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# ON THE MORGAN PROBLEM WITH STABILITY[1]

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The row-by-row decoupling of linear multivariable systems with stability is considered. We provide necessary and sufficient conditions for this problem to have a solution, based on the polynomial equations approach.

## 1. INTRODUCTION

We shall consider a linear time-invariant system $(C, A, B)$ governed by

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. All the matrices are over $\mathbb{R}$ and we assume that $rank\, B = m$, $rank\, C = p$, and that the system is controllable. As we are interested in the problem of decoupling, which will be precisely defined below, we also suppose that the system $(C, A, B)$ is right invertible, i.e. $rank\, T(s) = p$, where

$$T(s) = C(sI - A)^{-1}B \tag{1}$$

is the transfer function of $(C, A, B)$.

Let $T(s)$ be factorized as

$$T(s) = CN_1(s)\, D^{-1}(s) \tag{2}$$

where the polynomial matrices $N_1(s)$ and $D(s)$ form a normal external description (NED) of $(C, A, B)$; see [11]. The matrices $N(s) := CN_1(s)$ and $D(s)$ then form, not necessarily coprime, a matrix fraction description (MFD) of $(C, A, B)$. Recall that $N_1(s)$ and $D(s)$ are right coprime with $D(s)$ being column reduced. The column degrees, $c_i := deg_{ci} D(s)$, $i = 1, 2, \ldots, m$, are the controllability indices of $(C, A, B)$. We shall further assume (see [8] for more details) that $\mathcal{V}^* = \mathcal{R}^*$, where $\mathcal{R}^*$ denotes the maximal controllability subspace lying in $Ker\, C$ and $\mathcal{V}^*$ is the maximal $(A, B)$-invariant subspace contained in $Ker\, C$.

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It has already been mentioned that we are interested in the problem of decoupling, or more precisely, in the Morgan problem with stability (MPS), which is defined as follows.

**MPS.** Given a system $(C, A, B)$, does there exist a (nonregular) state feedback

$$u(t) = Fx(t) + Gv(t) \tag{3}$$

where $F \in \mathbb{R}^{m\times n}$ and $G \in \mathbb{R}^{m\times p}$ with $rank\, G = p$, such that the transfer function $T_{F,G}(s)$ of the closed-loop system $(C, A + BF, BG)$ is of the form

$$\begin{aligned} T_{F,G}(s) &= C(sI_n - A - BF)^{-1}BG \\ &= T(s)\,[I_m - FN_1(s)\,D^{-1}(s)]^{-1}G \\ &= diag\{w_1(s), \dots, w_p(s)\} =: W(s) \end{aligned} \tag{4}$$

where $w_i(s)$, $i = 1, 2, \dots, p$, are strictly proper and stable rational functions, and $(C, A + BF, BG)$ is internally stable, i.e. $(A + BF)$ is stable?

The Morgan problem has a long history going back into sixtieths when Morgan [13] gave the first precise definition of the problem. We do not want to review all the results concerning the problem; the reader is referred to the work of Morse and Wonham [14] for a more detailed historical background. But still it is necessary to mention at least the work of Falb and Wolovich [6] where necessary and sufficient conditions for solvability of decoupling in the case of square systems are given. Morse and Wonham [14] used the geometric approach to tackle the block decoupling problem, Descusse et al [2] established explicit necessary and sufficient conditions in the case of the so-called shifted systems and Zagalak et al [17] gave implicit necessary and sufficient conditions for there to exist a solution to the Morgan problem as defined above, where $w_i(s) = s^{-r_i}$ with $r_i \geq n_{ie}$, $i = 1, 2, \dots, p$, $n_{ie}$'s being the essential orders of (C,A,B). Thus the Morgan problem remains unsolved in its most general case.

The main purpose of this paper is to tackle the Morgan problem with stability. To this end, we shall generalize the approach of Zagalak et al [17] including the stability issue and we shall also show why it is so difficult to establish explicit necessary and sufficient conditions of solvability in this case.

As far as the notation of the paper is concerned , the symbols we use are defined where needed. Some of them, however, are used without extra definition, namely the symbols $:=$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}(s)$ and $\mathbb{R}[s]$ standing for the defining equality, the fields of real numbers, complex numbers, rational functions over $\mathbb{R}$, and the ring of polynomials over $\mathbb{R}$.

## 2. BACKGROUND

As we are going to deal with the stability issue, we introduce first some basic concepts concerning the ring of proper and stable rational functions over $\mathbb{R}$, hereafter denoted

$\mathbb{R}_{ps}(s)$, the basic mathematical tool we shall use. For this and other purposes, we define first the subset $\mathbb{R}_\Lambda(s)$ of the field $\mathbb{R}(s)$ by the following property. $\mathbb{R}_\Lambda(s)$ consists of all $f(s) \in \mathbb{R}(s)$ having its poles in a symmetric set $\Lambda \subset \mathbb{C} \cup \{\infty\}$, i.e. $\Lambda$ contains at least one point of the real axis, and if $\alpha \in \Lambda$ then $\alpha^*$ (complex conjugate) lies in $\Lambda$, too. As a special case we define $\Lambda = \{\infty\}$.

It is well known [15] that $\mathbb{R}_\Lambda(s)$ is an Euclidean domain with the degree function (denoted by $deg\, f$, $f \in \mathbb{R}_\Lambda(s)$) defined by

$deg\, f :=$ the number of zeros of $f$ lying outside of $\Lambda$ (including those at infinity),

which means, in other words, that $\mathbb{R}_\Lambda(s)$ has similar properties as the well-known ring of polynomials $\mathbb{R}[s]$. In fact, $\mathbb{R}[s]$ is a special case of $\mathbb{R}_\Lambda(s)$ for $\Lambda = \{\infty\}$. Now the ring $\mathbb{R}_{ps}(s)$ is just the ring $\mathbb{R}_\Lambda(s)$ with $\Lambda = \mathbb{C}^-$, where $\mathbb{C}^-$ denotes the open left half complex plane. In some literature, the symmetric set is called Hurwitz and enables us to introduce a fairly general notion of stability. We say that $f \in \mathbb{R}(s)$ is $\Lambda$-stable if $f \in \mathbb{R}_\Lambda(s)$.

