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AN AXIOMATIZATION OF EXTENSIONAL PROBABILITY MEASURES

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Replacing the demand of countable additivity (σ-additivity), imposed on probability measures by the classical Kolmogorov axiomatic, by a stronger axiom, and considering only probability measures taking their values in the Cantor subset of the unit interval of real numbers, we obtain such an axiomatic system that each probability measure satisfying these axioms is extensional in the sense that probability values ascribed to measurable unions and intersections of measurable sets are functions of probability values ascribed to particular sets in question. Moreover, each such probability measure can be set into a one-to-one correspondence with a boolean-valued probability measure taking its values in the set of all subsets of an infinite countable space, e. g., the space of all natural numbers.

1. INTRODUCTION

Since the last two centuries probability calculus has continually played the role of the most powerful and most often used tool for uncertainty quantification and processing in various theoretical as well as practical domains of human activities. The axiomatic setting of this calculus by Kolmogorov in 1933 [3] enabled (1) to transform this calculus into a rigorous mathematical theory, (2) to escape from some philosophical and methodological difficulties, and (3) to take profit of rich and powerful mathematical apparata offered by the theory of real functions and theory of measure. Moreover, Kolmogorov probability theory conserved the main philosophical idea of all former probability calculi according to which probability values (or degrees) are related to the corresponding relative frequences of occurrences of certain random events. Formalized mathematical expressions for this relation are the well-known laws of large numbers.

However, due to the laws of large numbers the natural intensional (i. e., non-extensional) character of relative frequences has been transformed into the intensionality of probability measures. This is to say, in particular, that there do not exist binary functions F, G, taking pairs of real numbers from the unit interval

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$\langle 0, 1\rangle$ into $\langle 0, 1\rangle$ and such that for all random events A, B, for which their probabilities $P(A)$, $P(B)$ are defined, the equalities $P(A \cap B) = F(P(A), P(B))$, and $P(A \cup B) = G(P(A), P(B))$ would hold. Just as an example of an extensional calculus let us recall the classical propositional calculus, where the truthvalues of composed statements like "A and B" or "A or B" are defined by the well-known simple functions of truthvalues ascribed to A and B.

At least for two reasons the intensionality of probability measures leads to difficulties in practical applications of probability theory. First, the computations of probabilities of composed random events are of high computational complexity even if the set of marginal and conditional probabilities being at our disposal is rich enough to enable such a computation. Second, in the case when these probabilities are not known a priori and have to be estimated from a collection of statistical data or on the ground of subjective opinions of experts, it may be rather difficult to obtain good estimation of conditional probabilities with complicated and rarely occurring conditioning events. It is just because of these reasons that various extensional calculi for uncertainty quantification and processing like, e. g., fuzzy sets, have attracted the attention of specialists dealing with application of uncertainty processing methods.

Consequently, it is perhaps not beyond any interest to ask, whether there exist probability measures which possess the property of extensionality, and, if the answer is affirmative, to characterize and investigate, in more detail, the class of such probability measures. In [5] we arrived at such probability measures by an appropriate numerical encoding of extensional boolean-valued probability measures taking their values in the set of all subsets of the set $\mathcal{N} = \{1, 2, \ldots\}$ of positive integers, and we stated and proved some simple results concerning the resulting extensional numerical probability measures. In what follows, we shall show that the class of such probability measures can be defined axiomatically, in a pattern strictly following that one applied in Kolmogorov axiomatic probability theory, just with the axiom of countable additivity (σ-additivity) replaced by a stronger one.

Having repeated, for the sake of reader's convenience, the classical definition of probability measure, we introduce, in Chapter 2, a nonstandard operation of addition in the Cantor subset of the unit interval of real numbers. Strong probability measure is then defined by a verbal rewriting of the classical definition, just replacing $\langle 0, 1\rangle$ by its Cantor subset and usual addition by its nonstandard version. A specific feature of strong probability measures consisting in the fact that the used nonstandard numerical operations are very close to the usual Boolean operations is investigated in Chapter 3. Chapter 4 then shows that strong probability measures are atomic and takes profit from this fact in order to prove that strong probability measures are σ-extensional – values for finite or countable unions and intersections of random events are defined by the values ascribed by the strong probability measure in question to the particular random events. Finally, Chapter 5 proves that each classical probability measure over a finite or countable space can be defined and processed using certain uniquely defined strong probability measures.

2. STRONG PROBABILITY MEASURES

First of all, let us briefly recall and discuss the classical definition of probability measure.

Definition 1. Let Ω be a nonempty set. A nonempty system $\mathcal{A}$ of subsets of Ω is called *σ-field*, if it is closed with respect to the set-theoretic operations of complement and countable union, i.e., if for each sequence A, A_1, $A_2, \dots$ of subsets of Ω which are in $\mathcal{A}$, also the sets $\Omega - A$ and $\bigcup_{i=1}^{\infty} A_i$ are in $\mathcal{A}$, The pair $\langle \Omega, \mathcal{A} \rangle$ is called *measurable space.*

Let $\mathcal{P}(\Omega) = \{S : S \subset \Omega\}$ denote the system of all subsets of Ω, i.e., the power-set over Ω (or: generated by Ω). As can be easily seen, $\{\emptyset, \Omega\}$ as well as $\mathcal{P}(\Omega)$ itself are σ-fields of subsets of Ω, here $\emptyset$ denotes the empty subset of Ω. Moreover, the inclusions $\{\emptyset, \Omega\} \subset \mathcal{A} \subset \mathcal{P}(\Omega)$ obviously hold for each σ-field $\mathcal{A}$ of subsets of Ω.

Let $\langle 0, 1 \rangle$ denote the closed unit interval of real numbers, let $\langle x_1, x_2, \dots \rangle$ be an infinite sequence of non-negative real numbers (not necessarily from $\langle 0, 1 \rangle$). Let $\sum_{i=1}^{\infty} x_i = \lim_{n \to \infty} \sum_{i=1}^{n} x_i$, if this limit value exists (i.e., if it is finite), let $\sum_{i=1}^{\infty} x_i = \infty$ otherwise, i.e., if $\sum_{i=1}^{n} x_i$ diverges. Obviously, $\sum_{i=1}^{\infty} x_i$ is defined for each sequence $\langle x_1, x_2, \dots \rangle$ of non-negative real numbers.

$\langle A_1, A_2, \dots \rangle$ is an infinite sequence of mutually disjoint sets, if $A_i \cap A_j = \emptyset$ holds for each i, $j \in \mathcal{N} = \{1, 2, \dots\}$, $i \neq j$.

Definition 2. Let $\langle \Omega, \mathcal{A} \rangle$ be a measurable space. A real-valued function P defined on $\mathcal{A}$ is called *probability measure*, if it satisfies the following conditions:

(a) $P : \mathcal{A} \to \langle 0, 1 \rangle$, hence, $0 \leq P(A) \leq 1$ for each $A \in \mathcal{A}$,

(b) $P(\Omega) = 1$,

(c) for each infinite sequence $\langle A_1, A_2, \dots \rangle$ of mutually disjoint sets from $\mathcal{A}$ the equality $\sum_{i=1}^{\infty} P(A_i) = P\left(\bigcup_{i=1}^{\infty} A_i\right)$ holds.

Obviously, if the infinite sum $\sum_{i=1}^{\infty} x_i$ is taken as undefined in the case when $\lim_{n \to \infty} \sum_{i=1}^{n} x_i = \infty$, (c) should be replaced by

(c1) for each ...from $\mathcal{A}$, $\sum_{i=1}^{\infty} P(A_i)$ *is defined* (hence, finite) and the equality ...holds.

Due to the condition that $\sum_{i=1}^{\infty} P(A_i) = P\left(\bigcup_{i=1}^{\infty} A_i\right)$, and due to the fact that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, so that $P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq 1$ follows from (a), both the definitions are equivalent. Let us present still another equivalent variant, perhaps less intuitive, but more close to the strong modification of (c), which will lead us to the notion of strong probability measure.

