leq N_{j+1}, \ldots, 0 \leq i_m \leq N_m$$

So, if we set these $(N_1 + 1) \ldots (N_{j-1} + 1)\,(N_{j+1} + 1)\,(N_m + 1)$ different values to $\tilde{z}$, we shall find the roots of $f(z_1, \ldots, z_m)$, which is now considered as an 1-D polynomial with respect to z_j (we note $f(z_j; z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m)$). As these roots are also the values of the supposed polynomials $p_k(\tilde{z})$, finally we can find the polynomials $p_k(\tilde{z})$. One can easily seen that $p_k(\tilde{z})$ can be found by the m-D Lagrange interpolation formula, [11] and [19].

However, the difficulty in finding $p_k(\tilde{z})$ is that we don't know exactly which root of the 1-D polynomial $f(z_j; z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m)$ corresponds to a particular $p_k(\tilde{z})$. For this reason, we set $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m$ at "small" values (absolutely). So, $p_k(\tilde{z})$, as it is proved, is now "very close" to $p_k(\tilde{0}) = c_k$, where c_k $(k = 1, \ldots, N_j)$ are the roots of $f(0, \ldots, 0, z_j, 0, \ldots, 0)$ which are assumed to be discrete, i.e. $c_k \neq c_l$ for every $k, l, (k, l = 1, \ldots, N_j)$. Therefore $p_k(\tilde{z})$'s which are now very close to the discrete $p_k(\tilde{0}) = c_k$ can be separated. Unfortunately, in this way, the case of multiple roots of $f(0, \ldots, 0, z_j, 0, \ldots, 0)$ i.e. the case where we have some k, l such that $c_k = c_l$, $(k, l = 1, \ldots, N_j)$ can not be faced.

More specifically, in the case of "simple" ("discrete") roots of $f(0, \ldots, 0, z_j, 0, \ldots \ldots 0)$ the following Theorem is proved, [11] and [19].

Theorem 1. If

$$|z_1| < \frac{1}{s}, \ldots, |z_{j-1}| < \frac{1}{s}, |z_{j+1}| < \frac{1}{s}, \ldots, |z_m| < \frac{1}{s} \tag{6}$$

where the positive real number s is calculated by

$$s = \frac{2(\tilde{A} + 1 + \max_k |c_k|)}{\min_{k,l\ k \neq l} |c_k - c_l|} \tag{7}$$

$k,\ l = 1, \ldots, N_j$ and $\tilde{A} = \max A_{i_j}$ $(i_j = 0, 1, \ldots, N_j - 1)$ where

$$A_{i_j} = \sum_{i_1=0}^{N_1} \cdots \sum_{i_{j-1}=0}^{N_{j-1}} \sum_{i_{j+1}=0}^{N_{j+1}} \cdots \sum_{i_m=0}^{N_m} |a(i_1, \ldots, i_{j-1}, i_j, i_{j+1}, \ldots, i_m)|$$

then

$$|p_k(\tilde{z}) - c_k| < \frac{1}{2} \min |c_k - c_l| \tag{8}$$

$k, l = 1, \ldots, N_j$ where $k \neq l$.

So, if we put appropriate values for $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m$, $(|z_1| < \frac{1}{s}, \ldots, |z_{j-1}| < \frac{1}{s}, |z_{j+1}| < \frac{1}{s}, \ldots, |z_m| < \frac{1}{s}$, where s is given by equation (7)), we know exactly which root of the 1-D polynomial $f(z_j; z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m)$ is the value of the particular $p_k(\tilde{z})$. A geometrical interpretation is given in Figure 1.