Another important concept is the notion of a unit of $\mathbb{R}_\Lambda(s)$, which is defined as follows

$h \in \mathbb{R}_\Lambda(s)$ is a unit of $\mathbb{R}_\Lambda(s)$ if the inverse of $h$ belongs to $\mathbb{R}_\Lambda(s)$.

As an immediate consequence of this definition, we have that $h \in \mathbb{R}_\Lambda(s)$ is a unit of $\mathbb{R}_\Lambda(s)$ if and only if $deg\, h = 0$. The set of all units of $\mathbb{R}_\Lambda(s)$ is a multiplicative subgroup of $\mathbb{R}_\Lambda(s)$. There are many other interesting properties of the ring $\mathbb{R}_\Lambda(s)$ and the reader is referred to [16] for more details.

Going back to the ring $\mathbb{R}_{ps}(s)$, the subsequent proposition summarizes some important facts about $\mathbb{R}_{ps}(s)$ (which can easily be generalized to $\mathbb{R}_\Lambda(s)$, too).

**Proposition 1.** Let the ring $\mathbb{R}_{ps}(s)$ be given. Then

(i) Any $f \in \mathbb{R}_{ps}(s)$ can be written in the form

$$f = h_f\, \frac{a_f}{\pi^{k_f}}$$

where $k_f := deg\, f$, $h_f$ is a unit of $\mathbb{R}_{ps}(s)$, $a_f \in \mathbb{R}[s]$ is an antistable polynomial (i.e. $a_f(\alpha) = 0 \Rightarrow Re\, \alpha \geq 0$), $\pi := (s + \beta)$, $\beta \in \mathbb{C}$ & $Re\, \beta > 0$. We say that $f$ is monic if $h_f = 1$.

(ii) Given $f, g \in \mathbb{R}_{ps}(s), g \neq 0$, we say that $g$ divides $f$ ($g \leq f$) if there exists $x \in \mathbb{R}_{ps}(s)$ such that $f = gx$.
Using the above notation, $g \leq f$ if and only if $a_f = a_g w$ for some $w \in \mathbb{R}[s]$ and $k_f \geq k_g + deg\, w$.

(iii) (Algorithm of division) Given $f, g \in \mathbb{R}_{ps}(s)$, $g \neq 0$, then there exist $q$, $r \in \mathbb{R}_{ps}(s)$ such that

$$f = gq + r$$

with $deg\ r, < deg\ g$ (or $r = 0$.)
(Notice that the division algorithm is defined for any two elements $f, g \in \mathbb{R}_{ps}(s), g \neq 0$.)

(iv) Any $r \in \mathbb{R}(s)$ can be written in the form $r = f/g,\ f, g \in \mathbb{R}_{ps}(s)$.

(v) A common divisor of any two elements $f, g \in \mathbb{R}_{ps}(s)$ is defined as an element $b \in \mathbb{R}_{ps}(s)$ such that $b \leq f$ and $b \leq g$. The greatest common divisor of $f$ and $g$ ($gcd(f, g)$) is an element $d \in \mathbb{R}_{ps}(s)$ such that $b \leq d$ where $b$ is a common divisor of $f$ and $g$. Clearly, we have

$$deg\ d \leq \min(deg\ f,\ deg\ g)$$

for $f, g \neq 0$.

As far as the matrices over $\mathbb{R}_{ps}(s)$ are concerned, the well-known properties of polynomial matrices will hold, with more or less obvious changes, in this case, too. We shall briefly describe those of them which will be subsequently used.

A matrix $B(s) \in \mathbb{R}_{ps}^{m\times m}(s)$ is called $\mathbb{R}_{ps}^{m\times m}(s)$-unimodular (or, shortly, $\mathbb{R}_{ps}(s)$-unimodular, biproper and bistable) if $B^{-1}(s)$ exists and $B^{-1}(s) \in \mathbb{R}_{ps}^{m\times m}(s)$, i.e. $\det B(s)$ is a unit of $\mathbb{R}_{ps}(s)$. The biproper and bistable matrices are units of the ring $\mathbb{R}_{ps}^{m\times m}(s)$ and are generated by a sequence of row or column elementary operations performed on $I_m$ over $\mathbb{R}_{ps}(s)$. (These operations are defined similarly as those over $\mathbb{R}[s]$, i.e. there are three types of them, namely

1. interchanging two rows (columns)
2. multiplying a row (column) by a unit of $\mathbb{R}_{ps}(s)$
3. multiplying the row (column) $i$ by a $g \in \mathbb{R}_{ps}(s), g \neq 0$, and adding it to the row (column) $j$, $i \neq j$).

**Proposition 2.** Given $T(s) \in \mathbb{R}_{ps}^{p\times m}(s)$ of *rank* $p$, there exists an $\mathbb{R}_{ps}(s)$-unimodular matrix $B(s)$ such that

$$T(s)\, B(s) = [\Phi_s^{-1}(s),\ 0], \tag{5}$$

$$\Phi_s^{-1}(s) := \begin{bmatrix} \varphi_{11}(s) & & (0) & \vdots & \\ \vdots & \ddots & & \vdots & (0) \\ \varphi_{p1} & \dots & \varphi_{pp}(s) & \vdots & \end{bmatrix}, \tag{6}$$

where

$$\begin{aligned} &\varphi_{ij}(s) \in \mathbb{R}_{ps}(s) \\ &\varphi_{ii}(s) := \tfrac{\epsilon_i(s)}{\pi^{k_i}}, \quad \epsilon_i(s) \in \mathbb{R}[s] \text{ is antistable} \\ &\varphi_{ij}(s) := \tfrac{a_{ij}(s)}{\pi^{k_{ij}}}, \quad a_{ij}(s) \in \mathbb{R}[s], \text{ and for } i > j,\ \ k_{ij} < k_i,\ j = 1, 2, \dots, i-1. \end{aligned} \tag{7}$$

The above introduced form is called the (right generalized, right $\pi$-) Hermite form of $T(s)$. This form depends on the term $\pi$; for a fixed $\pi$ it is unique. The matrix $\Phi_s(s)$ is called the (generalized, $\pi$-) interactor of $T(s)$.
Let $q_i$ denote the infinite zero order of the row $i$ of $T(s)$ and $z_i$ be the number of unstable zeros (with multiplicities included) of the same row of $T(s)$. Then, as $B(s)$ in (5) does not change these quantities, the numbers $q_i$ and $z_i$ are given by the greatest common divisors of $\varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{ii}$. Thus, if $g_i := gcd(\varphi_{i1}, \ldots, \varphi_{ii})$, we have

$$q_i + z_i = deg\, g_i. \tag{8}$$

The number $z_i$ is also called [12] the content of the unstable zeros of the row $i$ of $T(s)$.