Let $\langle 0, 1 \rangle^{\infty}$ denote the Cartesian product $\times_{i=1}^{\infty} H_i$, where $H_i = \langle 0, 1 \rangle$ for each $i \in \mathcal{N}$, hence, $\langle 0, 1 \rangle^{\infty}$ is the space of all infinite sequences of real numbers from the unit interval. Let $\sum^1$ be the partial mapping taking $\langle 0, 1 \rangle^{\infty}$ into $\langle 0, 1 \rangle$ and defined by $\sum^1 (\langle x_i \rangle_{i=1}^\infty) = \lim_{n\to\infty} \sum_{i=1}^n x_i = \lim_{n\to\infty}(x_1 + x_2 + \cdots + x_n)$ supposing that this limit value is in $\langle 0,1\rangle$, $\sum^1 (\langle x_i \rangle_{i=1}^\infty)$ being *undefined* otherwise, here $\langle x_i \rangle_{i=1}^\infty$ is a sequence from $\langle 0,1\rangle^\infty$. To make the notation more close to the classical one introduced above, we shall write $\sum_{i=1}^{1\,\infty} x_i$ instead of $\sum^1 (\langle x_i \rangle_{i=1}^\infty)$, leaving nevertheless the upper index 1 in $\sum^1$ to express explicitly the *partial* character in which this definition of countable addition on $\langle 0,1\rangle$ differs from the two ones mentioned above. The items (c) or (c1) should then be replaced by

(c2) for each...from $\mathcal{A}$, $\sum_{i=1}^{1\,\infty} P(A_i)$ *is defined* (hence, finite) and the equality $\sum_{i=1}^{1\,\infty} P(A_i) = P\left(\bigcup_{i=1}^\infty A_i\right)$ holds.

What makes this modification worth being formulated explicitly is that due to it, in order to define the classical probability measure, it is sufficient to define the countable addition just as a *partial* operation on $\langle 0,1\rangle^\infty$ with the values in $\langle 0,1\rangle$, completely neglecting as irrelevant the circumstance, that this operation can be (and is, in fact, in the mathematical analysis) defined more generally, either as a partial operation on $\langle 0,\infty)^\infty$ with values in $\langle 0,\infty)$, or even as a total operation on $\langle 0,\infty)^\infty$ with values in $\langle 0,\infty) \cup \{\infty\}$.

Let $\mathcal{C}$ be the well-known Cantor subset of $\langle 0,1\rangle$. Its informal definition reads as follows. Divide $\langle 0,1\rangle$ into three parts, $\langle 0,\ 1/3\rangle$, $(1/3,\ 2/3)$, $\langle 2/3,\ 1\rangle$, and erase the middle open interval $(1/3, 2/3)$ from $\langle 0,1\rangle$. Apply an analogous operation to the intervals $\langle 0,\ 1/3\rangle$ and $\langle 2/3,\ 1\rangle$, erasing from them the open intervals $(1/9,\ 2/9)$ and $(7/9,\ 8/9)$. And so on for the intervals $\langle 0,\ 1/9\rangle$, $\langle 2/9,\ 1/3\rangle$, $\langle 2/3,\ 7/9\rangle$, $\langle 8/9,\ 1\rangle$ *ad infinitum*. What rests is just the Cantor set $\mathcal{C}$. Formally defined, $\mathcal{C}$ is just the set of all real numbers from $\langle 0,1\rangle$, for which there exist triadic decompositions (decompositions to the base 3) containing just the numerals 0 and 2. Moreover, for each $x \in \mathcal{C}$ there exists just one triadic decomposition containing only 0's and 2's (obviously, the alternative decomposition for $x_1, \ldots, x_n$, 0, 2, 2, 2, ..., i.e., $x_1, \ldots, x_n$, 1, 0, 0, 0, ... contains the numeral 1). Let $\langle y_1(x),\ y_2(x), \ldots\rangle \in \{0,\ 2\}^\infty$ denote the triadic decomposition of a number $x \in \mathcal{C}$ not containing 1, let $d(i,x) = y_i/2$ for each $i \in \mathcal{N}$), so that $D(x) = \langle d(1,x),\ d(2,x), \ldots\rangle$ is a binary sequence from $\{0,1\}^\infty$. Set, for each $x \in \mathcal{C}$, $s(x) = \{i \in \mathcal{N} : d(i,x) = 1\}$ $(= \{i \in \mathcal{N} : y_i(x) = 2\})$, so that $D(x)$ can be taken as the characteristic function (identifier) of the subset $s(x)$ of $\mathcal{N}$. Obviously, D is a one-to-one mapping of $\mathcal{C}$ onto $\{0,1\}^\infty$, so that, for each $\langle x_1,\ x_2, \ldots\rangle \in \{0,1\}^\infty$, $D^{-1}(\langle x_1,\ x_2, \ldots\rangle)$ is defined, namely, $D^{-1}(\langle x_1,\ x_2 \ldots\rangle) = \sum_{i=1}^\infty 2x_i\, 3^{-i} \in \mathcal{C}$, where $\sum_{i=1}^\infty$ is the classical operation of countable addition.

Let $\mathcal{C}^\infty$ be the space of all infinite sequences of real numbers from the Cantor set, let $\sum^0$ be a partial operation defined on $\mathcal{C}^\infty$ in this way: if $\langle x_1,\ x_2 \ldots\rangle \in \mathcal{C}^\infty$, set $z_i = \operatorname{card}\{j \in \mathcal{N} : d(i,x_j) = 1\}$, then $\sum^0 (\langle x_i \rangle_{i=1}^\infty)$ (or $\sum_{i=1}^{0\,\infty} x_i$) is defined iff $\langle z_1,\ z_2, \ldots\rangle \in \{0,1\}^\infty$ (i.e., iff for each $i \in \mathcal{N}$ there exists at most one j such that $d(i,x_j) = 1$) and, if this is the case, $\sum_{i=1}^{0\,\infty} x_i = D^{-1}(\langle z_1,\ z_2, \ldots\rangle) = \sum_{i=1}^\infty 2z_i\, 3^{-i}$, this value obviously is in $\mathcal{C}$.

Lemma 1. Let $\langle x_1,\ x_2, \ldots\rangle \in \mathcal{C}^\infty$. If $\sum_{i=1}^{0\,\infty} x_i$ is defined, then the sets $s(x_i)$, $i = 1, 2, \ldots$, are mutually disjoint, hence, $s(x_i) \cap s(x_j) = \emptyset$ for each $i,\ j \in \mathcal{N}$, $i \neq j$.

Proof. By contradiction, let $i \in s(x_j) \cap s(x_k)$ for $j \neq k$. As $s(x_j) = \{\ell \in \mathcal{N} : d(\ell, x_j) = 1\}$, we obtain that $d(i, x_j) = d(i, x_k) = 1$, so that $z_i = \operatorname{card}\{\ell \in \mathcal{N} : d(i, x_\ell) = 1\} \geq 2$, hence, $\sum_{i=1}^{0\,\infty} x_i$ is not defined. □

So, we have arrived at the key definition of this paper.

Definition 3. Let $\langle \Omega, \mathcal{A} \rangle$ be a measurable space. A real-valued function π defined on $\mathcal{A}$ is called *strong probability measure*, if it satisfies the following conditions:

(a0) $\pi : \mathcal{A} \to \mathcal{C}$, hence, $0 \leq \pi(A) \leq 1$ for each $A \in \mathcal{A}$,

(b0) $\pi(\Omega) = 1$ ($1 = 0,2222\ldots$ is obviously in $\mathcal{C}$),

(c0) for each infinite sequence $\langle A_1, A_2, \ldots \rangle$ of mutually disjoint sets from $\mathcal{A}$, $\sum_{i=1}^{0\,\infty} \pi(A_i)$ is defined and the equality $\sum_{i=1}^{0\,\infty} \pi(A_i) = \pi\left(\bigcup_{i=1}^{\infty} A_i\right)$ holds.

Theorem 1. Each strong probability measure defined on a measurable space $\langle \Omega, \mathcal{A} \rangle$ is also a classical probability measure on $\langle \Omega, \mathcal{A} \rangle$ in the sense of Definition 2.