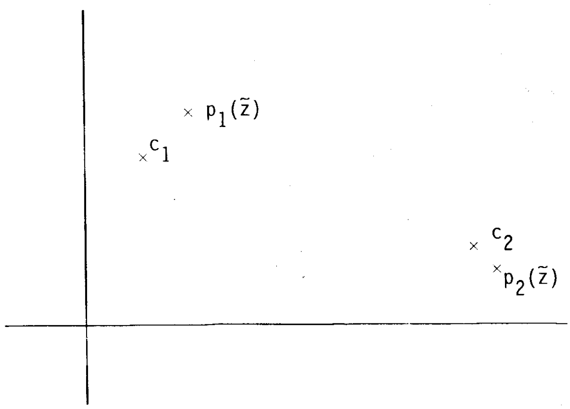

Fig. 1. In the case of discrete roots c_k, c_l of $f(z_j; , \ldots, 0)$, the values of $p_k(\tilde{z})$ can be separated.

Furthermore, the polynomial $p_k(\tilde{z})$ can be found using the m-D Lagrange interpolation formula. At this point, we recall, that if we know the values of an m-D polynomial $p(z_1, \ldots, z_m)$ for $N_1 + 1$ values of z_1, ..., $N_m + 1$ values of z_m, then this polynomial can be found by the m-D Lagrange interpolation formula.

$$p(z_1, \ldots, z_m) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_m=0}^{N_m} l_{i_1,\ldots,i_m}(z_1, \ldots, z_m) \cdot g(z_{1_{i_1}}, \ldots, z_{m_{i_m}}) \tag{9}$$

where the $g(z_{1_{i_1}}, \ldots, z_{m_{i_m}})$'s are the polynomial values of $p(z_1, \ldots, z_m)$ at the considered points $(0 \leq i_1 \leq N_1, \ldots, 0 \leq i_m \leq N_m)$ and:

$$l_{i_1,\ldots,i_m}(z_1, \ldots, z_m) = \frac{\prod_{k=0,\, k\neq i_1}^{N_1}(z_1 - z_{1_k}) \cdots \prod_{k=0,\, k\neq i_m}^{N_m}(z_m - z_{m_k})}{\prod_{k=0,\, k\neq i_1}^{N_1}(z_{1_{i_1}} - z_{1_k}) \cdots \prod_{k=0,\, k\neq i_m}^{N_m}(z_{m_{i_m}} - z_{m_k})}.$$

In the case of $p_k(\tilde{z})$, we have the variables $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m$ in (9) and (10).

Suppose now that a polynomial $p_k(\tilde{z})$ has been constructed by the Lagrange interpolation formula. Then, the m-D polynomial $z_j - p_k(\tilde{z})$ is tested as a factor of $f(z_1, \ldots, z_m)$ by applying the following theorem. Its proof can be found in [11], [12] and [18].

Theorem 2. Suppose that we have found $p_k(\tilde{z})$ from (9). Then a possible m-D: polynomial factor of $f(z_1, \ldots, z_m)$ is the polynomial $z_j - p_k(\tilde{z})$. *This is an m-D polynomial factor of* $f(z_1, \ldots, z_m)$ if and only if

$$f(z_1, \ldots, z_{j-1}, p_k(\tilde{z}), z_{j+1}, \ldots, z_m) \equiv 0. \tag{10}$$

Now, if we find one factor $z_j - p_k(\tilde{z})$, we carry out the 1-D algorithmic division $f(z_1, \ldots, z_m) : (z_j - p_k(\tilde{z})) = q(z_1, \ldots, z_m)$. The procedure of the m-D polynomial factorization might progress if the polynomial $q(z_1, \ldots, z_m)$ could be factorized by applying the same or some other method [11]–[19], [22].

3. THE CASE OF MULTIPLE ROOTS

In this paragraph, we exam the interesting case in which $f(0, \ldots, 0, z_j, 0, \ldots, 0)$ has multiple roots. First some useful preliminary results are presented: Let the 1-D polynomials

$$f(z) = \sum_{i=0}^{N} a_i z^i, \qquad g(z) = \sum_{i=0}^{N} b_i z^i. \tag{11}$$

If they are different polynomials, (we denote $f(z) \not\equiv g(z)$), i.e. $a_i \neq b_i$ at least for one i, then, except of N values of z, we have $f(z) \neq g(z)$. Therefore, one can write:

$$f(z) \neq g(z) \qquad \forall\, z \in C - \{w_1, \ldots, w_N\} \tag{12}$$

where $w_1, \ldots, w_N$ are the roots of the polynomial $f(z) - g(z)$ and C is the set of the complex numbers.