Using biproper and bistable matrices on both sides of $T(s)$, we can obtain the (generalized, $\pi$-) Smith form of $T(s)$, which again resembles that known from the theory of polynomial matrices.

**Proposition 3.** Given $P(s) \in \mathbb{R}_{ps}^{m\times n}(s)$ of *rank* $r$, then there exist biproper and bistable matrices $U(s)$ and $V(s)$ such that

$$P(s) = U(s) \begin{bmatrix} S_\pi(s) & 0 \\ 0 & 0 \end{bmatrix} V(s) \tag{9}$$

$$S_\pi(s) := diag\{f_1, \ldots, f_r\} \tag{10}$$

where $f_1 \leq f_2 \leq \cdots \leq f_r$ are unique (for a given $\pi$) monic elements of $\mathbb{R}_{ps}(s)$ of the form

$$f_i = \frac{a_i}{\pi^{k_i}}, \qquad i = 1, 2, \ldots, r \tag{11}$$

with $a_i$ being antistable.

The $\pi$-Smith form is an important tool when investigating the structure at infinity and the structure of unstable zeros of the system $(C, A, B)$. Indeed, bringing $T(s)$ into its $\pi$-Smith form, the numbers

$$n_i := k_i - deg\, a_i, \qquad i = 1, \ldots, p \tag{12}$$

define the infinite zero orders of $(C, A, B)$, while the polynomials $a_1 \leq a_2 \leq \cdots \leq a_p$ define the structure of the unstable zeros of $(C, A, B)$. The sums $C_\infty := \sum_{i=1}^{p} n_i$ and $C_+ := \sum_{i=1}^{p} deg\, a_i$ are called the content of $(C, A, B)$ at infinity and the content of the unstable zeros of $(C, A, B)$, respectively, elsewhere [12].

Generalizing the concept of a biproper and bistable matrix to $\mathbb{R}_\Lambda^{m\times m}(s)$, we can define a $\mathbb{R}_\Lambda^{m\times m}(s)$-unimodular matrix as a matrix $U(s)$ whose determinant is a unit of $\mathbb{R}_\Lambda(s)$. This enables us to define the following kinds of equivalences.

Let $\Lambda$ be fixed, $P(s)$, $Q(s) \in \mathbb{R}^{m\times n}(s)$, and let $U(s)$ be $\mathbb{R}_\Lambda^{m\times m}(s)$-unimodular and $V(s)$ be $\mathbb{R}_\Lambda^{n\times n}(s)$-unimodular. Then we shall say that

I. $P(s)$ is left equivalent to $Q(s)$ with respect to $\mathbb{R}_\Lambda(s)$ if $P(s) = U(s)\,Q(s)$,
II. $P(s)$ is right equivalent to $Q(s)$ with respect to $\mathbb{R}_\Lambda(s)$ if $P(s) = Q(s)\,V(s)$,
III. $P(s)$ is equivalent to $Q(s)$ with respect to $\mathbb{R}_\Lambda(s)$ if $P(s) = U(s)\,Q(s)\,V(s)$
(or shortly, $\mathbb{R}_\Lambda(s)$-(left/right) equivalent).

For example, if $\Lambda = \mathbb{C}$ then $\mathbb{R}_\Lambda(s)$ is the ring of proper rational functions $\mathbb{R}_p(s)$. Then $T(s) \in \mathbb{R}_p^{p\times m}(s)$, $rank\, T(s) = p$, is $\mathbb{R}_p(s)$-equivalent to the well-known Smith–McMillan form of $T(s)$ at infinity,

$$\begin{aligned} &T(s) \sim [S_\infty,\, 0], \\ &S_\infty := diag\{s^{-n_i}\}_{i=1}^{p} \end{aligned} \tag{13}$$

where the integers $0 < n_1 \le n_2 \le \cdots \le n_p$ are the infinite zero orders of $T(s)$.

If we put $\Lambda = \{\infty\}$ then $\mathbb{R}_\Lambda(s)$ is just the ring of polynomials $\mathbb{R}[s]$ and we get the Smith–McMillan form of $T(s)$, i.e.

$$\begin{aligned} &T(s) = U(s)\,[S_M,\, 0]V(s), \\ &S_M := diag\{\tfrac{\theta_i}{\psi_i}\}_{i=1}^{p} \end{aligned} \tag{14}$$

where $U(s)$, $V(s)$ are unimodular over $\mathbb{R}[s]$, $\theta_i$, $\psi_i \in \mathbb{R}[s]$, $i = 1,2,\ldots,p$, and $\theta_1 \le \theta_2 \le \cdots \le \theta_p$, $\psi_1 \ge \psi_2 \ge \cdots \ge \psi_p$. Based on (14) we can also write $S_M$ in the form

$$S_M(s) = \Theta(s)\,\Psi^{-1}(s)$$

where $\Theta(s) := diag\{\theta_i\}_{i=1}^{p}$, $\Psi(s) := diag\{\psi_i\}_{i=1}^{p}$.

The Smith–McMillan form reveals the so-called finite zero structure of $T(s)$, which is a synonym for the set of the elementary divisors of $\Theta(s)$. Similarly, we define the finite pole structure of $T(s)$ as the set of the elementary divisors of $\Psi(s)$.