Proof. The validity of (a) and (b) from Definition 2 for strong probability measures is obvious, the only we have to prove is that $\sum_{i=1}^{0\,\infty} \pi(A_i) = \sum_{i=1}^{\infty} \pi(A_i)$ holds for each sequence $\langle A_1, A_2, \ldots, \rangle$ of mutually disjoint sets from $\mathcal{A}$ and for each strong probability measure π on $\langle \Omega, \mathcal{A} \rangle$. Let $\langle A_1, A_2, \ldots \rangle$ be such a sequence of sets. By (c0), $\sum_{i=1}^{0\,\infty} \pi(A_i)$ is defined so that, by Lemma 1, the sets $s(\pi(A_i))$ are mutually disjoint. By definition of $\sum_{i=1}^{0\,\infty} \pi(A_i)$, $d\left(j, \sum_{i=1}^{0\,\infty} \pi(A_i)\right) = 1$ iff there exists (unique, if this is the case) $i \in \mathcal{N}$ such that $d(j, \pi(A_i)) = 1$, in other words, $j \in s\left(\sum_{i=1}^{0\,\infty} \pi(A_i)\right)$ iff $j \in s(\pi(A_i))$ for (just one) $i \in \mathcal{N}$, hence, $s\left(\sum_{i=1}^{0\,\infty} \pi(A_i)\right) = s\left(\pi\left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \bigcup_{i=1}^{\infty} s(\pi(A_i))$.
By definition of $d(j, \pi(A_i))$, $\pi(A_i) = \sum_{j=1}^{\infty} 2d(j, \pi(A_i))\, 3^{-j} = \sum_{j \in s(\pi(A_i))} 2 \cdot 3^{-j}$, as $d(j, \pi(A_i)) = 0$ for $j \notin s(\pi(A_i))$. As the sets $\pi(A_i)$ are mutually disjoint,

$$\begin{aligned} \sum_{i=1}^{\infty} \pi(A_i) &= \sum_{i=1}^{\infty} \sum_{j \in s(\pi(A_i))} 2 \cdot 3^{-j} = \sum_{j \in \bigcup_{i=1}^{\infty} \pi(A_i)} 2 \cdot 3^{-j} = \\ &= \sum_{j \in s(\pi(\bigcup_{i=1}^{\infty} A_i))} 2 \cdot 3^{-j} = \sum_{j=1}^{\infty} 2d\left(j, \left(\bigcup_{i=1}^{\infty} A_i\right)\right) 3^{-j} = \\ &= \pi\left(\bigcup_{i=1}^{\infty} A_i\right), \end{aligned} \tag{1}$$

by definition of $d\left(j, \pi\left(\bigcup_{i=1}^{\infty} A_i\right)\right)$. The theorem is proved. □

Lemma 2. Let π be a strong probability measure defined on a measurable space $\langle\Omega, \mathcal{A}\rangle$, let $\emptyset$ be the empty subset of Ω, let $A \in \mathcal{A}$. Then the equalities $\pi(\emptyset) = 0$ and $\pi(\Omega - A) = 1 - \pi(A)$ hold.

Proof. The assertion follows immediately from the fact that π is a classical probability measure, but let us introduce here detailed proofs copying these for classical probability measure. Let $\langle A_1, A_2, \ldots\rangle = \langle\Omega, \emptyset, \emptyset, \ldots\rangle$ be the sequence of mutually disjoint sets. Then $\sum_{i=1}^{0\,\infty} \pi(A_i)$ is defined, so that, by Lemma 1, $s(\pi(\Omega))$ and $s(\pi(\emptyset))$ are mutually disjoint subsets of $\mathcal{N}$. However, $\pi(\Omega) = 1 = 0,222\ldots$, so that $s(\pi(\Omega)) = \mathcal{N}$, consequently, $s(\pi(\emptyset)) = \emptyset$ and $\pi(\emptyset) = \sum_{j\in s(\pi(\emptyset))} 2\cdot 3^{-j} = 0$. Given $A \in \mathcal{A}$, take $\langle A_1, A_2, \ldots,\rangle = \langle A, \Omega - A, \emptyset, \emptyset, \ldots\rangle$. $\Omega - A \in \mathcal{A}$, the sets A_i are mutually disjoint, hence, $\sum_{i=1}^{0\,\infty} \pi(A_i)$ is defined, so that $s(\pi(A))$ and $s(\pi(\Omega - A))$ are mutually disjoint and such that $s(\pi(A)) \cup s(\pi(\Omega - A)) = s(\pi(A \cup (\Omega - A))) = s(\pi(\Omega)) = \mathcal{N}$. Consequently, $s(\pi(\Omega - A)) = \mathcal{N} - s(\pi(A))$. But

$$\begin{aligned} 1 &= \sum_{j=1}^{\infty} 2\cdot 3^{-j} = \sum_{j\in\mathcal{N}} 2\cdot 3^{-j} = \sum_{j\in s(\pi(A))} 2\cdot 3^{-j} + \\ &\quad + \sum_{j\in\mathcal{N}-s(\pi(A))} 2\cdot 3^{-j} = \sum_{j\in s(\pi(A))} 2\cdot 3^{-j} + \sum_{j\in s(\pi(\Omega-A))} 2\cdot 3^{-j} = \\ &= \sum_{j=1}^{\infty} 2d(j, \pi(A))\, 3^{-j} + \sum_{j=1}^{\infty} 2d(j, \pi(\Omega - A)) = \\ &= \pi(A) + \pi(\Omega - A), \end{aligned} \tag{2}$$

so that $\pi(\Omega - A) = 1 - \pi(A)$. Lemma 2 is proved. □

In the next chapter we shall prove that the homomorphism between set-theoretical operations and operations over strong probability measures, proved in Lemma 1 and Lemma 2 for disjoint sets, is valid in general, i.e., for each systems of sets. However, before closing this chapter, let us introduce a particular example of a strong probability measure to prove that this notion is not logically empty.

The example is very simple and it served, in fact, as a motivation for the theoretical approach explained above. Let $\Omega = \mathcal{N} = \{1, 2, \ldots\}$, let $\mathcal{A} = \mathcal{P}(\Omega) = \mathcal{P}(\mathcal{N})$, let $\pi(\{i\}) = 2\cdot 3^{-i}$ for each $i \in \mathcal{N}$, let $\pi(A)$, $A \subset \mathcal{N}$, be defined by σ-additivity, i.e., $\pi(A) = \sum_{i\in A} \pi(\{i\}) = \sum_{i\in A} 2\cdot 3^{-i}$. Obviously, $s(\pi(A)) = A$ for each $A \subset \mathcal{N}$, so that for each sequence $\langle A_1, A_2, \ldots\rangle$ of disjoint subsets of $\mathcal{N}$, the sets $s(\pi(A_1)), s(\pi(A_2)), \ldots$ are also mutually disjoint, hence,

$$\begin{aligned} \pi\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{i\in s(\pi(\bigcup_{i=1}^{\infty} A_i))} 2\cdot 3^{-i} = \sum_{j=1}^{\infty}\sum_{i\in A_j} 2\cdot 3^{-i} = \\ &= \sum_{j=1}^{0\,\infty} \sum_{i\in s(\pi(A_j))} 2\cdot 3^{-i} = \sum_{j=1}^{0\,\infty} \pi(A_j), \end{aligned} \tag{3}$$

so that σ-additivity of π is proved. The facts that $\pi(A) \in \mathcal{C}$ for all $A \subset \mathcal{N}$ and that $\pi(\mathcal{N}) = 1$ are obvious.

3. BOOLEAN-VALUED UNCERTAINTY QUANTIFICATIONS INDUCED BY STRONG PROBABILITY MEASURES

In the last chapter we proved, that for a sequence $\langle A_1, A_2, \ldots\rangle$ of mutually disjoint sets from $\mathcal{A}$ the corresponding support sets $s(\pi(A_1))$, $s(\pi(A_2)), \ldots$ are also mutually disjoint, in other words said, that for all pairs i, $j \in \mathcal{N}$, $i \neq j$, the equality $s(\pi(A_i)) \cap s(\pi(A_j)) = \emptyset = s(\pi(A_i \cap A_j)) = s(\pi(\emptyset)) = s(0)$. The following assertion proves, that this possibility to translate set-theoretic operations over sets from $\mathcal{A}$ into the same operations over the support sets of their strong probability values is more general and that it is not limited only to disjoint sets.

Theorem 2. Let π be a strict probability measure defined on a measurable space $\langle\Omega, \mathcal{A}\rangle$. Let $\langle A, A_1, A_2, \ldots\rangle$ be a sequence of sets from $\mathcal{A}$. Then the following set equalities hold in $\mathcal{P}(\mathcal{N})$:

(i) $s(\pi(\Omega - A)) = \mathcal{N} - s(\pi(A))$,

(ii) $s\left(\pi\left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \bigcup_{i=1}^{\infty} s(\pi(A_i))$,

(iii) $s\left(\pi\left(\bigcap_{i=1}^{\infty} A_i\right)\right) = \bigcap_{i=1}^{\infty} s(\pi(A_i))$.