Suppose now that we have n different 1-D, N degree, polynomials. Then, except of $\frac{N\cdot(n-1)n}{2}$ values of z, they all have different values.

In 2-D polynomials, we have the following. Suppose that the polynomials

$$f(z)=\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}a_{i_1,i_2}z_1^{i_1}z_2^{i_2},\quad g(z)=\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}b_{i_1,i_2}z_1^{i_1}z_2^{i_2}$$

are different from each other. We denote $f(z)\not\equiv g(z)$. That is to say $a_{i_1,i_2}\neq b_{i_1,i_2}$ at least for one pair (i_1,i_2). These polynomials can be written as

$$f(z_1,z_2)=\sum_{i_1=0}^{N_1}\left(\sum_{i_2=0}^{N_2}a_{i_1,i_2}z_2^{i_2}\right)z_1^{i_1} \tag{13}$$

$$g(z_1,z_2)=\sum_{i_1=0}^{N_1}\left(\sum_{i_2=0}^{N_2}b_{i_1,i_2}z_2^{i_2}\right)z_1^{i_1}. \tag{14}$$

Since $a_{i_1,i_2}\neq b_{i_1,i_2}$ for at least one pair (i_1,i_2), at least one pair of 1-D polynomials:

$$\sum_{i_2=0}^{N_2}a_{i_1,i_2}z_2^{i_2},\quad \sum_{i_2=0}^{N_2}b_{i_1,i_2}z_2^{i_2} \tag{15}$$

are different from each other. Then, except of N_2 values of z_2, these two polynomials have different values. So, for $z_2=w_2\in C-\{w_{2,1},\dots,w_{2,N_2}\}$ we obtain that $f(z_1,w_2)\not\equiv g(z_1,w_2)$. These are 1-D polynomials with respect to z_1, so, for $z_1=w_1\in C-\{w_{1,1}(w_2),\dots,w_{1,N_1}(w_2)\}$ we have that $f(w_1,w_2)\not\equiv g(w_1,w_2)$.

The conclusion is that: $f(z_1,z_2)\neq g(z_1,z_2)\ \ \forall(w_1,w_2)$ with $w_2\in C-\{w_{2,1},\dots$ $\dots,w_{2,N_2}\}$ and $w_1\in C-\{w_{1,1}(w_2),\dots,w_{1,N_1}(w_2)\}$.

Suppose now, that we have n different 2-D, N_1,N_2-degree, polynomials. Then, they are unequal (they have different values) for each (w_1,w_2) with

$$w_2\in C-\{w_{2,1},\dots,w_{2,N_2\cdot(n-1)n/2}\}$$

and

$$w_1\in C-\{w_{1,1}(w_2),\dots,w_{1,N_1\cdot(n-1)n/2}(w_2)\}.$$

Now, generalizing the above results, if we have n different m-D, $N_1,\dots,N_m$-degree polynomials, they are unequal (they have different values) for all $(w_1,\dots,w_m)$ with

$$w_m\in C-\{w_{m,1},\dots,w_{m,N_m\cdot(n-1)n/2}\}$$

$$\vdots$$

$$w_1\in C-\{w_{1,1}(w_2,\dots,w_m),\dots,w_{1,N_1\cdot(n-1)n/2}(w_2,\dots,w_m)\}.$$

Henceforth, the following notation is introduced

$$C_m^r=\{w_{m,1},\dots,w_{m,N_m\cdot(n-1)n/2}\}$$

$$\vdots$$

$$C_1^r=\{w_{1,1}(w_2,\dots,w_m),\dots,w_{1,N_1\cdot(n-1)n/2}(w_2,\dots,w_m)\}.$$

With the above preparations, we are ready to present the extension of the root perturbation method in the case in which $f(0,\ldots,0,z_j,0,\ldots,0)$ has multiple roots, i.e. if $\exists\ k_1,k_2,$ (with $k_1,k_2=1,\ldots,N_j$) such that $p_{k_1}(\tilde{0})=p_{k_2}(\tilde{0})$.