If we consider the same structures with respect to the system $(C, A, B)$, then they represent what is usually known as the transmission structures. However, when considering the infinite zero structure, which is defined via the Smith–McMillan form at infinity, then this structure of $T(s)$ and that of $(C, A, B)$ coincide.

The aforementioned unstable zero structure of $T(s)$, given by the antistable polynomials $a_i$, $i = 1,2,\ldots,p$, in (11) is of course nothing else than a substructure of the finite zero structure of $T(s)$ defined by $\theta_i$'s in (14) since $a_i \le \theta_i$, $i = 1,2,\ldots,p$. The unstable pole structure is defined in a similar way.

## 3. MAIN RESULTS

The relationship (4) is not very convenient for getting the matrices $F$ and $G$. Actually, this is not an equation since the functions $w_i(s)$'s are not known yet. However, we shall treat (4) as an equation and specify the functions $w_i(s)$ later on. Thus, write (4) in the form

$$T(s)\,C(s) = W(s) \tag{15}$$

where $C(s) := B(s)\,G$ with $B(s) := [I_m - F N_1(s)\,D^{-1}(s)]^{-1}$. The matrix $B(s)$ is clearly biproper ($\mathbb{R}_p(s)$-unimodular), which means that $C(s)$ has a left proper inverse. Such matrices will be called column biproper (column or right $\mathbb{R}_p(s)$-unimodular).

The main idea is to solve the equation (15) over the ring $\mathbb{R}_{ps}(s)$. This will guarantee the internal stability of $T(s)\,C(s)$ since all operations performed in this ring keep both the unstable pole and zero structures unchanged. There is no loss of generality if we suppose that $(C, A, B)$ is a stable system. If not, a preliminary stabilizing feedback $u(t) = Fx(t) + v(t)$ can be used (since $(C, A, B)$ is controllable) to ensure its stability. Then, $T(s)$ is a stable matrix and $C(s)$ is right $\mathbb{R}_{ps}(s)$-unimodular, or column biproper and bistable. The stability of $T(s)$ also implies that $D(s)$ has no unstable zero structure, i.e. $D^{-1}(s) \in \mathbb{R}_{ps}^{m\times m}(s)$.

Now, as *rank* $N(s) = p$, there exists a right inverse of $N(s)$, say $\hat{N}(s)$, which can be written in the form

$$\hat{N}(s) = \hat{N}_0(s) + \hat{N}_k(s)\,T_p(s) \tag{16}$$

where $\hat{N}_0(s)$ is a particular inverse and $\hat{N}_k(s)$ forms a basis for *Ker* $N(s)$. $T_p(s)$ stands for a rational parameter. Using (16), the equation (15) can be rewritten into the form

$$M(s)\,[\hat{N}_0(s) + \hat{N}_k(s)\,T_p(s)]\,W(s) = Z \tag{17}$$

where

$$M(s) = XD(s) + YN_1(s) \tag{18}$$

with $X \in \mathbb{R}^{m\times m}$ nonsingular, $Y \in \mathbb{R}^{m\times n}$, and $Z \in \mathbb{R}^{m\times p}$ monic such that

$$F = -X^{-1}Y \quad \text{and} \quad G = X^{-1}Z. \tag{19}$$

The relationships (17), (18) and (19) represent a basic formulation of the Morgan problem. Once we find $W(s)$, a monic $Z$, and $M(s)$, $M^{-1}(s) \in \mathbb{R}_{ps}^{m\times m}(s)$, that is column reduced with the same column degrees as $D(s)$, we are able to compute $F$ and $G$ using (18) and (19); see [10] for more details. The problem is how to find $M(s)$ and $W(s)$. To this end we shall use the concept of the extended system [17].

Let $U(s)$ be $\mathbb{R}[s]$-unimodular such that

$$N(s)\,U(s) = [Q(s),\ 0] \tag{20}$$

with a nonsingular $Q(s) \in \mathbb{R}^{p\times p}[s]$, and define

$$K(s) := \begin{bmatrix} Q(s) & 0 \\ 0 & I_{m-p} \end{bmatrix} U^{-1}(s). \tag{21}$$

It was shown in [17] that $U(s)$ can be chosen such that

$$T_e(s) = K(s)\,D^{-1}(s)$$

is strictly proper and, as $D^{-1}(s) \in \mathbb{R}_{ps}^{m\times m}(s)$, it follows that $T_e(s) \in \mathbb{R}_{ps}^{m\times m}(s)$, too.

The matrix $T_e(s)$ is called an extension of $T(s)$ and a realization of $T_e(s)$, with the same order as $(C, A, B)$, is called an extended system.

**Lemma 1.** Let $\Phi_e^{-1}(s)$ be the right $\pi$-Hermite form, see (5), of $T_e(s)$, i.e.

$$T_e(s)\,B_e(s) = \Phi_e^{-1}(s) \tag{22}$$

for some $\mathbb{R}_{ps}$-unimodular matrix $B_e(s)$. Then the matrix $\Phi_e(s)$ is called the extended $\pi$-interactor of $T(s)$ and is of the form

$$\Phi_e(s) = \begin{bmatrix} \Phi_1(s) & 0 \\ \Phi_2(s) & \Phi_3(s) \end{bmatrix}$$

where $\Phi_1(s)$ is the $\pi$-interactor of $T(s)$ and $\Phi_3(s) = diag\{\pi^{\sigma_i}\}_{i=1}^{m-p}$ where the list $\{\sigma_i\}_{i=1}^{m-p}$ is the Morse list $\mathbb{I}_2$, which is defined as the set of column minimal indices of the system matrix

$$P(s) := \begin{bmatrix} sI_n - A & -B \\ C & 0 \end{bmatrix}$$

Proof. Let $U(s)$ be the same as in (20) and define

$$[D_1(s),\, D_2(s)] := D(s)\,U(s)$$

$$[N_{11}(s),\, N_{12}(s)] := N_1(s)\,U(s).$$

Then

$$\begin{bmatrix} sI_n - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ D_1(s) & D_2(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ Q(s) & 0 \end{bmatrix}$$

which means that

$$\begin{bmatrix} N_{12}(s) \\ D_2(s) \end{bmatrix}$$

forms a basis for *Ker* $P(s)$. Making $D_2(s)$ column reduced, it implies that the basis is minimal. Hence, $\sigma_i = deg_{ci} D_2(s)$, $i = 1, 2, \dots, m-p$, form the Morse list $\mathbb{I}_2$.