Proof. (i) Take $\langle A_1, A_2, \ldots\rangle = \langle A, \Omega - A, \emptyset, \emptyset, \emptyset, \ldots\rangle$ as a sequence of mutually disjoint subsets from $\mathcal{A}$. Then $\sum_{i=1}^{0\,\infty} \pi(A_i) = \pi(\Omega) = 1$ is defined, hence, for each $j \in \mathcal{N}$, $d\left(j, \sum_{i=1}^{0\,\infty} \pi(A_i)\right) = d(j, \pi(\Omega)) = 1$. As $d(j, \emptyset) = 0$ for all $j \in \mathcal{N}$, we obtain that for all $j \in \mathcal{N}$ either $d(j, \pi(A)) = 1$, or $d(j, \pi(\Omega - A)) = 1$, but not both together. Consequently, $d(j, \pi(A)) = 1$ iff $d(j, \pi(\Omega - A)) = 0$ holds for each $j \in \mathcal{N}$, so that $s(\pi(\Omega - A)) = \mathcal{N} - s(\pi(A))$ and (i) is proved.

(ii) Let A, $B \in \mathcal{A}$, let $A \subset B$. Then $B - A \in \mathcal{A}$ and $\pi(B) = \sum_{i=1}^{0\,\infty} \pi(A_i)$, where $\langle A_1, A_2, \ldots\rangle = \langle A, B - A, \emptyset, \emptyset, \emptyset, \ldots\rangle$. Hence, for all $j \in \mathcal{N}$, $d(j, \pi(B)) = 1$ iff $d(j, \pi(A)) = 1$ or $d(j, \pi(B - A)) = 1$, and just one of these two possibilities holds. So, $s(\pi(B)) = s(\pi(A)) \cup s(\pi(B - A))$, consequently, $s(\pi(A)) \subset s(\pi(B))$ and $s(\pi(B - A)) \subset s(\pi(B))$. For no matter which $A, B \in \mathcal{A}$, $A \subset A \cup B$ and $B \subset A \cup B$ immediately yield that $s(\pi(A)) \subset s(\pi(A \cup B))$ and $s(\pi(B)) \subset s(\pi(A \cup B))$, hence, $s(\pi(A)) \cup s(\pi(B)) \subset s(\pi(A \cup B))$. Suppose, in order to arrive at a contradiction, that $s(\pi(A)) \cup s(\pi(B)) \neq s(\pi(A \cup B))$, hence, that there exists $i_0 \in \mathcal{N}$ such that $i_0 \in s(\pi(A \cup B))$, but $i_0 \notin s(\pi(A))$, $i_0 \notin s(\pi(B))$, in other words, $d(i_0, \pi(A \cup B)) = 1$ but $d(i_0, \pi(A)) = d(i_0, \pi(B)) = 0$. As $s(\pi(B - A)) \subset s(\pi(B))$ holds, $d(i_0, \pi(B - A)) = 0$ holds as well. But, $A \cup B = A \cup (B - A)$ also holds, so that $d(i_0, \pi(A)) = d(i_0, \pi(B - A)) = 0$ implies that also $d(i_0, \pi(A \cup B)) = 0$. So, we have arrived at a contradiction and the equality $s(\pi(A \cup B)) = s(\pi(A)) \cup s(\pi(B))$ is proved.

Let $\langle A_1, A_2 \ldots\rangle$ be a sequence of sets from $\mathcal{A}$, let us prove, first, that

$$s\left(\pi\left(\bigcup_{i=1}^{n} A_i\right)\right) = \bigcup_{i=1}^{n} s(\pi(A_i)) \tag{4}$$

holds for each $n \in \mathcal{N}$. For $n = 1$, the assertion is trivial. By induction, suppose that it holds for n. Setting $\bigcup_{i=1}^{n} A_i = B$, we obtain that

$$\begin{aligned} s\left(\pi\left(\bigcup_{i=1}^{n+1} A_i\right)\right) &= s\left(\pi\left(\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right)\right) = s(\pi(B \cup A_{n+1})) = \\ &= s(\pi(B)) \cup s(\pi(A_{n+1})) = s\left(\pi\left(\bigcup_{i=1}^{n} A_i\right)\right) \cup s(\pi(A_{n+1})) = \\ &= \bigcup_{i=1}^{n} s(\pi(A_i)) \cup s(\pi(A_{n+1})) = \bigcup_{i=1}^{n+1} s(\pi(A_i)). \end{aligned} \tag{5}$$

For all $n \in \mathcal{N}$, $\bigcup_{i=1}^{n} A_i \subset \bigcup_{i=1}^{\infty} A_i$, hence, $s(\pi(\bigcup_{i=1}^{n} A_i)) \subset s(\pi(\bigcup_{i=1}^{\infty} A_i))$, so that $\bigcup_{i=1}^{n} s(\pi(A_i)) \subset s(\pi(\bigcup_{i=1}^{\infty} A_i))$ holds for each $n \in \mathcal{N}$, consequently, the inclusion $\bigcup_{i=1}^{\infty} s(\pi(A_i)) \subset s(\pi(\bigcup_{i=1}^{\infty} A_i))$ also holds. Suppose, to arrive at a contradiction, that there exists $i_0 \in \mathcal{N}$ such that $i_0 \in (\pi(\bigcup_{i=1}^{\infty} A_i)) - \bigcup_{i=1}^{\infty} s(\pi(A_i))$. Then $i_0 \notin s(\pi(A_i))$ holds for each $i \in \mathcal{N}$, so that $d(i_0, \pi(A_i)) = 0$ for all $i \in \mathcal{N}$. Consequently, $d\left(i_0, \pi\left(A_i - \bigcup_{j=1}^{i-1} A_i\right)\right)$
$= 0$ for all $i \in \mathcal{N}$, so that

$$d\left(i_0, \sum_{i=1}^{0\,\infty} \pi\left(A_i - \bigcup_{j=1}^{i-1} A_j\right)\right) = d\left(i_0, \pi\left(\bigcup_{i=1}^{\infty} A_i\right)\right) = 0. \tag{6}$$

Hence, $i_0 \notin s(\pi(\bigcup_{i=1}^{\infty} A_i))$, and we have arrived at a contradiction. Consequently, $\bigcup_{i=1}^{\infty} s(\pi(A_i)) = s(\pi(\bigcup_{i=1}^{\infty} A_i))$ holds and (ii) is proved.
(iii) Let $A_1, A_2, \ldots \in \mathcal{A}$. Then

$$\begin{aligned} s\left(\pi\left(\bigcap_{i=1}^{\infty} A_i\right)\right) &= s\left(\pi\left(\Omega - \bigcup_{i=1}^{\infty}(\Omega - A_i)\right)\right) = \\ &= \mathcal{N} - s\left(\pi\left(\bigcup_{i=1}^{\infty}(\Omega - A_i)\right)\right) = \mathcal{N} - \bigcup_{i=1}^{\infty} s(\pi(\Omega - A_i)) = \\ &= \mathcal{N} - \bigcup_{i=1}^{\infty} (\mathcal{N} - s(\pi(A_i))) = \bigcap_{i=1}^{\infty} s(\pi(A_i)), \end{aligned} \tag{7}$$

and (iii), as well as Theorem 2 as a whole, are proved. □

For a number of reasons explained in more detail, e.g., in [1] or [4], also non-numerical measures of degrees of uncertainty are worth being considered. Among these measures, attention is often concentrated to the set-valued or, more generally, boolean-valued measures of uncertainty because of the possibility to understand these degrees as sets of possible worlds satisfying some conditions or verifying some assertion, and because of the possibility to take profit from a relatively rich apparatus of notions and results concerning Boolean algebras. Namely, we shall define Boolean-valued probability measures as presented below.

Boolean algebra $\mathcal{B}$ is a structure $\langle B, \vee, \wedge, \neg, \mathbf{0}, \mathbf{1}\rangle$, where B is a nonempty set, $\vee$ (supremum) and $\wedge$ (infimum) are two binary operations defined on $B \times B$ and taking their values in B, $\neg$ (complement) is a unary operation taking B into itself, $\mathbf{0} \in B$ is the zero element of $\mathcal{B}$ and $\mathbf{1} \in \mathcal{B}$ is the unit element of $\mathcal{B}$, in the case of necessity we can write $\mathbf{0}_{\mathcal{B}}$ and $\mathbf{1}_{\mathcal{B}}$. The operations $\vee$, $\wedge$, and $\neg$, as well as the elements $\mathbf{0}$ and $\mathbf{1}$ are supposed to satisfy the axioms of Boolean algebras which are known in various settings, e. g., that one presented in [7].