First, suppose that $p_k(\tilde{z})\not\equiv p_l(\tilde{z})\ \forall\, k,l\ (k,l=1,\ldots,N_j)$ i.e. $p_k(\tilde{z})$ are different from each other. In this case, based on the previous preliminary results, there exists a point $(w_1,\ldots,w_{j-1},w_{j+1},\ldots,w_m)$ for which these polynomials take different values, i.e. the values $p_k(w_1,\ldots,w_{j-1},w_{j+1},\ldots w_m)$ are different from each other. So, if one finds such a point $(w_1,\ldots,w_{j-1},w_{j+1},\ldots,w_m)$, then one can consider the transformation

$$\begin{aligned} z_1 &\longrightarrow z_1+w_1 \\ &\vdots \\ z_{j-1} &\longrightarrow z_{j-1}+w_{j-1} \\ z_{j+1} &\longrightarrow z_{j+1}+w_{j+1} \\ &\vdots \\ z_m &\longrightarrow z_m+w_m \end{aligned}$$

under which the polynomial

$$\begin{aligned} &\hat{f}(z_1,\ldots,z_{j-1},z_j,z_{j+1},\ldots,z_m) \\ = \;& f(z_1+w_1,\ldots,z_{j-1}+w_{j-1},z_j,z_{j+1}+w_{j+1},\ldots,z_m+w_m) \end{aligned} \tag{16}$$

results. Obviously $\hat{f}(z_1,\ldots,z_{j-1},z_j,z_{j+1},\ldots,z_m)$ is a factorizable polynomial if and only if $f(z_1,\ldots,z_{j-1},z_j,z_{j+1},\ldots,z_m)$ is factorizable. It is easy to verify that the roots of $\hat{f}(0,\ldots,0,z_j,0,\ldots,0)$, which are symbolized as $\hat{c}_i$, are $p_k(w_1,\ldots,w_{j-1},w_{j+1},\ldots,w_m)$. However, $p_k(w_1,\ldots,w_{j-1},w_{j+1},\ldots w_m)$ are different from each other. In other words, the roots of the new polynomial $\hat{f}(0,\ldots,0,z_j,0,\ldots,0)$ are now simple (discrete). Thus, the root perturbation method can be applied.

Furthermore, the factorization of $f(z_1,\ldots,z_m)$ results from the factorization of $\hat{f}(z_1,\ldots,z_{j-1},z_j,z_{j+1},\ldots,z_m)$ after the inverse transformation:

$$\begin{aligned} z_1 &\longrightarrow z_1-w_1 \\ &\vdots \\ z_{j-1} &\longrightarrow z_{j-1}-w_{j-1} \\ z_{j+1} &\longrightarrow z_{j+1}-w_{j+1} \\ &\vdots \\ z_m &\longrightarrow z_m-w_m \end{aligned}$$

Secondly, in the very special case where some of $p_k(\tilde{z})$ are identically equal, it is not possible to succeed to transform all the roots of $f(z_j;0,\ldots,0)$ into simple ones. However, in this case, the problem of separation of roots does not actually exist, since equal roots of the 1-D polynomial $f(z_j\,; z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m)$ correspond to identically equal $p_k(\tilde{z})$. Therefore the root perturbation method, Section 2, [11], [12] and [19] can be applied.