Now, from (22), we have

$$\Phi_e(s)K(s) = B_e^{-1}(s)\,D(s). \tag{23}$$

The matrix $B_e^{-1}(s)\,D(s)$ is clearly stable, which means that the matrix $\Phi_e(s)K(s)$ is stable, too. On the other hand, $\Phi_e(s)$ is a rational matrix having just unstable poles, and $K(s)$ is polynomial. This means that all the unstable poles of $\Phi_e(s)$ have to be canceled out and hence, the product $\Phi_e(s)K(s)$ is polynomial. Thus, $B_e^{-1}(s)\,D(s)$ is polynomial, too. Moreover, $deg_{ci}\, B_e^{-1}(s)\,D(s) = c_i$, $i = 1, 2, \dots, m$ since $B_e^{-1}(s)$ is biproper.

Next, postmultiplying (23) by $U(s)$, it follows that

$$deg_{ci}\, \Phi_3(s) = deg_{ci}\, D_2(s) = \sigma_i, \qquad i = 1, 2, \dots, m-p.$$

The diagonality of $\Phi_3(s)$ follows from the fact that $D_2(s)$ is column reduced. □

Now the assumption $\mathcal{V}^* = \mathcal{R}^*$ should become clearer. The role of the extended system is to reveal $\mathcal{R}^*$, the hidden part of $(C, A, B)$, and use it for decoupling.

Going back to the equation (17), the characterization of $M(s)$ seems to be much easier. Indeed, write first $\hat{N}(s)$ in the form

$$\hat{N}(s) = K^{-1}(s)\,[\bar{N}_0(s) + \bar{N}_k T_p(s)],$$

where

$$\bar{N}_0 = \begin{bmatrix} I_p \\ 0 \end{bmatrix}, \quad \text{and} \quad \bar{N}_k = \begin{bmatrix} 0 \\ I_{m-p} \end{bmatrix}$$

(see [17] for details), to get

$$M(s)K^{-1}(s)\,[\bar{N}_0 + \bar{N}_k T_p(s)]\,W(s) = Z. \tag{24}$$

Noticing further that

$$M(s) = [X + Y N_1(s)\,D^{-1}(s)]\,D(s),$$

where the matrix $[X + Y N_1(s)\,D^{-1}(s)]$ is $\mathbb{R}_{ps}(s)$-unimodular, we can write $M(s)$ in the form

$$M(s) = V(s)\Phi_e(s)K(s) \tag{25}$$

with a biproper and bistable $V(s)$ yielding $M(s)$ polynomial. Substituting (25) into (24) we arrive at the equation

$$V(s)\begin{bmatrix} \Phi_1(s)\,W(s) \\ \Phi_2(s)\,W(s) + \Phi_3(s)\,T_p(s) \end{bmatrix} = Z \tag{26}$$

with the constraint

$$V(s)\Phi_e(s)K(s) \in \mathbb{R}^{m\times m}[s] \tag{27}$$

The relationship (26) is nothing else than the Bezout identity for matrices over $\mathbb{R}_{ps}(s)$. Thus, if we find a diagonal matrix $W(s) \in \mathbb{R}_{ps}^{p\times p}(s)$ and a rational matrix $T_p(s)$ such that

$$Y(s) := \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix},$$

where $Y_1(s) := \Phi_1(s)\,W(s)$, $Y_2(s) := \Phi_2(s)\,W(s) + \Phi_3(s)\,T_p(s)$, is right $\mathbb{R}_{ps}(s)$-unimodular, then $V(s)$ will be $\mathbb{R}_{ps}(s)$-unimodular and (26) will hold. To achieve that, consider first the interactor $\Phi_1(s)$. We can factorize it as

$$\Phi_1(s) = \Gamma(s)\,diag\left\{\frac{1}{h_i(s)}\right\}_{i=1}^{p} \tag{28}$$

where $h_i(s) \in \mathbb{R}_{ps}(s)$, $i = 1, 2, \ldots, p$, and $\Gamma(s) \in \mathbb{R}_{ps}^{p\times p}(s)$. Let $h_i(s)$, $i = 1, 2, \ldots, p$ be the lowest-possible-degree functions satisfying (28). Certainly such $h_i(s)$'s exist and are unique up to units of $\mathbb{R}_{ps}(s)$. We shall call the degrees of $h_i(s)$ (which are indeed unique) the s-essential orders of $(C, A, B)$ and they will be denoted as $n_{ie,s}$.

The $s$-essential orders determine the lower bounds for the degrees of the functions $w_i(s) \in \mathbb{R}_{ps}(s)$ of $W(s)$ for which $\Phi_1(s)\,W(s) \in \mathbb{R}_{ps}^{p\times p}(s)$. Good candidates for such $w_i(s)$'s are just the functions $h_i(s)$'s, or the functions $w_i(s)$'s satisfying

$$h_i(s) \leq w_i(s), \qquad i = 1, 2, \ldots, p. \tag{29}$$

Without loss of generality we shall consider that all these functions are monic.

We shall say that $W(s) = diag\{w_i(s)\}_{i=1}^{p}$ is an admissible matrix if $w_i(s)$, $i = 1, 2, \ldots, p$ satisfy the conditions (29).

The problem now is if there exists a rational parameter $T_p(s)$ such that $Y(s)$ is right $\mathbb{R}_{ps}(s)$-unimodular.