It is a well-known fact, cf., e. g., again [7], that the binary relation $\leq$, defined on B in such a way that, for each $x, y \in B$, $x \leq y$ holds iff $x \wedge y = x$ or, what turns to be the same, iff $x \vee y = y$, is a partial ordering relation on B and $x \vee y$ ($x \wedge y$, resp.) is just the supremum (the infimum, resp.) of x and y with respect to this partial ordering relation. Using the associativity and commutativity properties of both the operations $\vee$ and $\wedge$, we can immediately deduce by induction, that supremum and infimum of each finite set of elements of a Boolean algebra is defined. For an infinite set of elements, in general, this need not be the case. A Boolean algebra is called *complete*, if for *each* set of elements their supremum and infimum are defined, a Boolean algebra is called *σ-complete*, if for each *countable* set of elements their supremum and infimum are defined.

Definition 4. Let $\langle \Omega, \mathcal{A}\rangle$ be a measurable space, let $\mathcal{B} = \langle B, \vee, \wedge, \neg, \mathbf{0}, \mathbf{1}\rangle$ be a σ-complete Boolean algebra. A mapping ρ, defined on $\mathcal{A}$ and taking its values in B, is called *$\mathcal{B}$-valued Boolean probability measure* (or: Boolean-valued probability measure taking its values in the Boolean algebra $\mathcal{B}$), if

(i) $\rho(\Omega) = \mathbf{1}$

(ii) for each infinite sequence $\langle A_1, A_2, \ldots\rangle$ of mutually disjoint sets from $\mathcal{A}$, the subset $\{\rho(A_i) : i \in \mathcal{N}\}$ of $\mathcal{B}$ is a decomposition of the element $\rho\left(\bigcup_{i=1}^{\infty} A_i\right)$ of B. (In other words, for each $i, j \in \mathcal{N}$, $i \neq j$ implies that $\rho(A_i) \wedge \rho(A_j) = \mathbf{0}$ and $\bigvee_{i=1}^{\infty} \rho(A_i) = \rho\left(\bigcup_{i=1}^{\infty} A_i\right)$.)

The following assertion can be seen rather as a re-interpretation of the results of Theorem 2.

Theorem 3. Let π be a strict probability measure defined on a measurable space $\langle \Omega, \mathcal{A}\rangle$, let $\mathcal{B}_{\mathcal{N}} = \langle \mathcal{P}(\mathcal{N}), \cup, \cap, {}^c, \emptyset, \mathcal{N}\rangle$ be the complete Boolean algebra of all subsets of the set of all positive integers, where $\cap$, $\cup$ and ^{c} are the usual set-theoretic operations of union, intersection and complement. Then the mapping $\rho : \mathcal{A} \to \mathcal{P}(\mathcal{N})$, defined by $\rho(A) = s(\pi(A))$ for each $A \in \mathcal{A}$, is a $\mathcal{B}_{\mathcal{N}}$-valued Boolean probability.

P r o o f. Follows immediately from (b0) of Definition 2 and from Theorem 2. □

4. ATOMIC AND EXTENSIONAL PROPERTIES OF STRONG PROBABILITY MEASURES

When investigating and processing a classical probability measure the situation turns out to be much simplified, if the measure in question is atomic with a finite or countable space of atoms. Or, in such a case, the probability of each random event can be computed as a sum of positive probabilities of a finite or countable set of atomic random events corresponding to the random event in question. We shall prove that strong probability measures are atomic in this sense and we shall use this property in order to prove that strong probability measures are extensional. For the sake of reader's convenience let us begin with the classical definition.

Definition 5. Let $\langle\Omega,\mathcal{A}\rangle$ be a measurable space, let P be a (classical) probability measure defined on $\langle\Omega,\mathcal{A}\rangle$. A set $A\in\mathcal{A}$ is called an *atom* of $\mathcal{A}$ with respect to P, if $P(A)>0$ and for each $B\in\mathcal{A}$, $B\subset A$, either $P(B)=0$ or $P(B)=P(A)$. Measure P is called *atomic* with respect to a set $At\subset\mathcal{A}$ of atoms of $\mathcal{A}$ with respect to P, if for each $B\in\mathcal{A}$ such that $P(B)>0$ there exists an atom $A\in At$ such that $A\subset B$.

Given a strong probability measure π defined on a measurable space $\langle\Omega,\mathcal{A}\rangle$, we shall construct, in the sequel, the set of all atoms of $\mathcal{A}$ with respect to π. The following lemma will be useful for these sakes.

Lemma 3. Let π be a strong probability measure defined on a measurable space $\langle\Omega,\mathcal{A}\rangle$, let $\emptyset\neq\mathcal{S}\subset\mathcal{A}$ be a system of sets from $\mathcal{A}$. Then there exists a sequence $A_1^{\mathcal{S}}, A_2^{\mathcal{S}},\dots$ of sets from $\mathcal{S}\cup\{\emptyset\}$ such that

$$\bigcup_{A\in\mathcal{S}} s(\pi(A))=\bigcup_{j=1}^{\infty} s(\pi(A_j^{\mathcal{S}}))=s\left(\pi\left(\bigcup_{j=1}^{\infty}A_j^{\mathcal{S}}\right)\right) \tag{8}$$

$$\bigcap_{A\in\mathcal{S}} s(\pi(A))=\bigcap_{j=1}^{\infty} s(\pi(A_j^{\mathcal{S}}))=s\left(\pi\left(\bigcap_{j=1}^{\infty}A_j^{\mathcal{S}}\right)\right). \tag{9}$$

Proof. Take $i\in\mathcal{N}$, if there exists $A\in\mathcal{S}$ such that $i\in s(\pi(A))$, choose one such set and denote it by $A_i^{\mathcal{S}}$, hence, axiom of choice is applied to the nonempty system $\{A: A\in\mathcal{S},\, i\in s(\pi(A))\}$ of sets. If there is no such A in $\mathcal{S}$, take $A_i^{\mathcal{S}}=\emptyset$ (the empty subset of Ω). We shall prove that $\bigcup_{i=1}^{\infty}s(\pi(A_i^{\mathcal{S}}))=\bigcup_{A\in\mathcal{S}}s(\pi(A))$. As $A_i^{\mathcal{S}}\in\mathcal{S}\cup\{\emptyset\}$ holds for each $i\in\mathcal{N}$, the inclusion $\bigcup_{i=1}^{\infty}s(\pi(A_i^{\mathcal{S}}))\subset\bigcup_{A\in\mathcal{S}}s(\pi(A))$ is obvious. Let $i\in\bigcup_{A\in\mathcal{S}}s(\pi(A))$. Then there exists $A_0\in\mathcal{S}$ such that $i\in s(\pi(A_0))$ holds. Hence, $\{A\in\mathcal{S}: i\in s(\pi(A))\}\neq\emptyset$, so that $i\in s(\pi(A_i^{\mathcal{S}}))$ by the definition of $A_i^{\mathcal{S}}$. So, $i\in\bigcup_{j=1}^{\infty}s(\pi(A_j^{\mathcal{S}}))$, consequently, $\bigcup_{A\in\mathcal{S}}s(\pi(A))\subset\bigcup_{j=1}^{\mathcal{S}}s(\pi(A_j^{\mathcal{S}}))$ and the first equality in (8) is proved. The dual equality in (9) follows by de Morgan rules, as

$$\bigcap_{A\in\mathcal{S}} s(\pi(A)) \;=\; \mathcal{N}-\bigcup_{A\in\mathcal{S}}(\mathcal{N}-s(\pi(A)))= \tag{10}$$

$$= \mathcal{N} - \bigcup_{A \in \mathcal{S}} s(\pi(\Omega - A)) = \mathcal{N} - \bigcup_{j=1}^{\infty} s(\pi(\Omega - A_j^{\mathcal{S}})) =$$
$$= \mathcal{N} - \bigcup_{j=1}^{\infty} (\mathcal{N} - s(\pi(A_j^{\mathcal{S}}))) = \bigcap_{j=1}^{\infty} s(\pi(A_j^{\mathcal{S}})).$$

The right-hand side inequalities in (8) and (9) follow directly from Theorem 2. □