4. THE PROPOSED ALGORITHM

Arranging the above ideas in a logical order, the following algorithm can be constructed.

Step 1: Find the roots of $f(0, \ldots, 0, z_j, 0, \ldots, 0)$. If they are discrete (simple), then apply the root perturbation method $\longrightarrow$ END. If there exist multiple roots, then follow the next steps.

Step 2: Set arbitrary values to $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots z_m$, say $w_1, \ldots, w_{j-1}, w_{j+1}, \ldots$ $\ldots, w_m$ and find the roots of $f(w_1, \ldots, w_{j-1}, z_j, w_{j+1}, \ldots, w_m)$.
If some of them continue to be multiple, then vary z_1 where
$z_1 = w_{1,1}, \ldots, w_{1,(1+N_j(N_j-1)/2)\cdot N_1}$.
If, after all, the multiplicity of roots remains, we conclude that some of $w_{2,1}, \ldots$ $\ldots, w_{j-1}, w_{j+1}, \ldots, w_{m,1}$ belong to $C_2^r, \ldots, C_{j-1}^r, C_{j+1}^r, \ldots, C_m^r$. In this case, vary z_2. Proceeding this procedure, finally, one can find a point such that:

$$(w_1, \ldots, w_{j-1}, w_{j+1}, \ldots w_m) \in (C - C_1^r) \times \ldots \times (C - C_{j-1}^r) \times (C - C_{j+1}^r) \times \ldots \times (C - C_m^r)$$

for which $f(w_1, \ldots, w_{j-1}, z_j, w_{j+1}, \ldots w_m)$ has simple roots. If this is not possible, go to Step 4.

Step 3: Find the polynomial

$$\begin{aligned} & \hat{f}(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_m) \\ = \; & f(z_1 + w_1, \ldots, z_{j-1} + w_{j-1}, z_j, z_{j+1} + w_{j+1}, \ldots, z_m + w_m) \end{aligned}$$

Find the roots of $\hat{f}(0, \ldots, 0, z_j, 0, \ldots, 0)$ (equivalently the roots of $f(z_j\,; w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_m)$). These will be simple. Afterwards, apply the root perturbation method.
If $\hat{f}(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_m)$ is factorized, then $f(z_1, \ldots, z_m)$ is factorized too. The second can be found in its factorized form since:
$f(z_1, \ldots, z_m) = \hat{f}(z_1 - w_1, \ldots, z_{j-1} - w_{j-1}, z_j, z_{j+1} - w_{j+1}, \ldots, z_m - w_m)$
$\longrightarrow$ END.

Step 4: Some of $p_k(\tilde{z})$ are identically equal. Their number has been known from Step 2: *It is the minimum number of multiple roots that results in the various changes of* $w_1, \ldots, w_{j-1}, w_{j+1}, \ldots w_m$ *following the procedure of* Step 2. Apply the root perturbation method (since equal $p_k(\tilde{z})$ correspond to equal c_k)$\longrightarrow$ END.

Remark. In *Step 2* of the previous algorithm, note that:

$$C_m^r = \{w_{m,1}, \ldots, w_{m,N_m \cdot n(n-1)/2}\}$$

$$\vdots$$

$$C^{r}_{j+1} = \{w_{j+1,1}(w_{j+2},\dots,w_m),\dots,w_{j+1,N_{j+1}\cdot n(n-1)/2}(w_{j+2},\dots,w_m)\}$$
$$C^{r}_{j-1} = \{w_{j-1,1}(w_{j+1},\dots,w_m),\dots,w_{j-1,N_{j-1}\cdot n(n-1)/2}(w_{j+1},\dots,w_m)\}$$
$$\vdots$$
$$C^{r}_{1} = \{w_{1,1}(w_2,\dots w_{j-1},w_{j+1},\dots,w_m),\dots,$$
$$w_{1,N_m\cdot n(n-1)/2}(w_2,\dots w_{j-1},w_{j+1}\dots,w_m)\}$$

where $n = N_j$.