**Lemma 2.** There exists $T_p(s) \in \mathbb{R}^{(m-p)\times p}(s)$ such that $Y(s)$ is right $\mathbb{R}_{ps}(s)$-unimodular if and only if there exists an admissible $W(s)$ such that the number of the unit invariant factors of $\Phi_1(s)\,W(s)$, say $k$, satisfies

$$m \geq 2p - k \tag{30}$$

Proof. (Neccesity.) Let $Y(s)$ be right $\mathbb{R}_{ps}(s)$-unimodular. Then $\Phi_1(s)\,W(s) \in \mathbb{R}_{ps}^{p\times p}(s)$, which implies that $W(s)$ is admissible. Thus, we have to show that $m \geq 2p - k$. To that end, let $B_1(s)$ and $B_2(s)$ be $\mathbb{R}_{ps}(s)$-unimodular matrices that bring $Y_1(s)$ into its $\pi$-Smith form, i.e.

$$\begin{bmatrix} B_1(s) & 0 \\ 0 & I_{m-p} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} B_2(s) = \begin{bmatrix} S_\pi(s) \\ Y_2(s)\,B_2(s) \end{bmatrix}.$$

Without any loss of generality, we may suppose that

$$S_\pi(s) = \begin{bmatrix} S_1(s) & 0 \\ 0 & S_2(s) \end{bmatrix}$$

where $S_1(s) := I_k$ and $S_2(s)$ contains the nonunit factors. Partition $Y_2(s)\,B_2(s)$ as $[L_1(s)\;\; L_2(s)]$, compatible with $S_\pi(s)$. This implies that $\begin{bmatrix} S_2(s) \\ L_2(s) \end{bmatrix}$ is right $\mathbb{R}_{ps}(s)$-unimodular. The claim now easily follows. Since $rank\, S_2(s) = p - k$ and $L_2(s)$, which is of the dimensions $(m - p) \times (p - k)$, has $p - k$ unit invariant factors, it follows that $m - p \geq p - k$.

(Sufficiency.) Put

$$T_p(s) = \Phi_3^{-1}(s)\,[E B_2^{-1}(s) - \Phi_2(s)\,W(s)]$$

where

$E = [0\ E_0]$, $\quad E_0 = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \in \mathbb{R}^{(m-p)\times(p-k)}$, and $E_1 \in \mathbb{R}^{(p-k)\times(p-k)}$ is nonsingular.

Then
$$\begin{bmatrix} S_\pi(s) \\ E \end{bmatrix} = \begin{bmatrix} B_1(s) & 0 \\ 0 & I_{m-p} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} B_2(s)$$
is clearly right $\mathbb{R}_{ps}(s)$-unimodular and the claim follows. □

It is to be noted that the above matrix $E_1$ is not the only one we can choose. Generally, for $E_1$ we can take any stable matrix such that $S_2(s)$ and $E_1$ are right coprime.

Once the existence of $T_p(s)$ such that (26) holds is guaranteed, we must consider the constraint (27), too. The first observation we can do is that the matrix $\Phi_e(s)K(s)$ can be written in the form

$$\Phi_e(s)K(s) = B_0(\pi)\, diag\{\pi^{c_i}\}_{i=1}^m \tag{31}$$

where $c_i$, $i = 1, 2, \ldots, m$, are the controllability indices of $(C, A, B)$. This immediately follows from using the transformation $s = \pi - \beta$ and from the fact that $\Phi_e(s)K(s)$ is column reduced with column degrees $c_i$, $i = 1, 2, \cdots, m$. The matrix $B_0(\pi)$ is a $\mathbb{R}_p(\pi)$-unimodular matrix with entries of the form $\frac{a_{ij}(\pi)}{\pi^{\alpha_{ij}}}$ (named as the $\pi$-form) where $a_{ij} \in \mathbb{R}[\pi]$.

Next, considering monic functions $w_i(s)$ and the entries of $Y_2(s)$ in the form $\frac{b_{ij}(s)}{\pi^{\gamma_{ij}}}$, they can always be chosen in these forms, we can bring all the elements in the $\pi$-form. Then the entries of $V$ will also be in the $\pi$-form. Hence, the product $V'(\pi) = V(\pi)\, B_0(\pi)$ is $\mathbb{R}_p(\pi)$-unimodular having all its entries in the $\pi$-form $\frac{f_{ij}(\pi)}{\pi^{d_{ij}}}$.

Let $d_i := \max\{d_{ji},\ j = 1, 2, \ldots, m\}$. Then, the polynomiality condition (27) reduces to the form
$$d_i \le c_i, \quad i = 1, 2, \ldots, m. \tag{32}$$

Based on (32) we can state the following.

**Theorem 1.** Given a right invertible, stable, and controllable system $(C, A, B)$, then (with the above notation) there exists a solution to MPS if and only if there exists an admissible $W(s)$ such that (30) and (32) hold.

Some remarks are in order now.

**Remark 1.** Theorem 1 parallels that stated in [17], which is valid for the case without stability. This fact is not surprising since the basic algebraic properties of the rings $\mathbb{R}_p(s)$ and $\mathbb{R}_{ps}(s)$ are the same.

The most difficult question here is the problem of a realization of a given precompensator $C(s)$, which is right $\mathbb{R}_{ps}(s)$-unimodular (or right $\mathbb{R}_p(s)$-unimodular if we consider the Morgan problem without stability), by state feedback (3). Consider the polynomiality condition (27). This condition is equivalent to the existence of some constant matrices $\bar{X} \in \mathbb{R}^{m\times m}$, $rank\,\bar{X} = m$, and $\bar{Y} \in \mathbb{R}^{m\times n}$ such that

$$[\bar{X} + \bar{Y} N_1(s)\, D_\phi^{-1}(s)] = V(s) \tag{33}$$

where $D_\phi(s) := \Phi_e(s) K(s)$. Substituting now (33) into (26), we get

$$[\bar{X} + \bar{Y} N_1(s)\, D_\phi^{-1}(s)]\, Y(s) = \begin{bmatrix} I_p \\ 0 \end{bmatrix},$$

which decomposes as

$$\bar{X} Y_0 = \begin{bmatrix} I_p \\ 0 \end{bmatrix} \tag{34}$$

and

$$[\bar{Y} N_1(s)\, D_\phi^{-1}(s)]\, Y(s) = -\bar{X}\hat{Y}(s) \tag{35}$$

where $Y_0$ denotes the constant part of $Y(s)$, i.e. $Y_0 := \lim_{s\to\infty} Y(s)$, and $\hat{Y}(s)$ is the strictly proper part of $Y(s)$, $\hat{Y}(s) = Y(s) - Y_0$.