Set, for each $i \in \mathcal{N}$,

$$S_i = \bigcap_{A \in \mathcal{A},\, i \in s(\pi(A))} s(\pi(A)). \tag{11}$$

Obviously, $i \in S_i$ for each $i \in N$. Let $j \in S_i$ for some $j \in \mathcal{N}$, $j \neq i$, let $i \notin S_j$. Then there exists $A \in \mathcal{A}$ such that $j \in s(\pi(A))$, $i \notin s(\pi(A))$, hence, $j \notin s(\pi(\Omega - A)) = \mathcal{N} - s(\pi(A))$, but $i \in s(\pi(\Omega - A))$, so that $j \notin S_i$. We have arrived at a contradiction proving that $j \in S_i$ implies $i \in S_j$. But, $j \in S_i$ means by definition of S_i, that $(\forall A \in \mathcal{A})\ (i \in s(\pi(A))) \Rightarrow j \in s(\pi(A))$, in other terms, $\{A \in \mathcal{A} : i \in S(\pi(A))\} \subset \{A \in \mathcal{A} : j \in s(\pi(A))\}$, consequently, $S_j \subset S_i$. As $j \in S_i$ implies that $i \in S_j$, we obtain by the same way of reasoning that $S_i \in S_j$, hence, $j \in S_i$ implies that $S_j = S_i$. So, if $S_i \cap S_j \neq \emptyset$ for $i, j \in \mathcal{N}$, i. e., if there exists $k \in S_i \cap S_j$, then $S_i = S_k = S_j$. We can conclude that for each $i, j \in \mathcal{N}$ either $S_i \cap S_j = \emptyset$ or $S_i = S_j$, hence, the system $\mathcal{S}^* = \{S_1, S_2, \ldots\}$ of sets is a decomposition of the set $\mathcal{N}$ of all positive integers (let us recall that $\mathcal{S}^*$ is taken as a *set* of sets, so that repeated occurrences of some $S \subset \mathcal{N}$ in the *sequence* $\langle S_1, S_2 \rangle$ are not taken into consideration).

Let $\mathcal{S} \subset \mathcal{A}$ be a nonempty system of subsets of Ω. A sequence $A_1^{\mathcal{S}}, A_2^{\mathcal{S}}, \ldots$ of sets from $\mathcal{S}$ is called a *representation* of $\mathcal{S}$ (with respect to π), if $i \in s(\pi(A_i^{\mathcal{S}}))$ for each $i \in \bigcup_{A \in \mathcal{S}} s(\pi(A))$ (consequently, $\bigcup_{i=1}^{\infty} s(\pi(A_i^{\mathcal{S}})) = \bigcup_{A \in \mathcal{S}} s(\pi(A))$). A representation $\langle A_1^{\mathcal{S}}, A_2^{\mathcal{S}}, \ldots \rangle$ of $\mathcal{S}$ is called *minimal*, if $j \notin s(\pi(A_i^{\mathcal{S}}))$ holds for all $i, j \in \mathcal{N}$ such that $s(\pi(A_i^{\mathcal{S}})) \neq s(\pi(A_j^{\mathcal{S}}))$. We proved in the proof of Lemma 3 above, that for each $\emptyset \neq \mathcal{S} \subset \mathcal{A}$ a representation of $\mathcal{S}$ exists. Let $\alpha(\mathcal{S})$ denote the set of all representations of $\mathcal{S}$ corresponding to different choices from the sets $\{A \in \mathcal{S} : i \in s(\pi(A))\}$ supposing that these sets are not singletons. In particular, let $\alpha^i = \alpha(\mathcal{S}_i)$, where $\mathcal{S}_i = \{A \in \mathcal{A} : i \in s(\pi(A))\}$.

Define, for each $i \in \mathcal{N}$,

$$At_i(\pi, \mathcal{A}) = \{A \in \mathcal{A} : s(\pi(A)) = S_i\}, \tag{12}$$

$$At(\pi, \mathcal{A}) = \bigcup_{i=1}^{\infty} At_i(\pi, \mathcal{A}). \tag{13}$$

Theorem 4. Let π be a strong probability measure defined on a measurable space $\langle \Omega, \mathcal{A} \rangle$. Then

(i) $At(\pi, \mathcal{A})$ is the set of all atoms of $\mathcal{A}$ with respect to π.

(ii) Each representation of $At(\pi, \mathcal{A})$ is minimal.

(iii) Measure π is atomic with respect to At.

(iv) Let $\langle B_1^{At}, B_2^{At}, \dots \rangle$ be a representation of At. Then, for each $A \in \mathcal{A}$,

$$\pi(A) = \sum_{\substack{j \in \mathcal{N},\ s(\pi(B_j^{At})) \subset s(\pi(A)), \\ s(\pi(B_j^{At})) \neq s(\pi(B_k^{At})) \text{ for all } k<j}} \pi(B_j^{At}). \tag{14}$$

Proof. ad (i) Let $A \in \mathcal{A}$, let $s(\pi(A)) = S_i$ for some $i \in \mathcal{N}$. As $S_i \neq \emptyset$, $\pi(A) = \sum_{j \in S_i} 2 \cdot 3^{-j} > 0$. Let $B \in \mathcal{A}$ be such that $B \subset A$, $0 < \pi(B) < \pi(A)$. Then $\emptyset \neq s(\pi(B)) \subsetneq S_i$ holds, so that there are $j_1, j_2 \in S_i$ such that $j_1 \in s(\pi(B))$ and $j_2 \in S_i - s(\pi(B))$. Hence, $B \in \mathcal{A}$ separates j_1 from j_2, so that j_1, j_2 cannot be in the same S_i and we have arrived at a contradiction. Consequently, such a $B \subset \mathcal{A}$ cannot exist, so that each A such that $s(\pi(A)) = S_i$ must be an atom. If $A \in \mathcal{A}$ is such that $\emptyset \neq s(\pi(A)) \neq S_i$ for all $i \in \mathcal{N}$, then the inclusion $S_i \subset s(\pi(A))$ must hold for at least one $i \in \mathcal{N}$. Let $\langle A_1^i, A_2^i, \dots \rangle$ be a representation of $\mathcal{S}_i = \{A \in \mathcal{A} : i \in s(\pi(A))\}$, so that $s\left(\pi\left(\bigcap_{j=1}^{\infty} A_j^i\right)\right) = s\left(\pi\left(\bigcap_{A \in \mathcal{S}_i} A\right)\right) = S_i$. Denote $\bigcap_{j=1}^{\infty} A_j$ by A^i ($\in \mathcal{A}$, obviously), and set $A_0 = A^c \cap A$. Then $s(\pi(A_0)) = S_i$, so that $0 < \pi(A_0) = \sum_{j \in S_i} 2 \cdot 3^{-j} < \pi(A)$ and $A_0 \subset A$ obviously hold. Hence, A is not an atom of $\mathcal{A}$ with respect to π and (i) is proved.

ad (ii) As for each $i, j \in \mathcal{N}$ either $S_i = S_j$, or $S_i \cap S_j = \emptyset$, and $S_i = s(\pi(A_i^{At}))$ for each representation $\langle A_1^{At}, A_2^{At}, \dots \rangle$ of At, the relation $j \notin s(\pi(A_i^{At}))$ for each i, j such that $s(\pi(A_i^{At})) \neq s(\pi(A_j^{At}))$ is obvious. Hence, representation $\langle A_1^{At}, A_2^{At}, \dots \rangle$ is minimal and (ii) is proved.

ad (iii) Let $\langle A_1^{At}, A_2^{At}, \dots \rangle$ be a representation of At, let $B \in \mathcal{A}$. If B is an atom of $\mathcal{A}$ with respect to π, let $s(\pi(B)) = S_i$. As $s(\pi(A_i^{At})) = S_i$ as well, we obtain that $s(\pi(B \cap A_i^{At})) = S_i$, hence, $0 < \pi(B \cap A_i^{At}) = \pi(B)$, so that A_i^{At} satisfies the demands. If B is not an atom of $\mathcal{A}$, and $\pi(B) > 0$, then $s(\pi(B)) \supset S_i$ for some $i \in \mathcal{N}$, so that A_i^{At} is such that $s(B \cap A_i^{At}) = S_i$ and $0 < \pi(B \cap A_i^{At}) < \pi(B)$, hence, $B \cap A_i^{At}$ is an atom of $\mathcal{A}$ with respect to π such that $B \cap A_i^{At} \subset B$. Consequently, π is atomic and (iii) holds.

ad (iv) Let $A \in \mathcal{A}$. Then for each $i \in \mathcal{N}$ either $S_i \subset s(\pi(A))$ or $S_i \cap s(\pi(A)) = \emptyset$ (if $S_i \cap s(\pi(A)) \subsetneq S_i$ is nonempty, then A separates at least two elements of S_i from each other). Hence, $s(\pi(A)) = \bigcup_{S_i \subset s(\pi(A))} S_i$. Let $\langle B_1^{At}, B_2^{At}, \dots, \rangle$ be a representation of At. Then $s(\pi(A)) = \bigcup_{s(\pi(B_j^{At})) \subset s(\pi(A))} s(\pi(B_j^{At}))$, hence,

$$\begin{aligned} \pi(A) &= \sum_{j \in s(\pi(A))} 2 \cdot 3^{-j} = \\ &= \sum_{\substack{s(\pi(B_j^{At})) \subset s(\pi(A)), \\ s(\pi(B_j^{At})) \neq s(\pi(B_k^{At})) \text{ for all } k<j}} \ \sum_{i \in s(\pi(B_j^{At}))} 2 \cdot 3^{-i} = \end{aligned} \tag{15}$$

probability values ascribed to the particular sets, in other words said, probabilities of at most countable unions and intersections are *functions* of probabilities of particular sets. Interesting enough, perhaps, in this paper we have arrived at a sub-class of probabilistic measures possessing the property of extensionality, and we have arrived at this sub-class of probability measures in a purely axiomatic way, just replacing the set $\langle 0,1\rangle$ of probability values by its Cantor subset $\mathcal{C}$ and the usual addition operation $\sum$ by its strengthened version $\sum^0$.