5. EXAMPLE

Let the polynomial $f(z_1,z_2,z_3) = z_1^2 + 4z_1z_2 + 3z_2^2 + 2z_1z_2^2 + 2z_2^3 + 4z_1z_3 + 12z_2z_3 + 8z_2^2z_3$.

The above algorithm can be applied as follows. Select $z_j = z_1$ (i.e. $j = 1$).

Step 1: First, we find the roots of $f(z_1,0,0)$. So, $f(z_1,0,0) = z_1^2$. We obtain the double root 0.

Step 2: Setting $z_2 = z_3 = 1$, the polynomial $f(z_1,1,1) = z_1^2 + 10z_1 + 25$ is found which also has the double root -5. Setting $z_2 = 0, z_3 = 1$, the resultant polynomial $f(z_1,0,1) = z_1^2 + 4z_1$ has the discrete roots $0,-4$.

Step 3: Under the transformation

$$z_2 \longrightarrow z_2 + 0$$
$$z_3 \longrightarrow z_3 + 1$$

the polynomial $f(z_1,z_2,z_3)$ is transformed into $\hat{f}(z_1,z_2,z_3) = f(z_1,z_2+0,z_3+1) = 4z_1 + z_1^2 + 12z_2 + 4z_1z_2 + 11z_2^2 + 2z_1z_2^2 + 2z_2^3 + 4z_1z_3 + 12z_2z_3 + 8z_2^2z_3$.
The roots of the polynomial $\hat{f}(z_1,0,0)$ are $-4,0$. Applying the root perturbation method, one can find: $A_0 = 45, A_1 = 14$. Therefore, following Theorem 1 we have $\tilde{A}$=45 and $s = 25$. Setting $(N_2+1)(N_3+1) = 4\times 2 = 8$ values (z_2,z_3) in $f(z_1;z_2,z_3)$ such that $|z_2| < \frac{1}{s}, |z_3| < \frac{1}{s}$, we find the corresponding roots – with respect to z_1– of $\hat{f}(z_1;z_2,z_3)$. For example, for $z_2 = -0.02$ and $z_3 = -0.02$, we find the roots 0.0592 and -3.9. Finally, using the Lagrange interpolation formula and Theorem 2, we find after some manipulation

$$\hat{f}(z_1,z_2,z_3) = (z_1 + 3z_2 + 2z_2^2)\cdot(z_1 + z_2 + 4z_3 + 4) \tag{17}$$

Step 4: Under the inverse transformation

$$z_2 \longrightarrow z_2 - 0$$
$$z_3 \longrightarrow z_3 - 1$$

(17) yields

$$f(z_1,z_2,z_3) = (z_1 + 3z_2 + 2z_2^2)\cdot(z_1 + z_2 + 4z_3) \tag{18}$$

since $\hat{f}(z_1,z_2-0,z_3-1) = f(z_1,z_2,z_3)$. Thus, the factorization of the given polynomial is achieved.

6. CONCLUSION

The factorization algorithm, presented in this paper, can be applied in a wide class of m-D polynomials. The method can be applied in such a way that, if the complete decomposition into N_1 factors $z_j - p_k(\tilde{z})$ is impossible, then, at least the evaluation of an eventual factor $z_j - p_k(\tilde{z})$ to be possible. So – as in the case of the simple root perturbation method [11], [19] – if one factor of $f(\tilde{z})$ is found, the polynomial $f(\tilde{z})/(z_j - p_k(\tilde{z}))$ may be factorized by some other method of the literature [11], [13]–[19], [22].

The extension of the root perturbation method to polynomial matrices may be an other interesting mathematical extension with many applications in *physics* (optical systems, biophysics), in *electrical engineering* (multidimensional circuits and systems, automatic control) as well as in *computer science* (digital image processing, theory of algorithms [8], [10], information theory). However, this is left for future research.

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