The equations (34) and (35) describe just the situation in which the compensator $Y(s)$ is realizable by the static state feedback (3) if we put $F = -X^{-1}Y$ and $G = X^{-1}\begin{bmatrix} I_p \\ 0 \end{bmatrix}$; see [7] for details.

**Remark 2.** Lemma 2 guarantees, for an admissible $W(s)$, the existence of $T_p(s)$ such that (26) holds if (30) is satisfied. However, the construction of $V(s)$ satisfying (27) is not clear. What one should do is to construct first a right $\mathbb{R}_{ps}(s)$-unimodular $Y(s)$ and $\mathbb{R}_{ps}(s)$-unimodular $V(s)$ satisfying (26). Then, the equation (35) can be used to check if $V(s)$ satisfies (27). If not, then $Y(s)$ and $V(s)$ must be reconstructed again. Unfortunately, no explicit procedure for obtaining $Y(s)$ and $V(s)$ satisfying both (26) and (27) is available yet and this calls for further study.

**Remark 3.** As far as decoupling with stability by dynamic state feedback is concerned, the approach developed in this paper, which works within the ring $\mathbb{R}_{ps}(s)$, leads naturally to the so-called "strong stability" case where both the feedback system and the feedback compensator $F(s)$ are required to be stable. As a result, the feedback term $[I_m - F(s)\, N_1(s)\, D^{-1}(s)]^{-1}$ is biproper and bistable since the system is supposed to be stable.

Necessary and sufficient conditions for the dynamic state feedback

$$u(s) = F(s)\, x(s) + Gv(s), \qquad F(s) \in \mathbb{R}_{ps}^{m\times n}(s)$$

to decouple the system $(C, A, B)$ with stability are given by (30). The procedure to find such a feedback is the following.

i) Complete $Y(s) := \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}$ to be right $\mathbb{R}_{ps}(s)$-unimodular and calculate a biproper and bistable $V(s)$ satisfying that $V(s)\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$.

ii) Construct $M(s) = V(s)\Phi_e(s)K(s)$, which is a rational and stable matrix.

iii) Premultiply $M(s)$ by a constant nonsingular matrix $X \in \mathbb{R}^{m\times m}$ such that $(XM(s))_{hc} = D_{hc}$, where $(XM(s))_{hc}$ $(D_{hc})$ denotes the highest column-degree coefficient matrix of $XM(s)$ $(D(s))$.

iv) Find a proper and stable solution to the equation

$$\bar{Y}(s)\, N_1(s) = \hat{M}(s)$$

where $\hat{M}(s) = XM(s) - D(s)$.
Observe that there always exists such a solution. For instance, write $\hat{M}(s) = M^+(s) + M^-(s)$, where $M^+(s)$ denotes the polynomial part of $\hat{M}(s)$ and $M^-(s)$ is the strictly proper and stable part of $\hat{M}(s)$. One solution is given by

$$\bar{Y}(s) = Y_0 + [M^-(s)\, R(s),\ 0]\, B_1(s)$$

where $Y_0$ is a constant solution to the equation $Y_0 N_1(s) = M^+(s)$ ($Y_0$ always exists since $deg_{ci}\, N_1(s) \geq deg_{ci}\, M^+(s)$ and $N_1(s)$ forms a polynomial basis), $R(s) \in \mathbb{R}_{ps}^{m\times m}(s)$ is a diagonal matrix such that $N_1(s)\, R(s)$ is column biproper and bistable, and $B_1(s)$ is a unit of $\mathbb{R}_{ps}^{n\times n}(s)$ such that $B_1(s)\, N_1(s)\, R(s) = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$.

v) Put $F(s) = \bar{Y}(s)$ and $G = X \begin{bmatrix} I_p \\ 0 \end{bmatrix}$.

On the other hand, if only the feedback system is required to be stable, but not necessarily $F(s)$, which is of course proper, then $[I + F(s)\, N_1(s)\, D^{-1}(s)]^{-1}$ is biproper and stable, but not bistable. The solution for decoupling by dynamic state feedback in this case ("weak stability") is the well known condition

$$m \geq 2p - k$$

where $k$ is the rank at infinity of $\Phi(s)W(s)$, and $\Phi(s)$ is the system interactor (see [3]).

The previously indicated procedure can also be applied here, with the difference that $V(s)$ is a unit of $\mathbb{R}_p^{m\times m}(s)$ and $\bar{Y}(s)$ in step iv) is a proper matrix (not necessarily stable).

**Remark 4.** The distinction between weak and strong stability is also evident when dynamic output feedback is used for decoupling. In this case, the control law is given by

$$u(s) = F(s)\, y(s) + Gv(s)$$

where $F(s)$ is proper and $G$ is constant.

As it can be expected, the problem here is more involved than the case of decoupling by dynamic state feedback. In order to show that, let us consider equation (35), which under dynamic output feedback becomes

$$\bar{Y}(s)\, N(s)\, D_\phi^{-1}(s)\, Y(s) = -\bar{X}\hat{Y}(s).$$

If we express $Y(s)$ as $Y(s) = \Phi_e(s) \begin{bmatrix} I \\ T_p \end{bmatrix} W(s)$ then, since $N(s)\, K^{-1}(s) = [I,\ 0]$, the above equation can be rewritten as

$$\bar{Y}(s) = -\bar{X}\hat{Y}(s)\, W^{-1}(s).$$

Since we are seeking for a proper $\bar{Y}(s)$ (or equivalently, a proper $F(s)$), the choice for $W(s)$ and $T_p(s)$ should be such that $\bar{X}\hat{Y}(s)\, W^{-1}(s)$ is proper. This condition is similar to the results given in [4]. In fact, it is equivalent to finding $G$ and $W(s)$ such that

$$\Phi_1(s) - \Phi_1(s)\, T(s)\, G W^{-1}(s) \tag{36}$$

is proper. The details for the construction of $G$ and $W(s)$ are given in [5].