And it is just the extensionality property which motivates our choose of the Cantor set $\mathcal{C}$ as the space of possible probability values. Or, to each real number x from $\mathcal{C}$ just one binary sequence can be uniquely ascribed, taking simply the unique $0-2$ sequence of ternary digits corresponding to this number x and dividing each digit by two, we obtain the sequence denoted above by $\langle d(1,x), d(2,x), \ldots\rangle$. This does not hold true in the case of binary decomposition, when, e. g., $0111\ldots$ and $1000\ldots$ correspond to the same real number $1/2$. A priori, two solutions are possible, but both of them can be easily proved to be unsatisfactory.

(I) We shall accept both the decompositions and we shall modify the σ-additivity axiom for strong probability measures in this way:

For each sequence $\langle A_1, A_2, \ldots\rangle$ of mutually disjoint sets from $\mathcal{A}$ and for each $i \in \mathcal{N}$ *there exists* a binary decomposition of the value $\pi(A_i)$ ascribed to A_i by the strong probability measure π such that $\sum_{i=1}^{0\,\infty} \pi(A_i)$ is defined and equals to $\pi\left(\bigcup_{i=1}^{\infty} A_i\right)$.

Now, we may ascribe the value $\pi(A) = 1/2$ to a set $A \in \mathcal{A}$ and the same value $\pi(\Omega - A) = 1/2$ to its complement, taking $\pi(A) = 1000,\ldots,$ $\pi(\Omega - A) = 0111\ldots$, and obtaining $\pi(A \cup (\Omega - A)) = \sum^0_{B\in\{A,\Omega-A\}} \pi(B) = 111\ldots = 1$. However, for $\pi(A\cup A)$ we obtain $1000\ldots = 1/2$ so that, for $A_1 = A$, $B_1 = \Omega - A$, $A_2 = A$, $B_2 = A$ we obtain that $\pi(A_1) = \pi(A_2)$, $\pi(B_1) = \pi(B_2)$, but $\pi(A_1 \cup B_1) \neq \pi(A_2 \cup B_2)$, so that the extensionality property is violated.

(II) We shall accept only one binary decomposition of each real number from $\langle 0,1\rangle$, namely this one containing infinitely many zeros. Evidently, just one binary decomposition satisfying this property corresponds to each $x \in \langle 0,1\rangle$. However, ascribing $\pi(A) = 1/2 = 1000\ldots$ to a set $A \in \mathcal{A}$, $\pi(\Omega - A)$ cannot be defined in such a way that $\sum^0_{B\in\{A,\Omega-A\}} \pi(B) = 111\ldots = 1$ would hold, as $0111\ldots$ is not the acceptable decomposition of the value $1/2$. Preferring the other decomposition obviously does not solve the problem, hence, the equality $\pi(A) = 1 - \pi(\Omega - A)$ is violated and this fact is also inacceptable.

An open question remains, which is the role of the Cantor set of real numbers in our constructions and considerations, and whether this role can be played by another subset of the unit interval of reals. Perhaps a more simple case is to find whether, using decomposition of real numbers to the base $n > 3$, e. g., to the common decadic base, there exists a subset of $\langle 0,1\rangle$ possessing the same advantageous properties (at least from the point of view of our constructions), as the Cantor set based on the decomposition to the base 3. A more general problem reads as follows: it is possible to obtain some explicit conditions imposed to subsets of $\langle 0,1\rangle$ such that every subset of $\langle 0,1\rangle$ satisfying these conditions can be used instead of the Cantor set when defining axiomatically extensional probability measures? Both these questions remain still unanswered and will be perhaps investigated at some appropriate future occasion.

5. DEFINITION OF CLASSICAL PROBABILITY MEASURES OVER COUNTABLE SPACES BY STRONG PROBABILITY MEASURES

Consider the following partial mapping w which takes the Cantor set $\mathcal{C}$ into the unit interval of real numbers as follows. If $x \in \mathcal{C}$, set

$$w(x) = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} d(i,x) \tag{23}$$

supposing that this limit values is defined, $w(x)$ being undefined otherwise. Hence, $w(x)$ is the limit value (if exists) of the relative frequences of the occurrences of the digit 2 in the initial segments of the unique $0-2$ ternary decomposition of the number x.

Theorem 6. Let π be a strong probability measure defined on a measurable space $\langle\Omega, \mathcal{A}\rangle$. Then the system $\mathcal{A}_0 \subset \mathcal{A}$ of subsets of $\mathcal{A}$ for which the value $w(\pi(A))$ is defined is closed with respect to complements and to finite unions of mutually disjoint sets and $w(\pi(\cdot))$ is a non-negative, normalized and finitely additive real-valued measure defined on $\mathcal{A}_0$.

Proof. The relation $w(\pi(A)) \in \langle 0,1\rangle$ obviously holds for each $A \in \mathcal{A}_0$. It is also obvious that $\emptyset \in \mathcal{A}_0$, $\Omega \in \mathcal{A}_0$, and $w(\pi(\emptyset)) = 0$, $w(\pi(\Omega)) = 1$. Let $A \in \mathcal{A}_0$, so that $w(\pi(A))$ is defined. Then

$$\begin{aligned} w(\pi(A)) &= \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} d(i,\, \pi(A)) = \\ &= \lim_{n\to\infty} n^{-1} \operatorname{card}(s(\pi(A)) \cap \{1,2,\dots,n\}) = \\ &= \lim_{n\to\infty} n^{-1} [n - \operatorname{card}(s(\pi(\Omega - A)) \cap \{1,2,\dots,n\})] = \\ &= 1 - \lim_{n\to\infty} n^{-1} \operatorname{card}(s(\pi(\Omega - A)) \cap \{1,2,\dots,n\}) = \\ &= 1 - \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} d(i,\, \pi(\Omega - A)), \end{aligned} \tag{24}$$

as $s(\pi(\Omega - A)) = \mathcal{N} - s(\pi(A))$. Hence, $w(\pi(\Omega - A)) = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} d(i,\, \pi(\Omega - A))$ is defined, so that $\Omega - A \in \mathcal{A}_0$ and $w(\pi(\Omega - A)) = 1 - w(\pi(A))$.

Let $A,\, B \in \mathcal{A}_0$, let $A \cap B = \emptyset$. Then $s(\pi(A)) \cap s(\pi(B)) = \emptyset$ and $s(\pi(A \cup B)) = s(\pi(A)) \cup s(\pi(B))$. Consequently,

$$w(\pi(A\cup B)) = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} d(i,\, \pi(A\cup B)) = \tag{25}$$

$$\begin{aligned}
&= \lim_{n\to\infty} n^{-1} \operatorname{card}\,(s(\pi(A\cup B))\cap\{1,2,\dots,n\}) = \\
&= \lim_{n\to\infty} n^{-1} \operatorname{card}\,((s(\pi(A))\cup s(\pi(B)))\cap\{1,2,\dots,n\}) = \\
&= \lim_{n\to\infty} \left[n^{-1} \operatorname{card}\,(s(\pi(A))\cap\{1,2,\dots,n\}) + \right.\\
&+ \left. n^{-1} \operatorname{card}\,(s(\pi(B))\cap\{1,2,\dots,n\})\right] = \\
&= \lim_{n\to\infty} n^{-1} \operatorname{card}\,(s(\pi(A))\cap\{1,2,\dots,n\}) + \\
&+ \lim_{n\to\infty} n^{-1} \operatorname{card}\,(s(\pi(B))\cap\{1,2,\dots,n\}) = \\
&= w(\pi(A)) + w(\pi(B)),
\end{aligned}$$

so that $A\cup B\in\mathcal{A}_0$ and the finite additivity of the measure $w(\pi(\cdot))$ is proved. □

As can be easily demonstrated, the measure $w(\pi(\cdot))$ is not, in general, σ-additive. Or, take $\langle\Omega,\mathcal{A}\rangle = \langle\mathcal{N},\mathcal{P}(\mathcal{N})\rangle$, and take $\pi(A)=\sum_{i\in A} 2\cdot 3^{-i}$. For each $i\in\mathcal{N}$ and for $A_i=\{i\}$ we obtain that $s(A_i)=\{i\}$, hence,

$$\begin{aligned}
w(\pi(A_i)) &= \lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} d(j(A_i)) = \qquad (26)\\
&= \lim_{n\to\infty} n^{-1} \operatorname{card}\,(s(\pi(A_i))\cap\{1,2,\dots,n\}) = \\
&= \lim_{n\to\infty} n^{-1} \operatorname{card}\,(\{i\}\cap\{1,2,\dots,n\}) = \lim_{n\to\infty} n^{-1} = 0,
\end{aligned}$$

so that $\sum_{i=1}^{\infty} w(\pi(A_i)) = 0$, however,

$$w\left(\pi\left(\bigcup_{i=1}^{\infty} A_i\right)\right) = w(\pi(\mathcal{N})) = w(1) = 1, \qquad (27)$$

hence, σ-additivity is violated.

The following question can perhaps arise: how large is the class of measures definable on $\langle\Omega,\mathcal{A}\rangle$, which can be defined by an appropriate strong probability measure π on $\langle\Omega,\mathcal{A}\rangle$ and by the mapping w in such a way that $P(A)=w(\pi(A))$ for each $A\in\mathcal{A}$? The following assertion proves that if the space Ω is at most countable, this class contains all the classical probability measures defined on $\langle\Omega,\mathcal{A}\rangle$.

Theorem 7. Let $\langle\Omega,\mathcal{A}\rangle$ be a measurable space such that Ω is at most countable, let P be a classical probability measure π defined on $\langle\Omega,A\rangle$. Then there exists a strong probability measure defined on $\langle\Omega,\mathcal{A}\rangle$ such that, for all $A\in\mathcal{A}$, $P(A)=w(\pi(A))$.

Proof. Obviously, we may limit ourselves to the case when $\langle\Omega,A\rangle=\langle\mathcal{N},\mathcal{P}(\mathcal{N})\rangle$. If Ω is countable, an enumeration of elements of Ω suffices, if Ω is finite, we ascribe zero probabilities to all i's from $\mathcal{N}$ such that $i>\operatorname{card}(\Omega)$. If $\mathcal{A}\neq\mathcal{P}(\Omega)$, we can consider, instead of Ω, the set Ω_0 of all atoms of $\mathcal{A}$ with respect to π and we shall transform the assertion to the same assertion for $\langle\Omega_0,\mathcal{P}(\Omega_0)\rangle$.

Let P be a probability measure on $\langle \mathcal{N}, \mathcal{P}(\mathcal{N})\rangle$, hence, P is uniquely defined by the values $p_i = P(\{i\})$ for all $i \in \mathcal{N}$. Ascribe, to each $i \in \mathcal{N}$, a binary sequence $x^i = \langle x^i_j\rangle_{j=1}^{\infty} \in \{0,1\}^{\infty}$ in this way:

If $p_i = 0$, them $x^i_j = 0$ for all $j \in \mathcal{N}$, if $p_i = 1$, then $x^i_j = 1$ for all $j \in \mathcal{N}$. Otherwise, set $x^i_1 = 0$, $x^i_2 = 1$. If $x^i_1, x^i_2, \ldots, x^i_n$ are already defined, set $x^i_{n+1} = 1$, if $n^{-1}\sum_{j=1}^{n} x^i_j \leq r_i$ holds, set $x^i_{n+1} = 0$ otherwise, here $r_i = p_i\left(1 - \sum_{j=1}^{i-1} p_j\right)^{-1} = p_i\left(\sum_{j=i}^{\infty} p_j\right)^{-1}$. As can be easily proved (here and below in this proof we omit the routine details), $\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} x^i_j = r_i$.

Having defined x^i for all $i \in \mathcal{N}$, we set $y^1 = \langle y^1_j\rangle_{j=1}^{\infty} = x^1$. Having already defined $y^i = \langle y^i_j\rangle_{j=1}^{\infty}$ for all $i \leq n$, we shall define $\langle y^{n+1}_j\rangle_{j=1}^{\infty}$ in this way: first of all, we define an auxiliary sequence $\langle z^{n+1}_j\rangle_{j=1}^{\infty}$ such that $z^{n+1}_j = 0$, if $\sum_{k=1}^{n} y^k_j \geq 1$, i.e., if there exists $k \leq n$ such that $y^k_j = 1$, and $z^{n+1}_j = 1$ otherwise, i.e., if $y^k_j = 0$ for all $j \leq n$. Now, we set $y^{n+1}_j = z^{n+1}_j$, if $z^{n+1}_j = 0$ and $y^{n+1}_j = x^{n+1}_\ell$, if z^{n+1}_j is the ℓ-th occurrence of 1 in $\langle z^{n+1}_j\rangle_{j=1}^{\infty}$ from the left, i.e., if $\sum_{s=1}^{j} z^{n+1}_s = \ell$. After a calculation the technical details of which can be found in [5] we prove that $\lim_{n\to\infty}\sum_{j=1}^{n} y^i_j = r_i\left(1 - \sum_{j=1}^{i-1} p_j\right) = p_i$ for all $i \in \mathcal{N}$.

Set, now, $\pi(\{i\}) = \sum_{j=1}^{\infty} 2y^i_j\, 3^{-j}$ for each $i \in \mathcal{N}$, obviously, $w(\pi(i)) = p_i$. Moreover, due to the way in which the sequences y^i have been built, for each $i \in \mathcal{N}$ there exists just one $i \in \mathcal{N}$ such that $y^i_j = 1$, so that, for each $A \subset \mathcal{N}$, $\pi(A) = \sum^{0}_{i\in A} \pi(\{i\})$ is defined, consequently, $P(A) = w(\pi(A)) = w\left(\sum^{0}_{i\in A}\pi(i)\right) = \sum_{i\in A} w(\pi(i)) = \sum_{i\in a} p_i$ holds for each $A \subset \mathcal{N}$. □

The following remarks concerning the relation between classical probability measures and strong probability measures are perhaps worth being stated explicitly. If a strong probability measure π on $\langle \Omega, A\rangle$ is given, then $w(\pi(A))$ is defined uniquely for each $A \in \mathcal{A}$ for which the corresponding limit value exists. On the other side, given a classical probability measure P on $\langle \mathcal{N}, \mathcal{P}(\mathcal{N})\rangle$, the induced strong probability measure π on $\langle \mathcal{N}, \mathcal{P}(\mathcal{N})\rangle$ is defined uniquely only supposing that the sequences $x^i = \langle x^i_j\rangle_{j=1}^{\infty}$ are constructed according to the way described in the proof of Theorem 7. However, if $0 < r_i = p_i\left(\sum_{j=i}^{\infty} p_j\right)^{-1} < 1$, then there exists an infinite (uncountable, in fact) number of sequences $v^i = \langle v^i_j\rangle_{j=1}^{\infty} \in \{0,1\}^{\infty}$ such that $\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} v^i_j = r_i$, and each of them can be used instead of x^i when constructing the sequence y^i. Obviously, replacing x^i by v^i will result in different value of $\pi(\{i\})$, even if the equality $w(\pi(\{i\})) = p_i$ remains valid. When extending the proof of Theorem 7 to another countable set Ω (instead of $\mathcal{N}$) we must keep in mind, of course, that different enumeration of elements of Ω by positive integers will also lead to different strong probability measure π, as in this case already the measure P induced on $\langle \mathcal{N}, \mathcal{P}(\mathcal{N})\rangle$ will depend on this enumeration.

As far as the items [2] and [6] in the list below are concerned, [2] can serve as a source dealing with discrete elementary combinatoric probabilities over finite or countable spaces, on the other side, [6] is a classical monograph written on very high and abstract level and dealing with the axiomatic Kolmogorov probability theory in its most general setting.

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