If the same problem is considered under the constraint of weak stability, then it is necessary and sufficient that (36) holds and $F(s)[I - T(s)\, F(s)]^{-1}$ is proper and stable. If we let $F(s) := [Y_0 - B_e(s)\, Y(s)]\, W^{-1}(s)$, then the latter condition becomes

$$F(s)\,[I - T(s)\, F(s)]^{-1} = [Y_0 - B_e(s)\, Y(s)]\,[T(s)\, Y_0]^{-1}$$

as $T(s)\, B_e(s)\, Y(s) = W(s)$. Thus, $T_p(s)$ and $W(s)$ are to be chosen such that $[Y_0 - B_e(s)\, Y(s)]\,[T(s)\, Y_0]^{-1}$ is proper and stable. It can be shown that this is equivalent to

$$\Phi_1(s) - \Phi_1(s)\, W(s)\,[T(s)\, G]^{-1}$$

being proper and stable (see [1]).

If the stability of $F(s)$ is further required (the so-called strong stability case), then the necessary and sufficient condition for the solvability of this problem is the existence of $G$ and $W(s)$ such that both

$$\Phi_1(s) - \Phi_1(s)\, T(s)\, G W^{-1}(s)$$

and

$$\Phi_1(s) - \Phi_1(s)\, W(s)\,[T(s)\, G]^{-1}$$

are proper and stable. For the details the reader is again referred to [1].

**Remark 5.** If the regular static state feedback $u(t) = Fx(t) + Gv(t)$, $G \in \mathbb{R}^{m\times m}$, *rank* $G = m$, is used for decoupling, we can derive explicit necessary and sufficient conditions for the existence of a solution. To this end, notice that the nonsingularity of $G$ implies that $\Phi_1(s)$ is $m \times m$, which means that $\Phi_1(s)\, W(s)$ must be $\mathbb{R}_{ps}(s)$-unimodular. From that, $\Phi_1(s)$ is diagonal, we conclude. Thus, the proper and stable rational functions $h_i(s)$ in (28) are just the inverses of the diagonal terms of $\Phi_1(s)$, i.e.

$$h_i(s) = \frac{a_i(s)}{\pi^{n_{ie,s}}}, \quad i = 1, 2, \dots, m \tag{37}$$

where $a_i(s) \in \mathbb{R}[s]$, $i = 1, 2, \dots, m$, are antistable polynomials. As the matrix $N(s) \in \mathbb{R}^{m\times m}[s]$, with $rank\, N(s) = m$, the product $\Phi_1(s)\, N(s) \in \mathbb{R}^{m\times m}[s]$, which means that all unstable zeros of $N(s)$ are canceled out. Hence, the polynomials $a_i(s)$ describe the same unstable zero structure as the unstable parts of the invariant factors of $N(s)$. We have proved the following result.

**Theorem 2.** A square, invertible and stable system $(C, A, B)$ is decouplable with stability by a regular static state feedback if and only if its associated $\pi$-interactor $\Phi(s)$ is diagonal.

The problem of decoupling a system with stability by regular static state feedback is also solved in [12] using a geometric approach. The authors provide necessary and sufficient conditions for this problem in terms of the infinite and unstable contents of the system. The following result shows that Theorem 2 of this paper and Theorem 6 of [12] are equivalent.

**Theorem 3.** The $\pi$-interactor of the system $(C, A, B)$ is diagonal if and only if the following two conditions hold

$$C_\infty(C, A, B) = \sum_{i=1}^{p} C_\infty(c_i, A, B) \tag{38}$$

and

$$C_+(C, A, B) = \sum_{i=1}^{p} C_+(c_i, A, B). \tag{39}$$

Proof. (Necessity.) From (5) and (9) it follows that

$$deg[\det\, S_\pi(s)] = deg[\det\, \Phi_s^{-1}(s)]. \tag{40}$$

Then, supposing that $\Phi_s(s)$ is diagonal, we have

$$\sum_{i=1}^{p}(n_i + r_i) = \sum_{i=1}^{p}(q_i + z_i) \tag{41}$$

where $n_i$ ($q_i$) are the infinite (row-infinite) zero orders of the system, $z_i$ is the number of row-unstable zeros, and $r_i := deg\, a_i(s)$ where $a_i(s)$ are the antistable polynomials in (11).

Since the system is decouplable with stability ($\Phi_s(s)$ is diagonal), then it is also decouplable without stability, i.e.

$$\sum_{i=1}^{p} n_i = \sum_{i=1}^{p} q_i \tag{42}$$

and, from that and (41),

$$\sum_{i=1}^{p} r_i = \sum_{i=1}^{p} z_i. \tag{43}$$

It readily follows that (42) and (43) are equivalent to (38) and (39), respectively.

(Sufficiency.) From (38) and (39), we have

$$\sum_{i=1}^{p}(n_i + r_i) = \sum_{i=1}^{p}(q_i + z_i).$$

Next, by (8),

$$q_i + z_i = deg\, g_i$$

where $g_i = gcd(\varphi_{i1}, \ldots, \varphi_{ii})$.

Then, because of the property (7), it is clear that

$$q_i + z_i \leq deg\, \varphi_{ii}(s)$$

with equality holding just if $\varphi_{ii}(s)$ is the only nonzero element in the row $i$ of $\Phi_s^{-1}(s)$.

Let us suppose that $\Phi_s^{-1}(s)$ is not diagonal. This means that

$$\sum_{i=1}^{p}(q_i + z_i) < \sum_{i=1}^{p} deg\, \varphi_{ii}(s),$$

i.e.

$$\sum_{i=1}^{p}(q_i + z_i) < \sum_{i=1}^{p}(n_i + r_i)$$

contradicting our assumption. □

## 4. CONCLUSIONS

Implicit necessary and sufficient conditions for the solution to the Morgan problem with stability are obtained. The approach used here takes advantage of the properties of $\mathbb{R}_{ps}(s)$. Connections with some other well known results are also mentioned.

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