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APPROXIMATE STABLE MULTIDIMENSIONAL POLYNOMIAL FACTORIZATION INTO LINEAR m–D POLYNOMIAL FACTORS

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In this paper, a solution to the approximate factorization problem of an m-D polynomial into stable m-D factors that are linear in all variables is presented. A non-factorizable multidimensional polynomial is factorized approximately into linear factors in the sense of the least squares approach. The constraints for the minimization problem are imposed by the stability conditions. The results are illustrated by means of a 2-D numerical example.

1. INTRODUCTION

A growing research effort has been obtained in the recent years in problems involving multidimensional (m-D) signals and multidimensional (m-D) systems [5, 23], due to their various applications and to their increasing mathematical interest. Multidimensional systems theory has attracted attention in areas such as stability, filter realization, analysis, synthesis, error analysis, feedback control and network analysis. A multidimensional (m-D) system is described by a transfer function which is a ratio of two m-D polynomials. If an m-D polynomial is written as a product of other lower degree polynomials , then it is said to be factorizable. Transfer function factorization, that is numerator and denominator factorization, enables a cascade realization of the corresponding m-D system. Factorization also helps among others in performing simpler stability tests and simpler controllers. More specifically, if the numerator and denominator of the transfer function $G(z_1,\dots,z_m) = g(z_1,\dots,z_m)/f(z_1,\dots,z_m)$ are factorized as

$$g(z_1,\dots,z_m) = g_1(z_1,\dots,z_m)\cdots g_N(z_1,\dots,z_m)$$
$$f(z_1,\dots,z_m) = f_1(z_1,\dots,z_m)\cdots f_N(z_1,\dots,z_m)$$

one has to realize the simpler m-D transfer functions: $\frac{g_1(z_1,\dots,z_m)}{f_1(z_1,\dots,z_m)},\dots,\frac{g_N(z_1,\dots,z_m)}{f_N(z_1,\dots,z_m)}$. As the stability tests are in the form: check if $f(z_1,\dots,z_m) \neq 0$ (in appropriate regions of $z_1,\dots,z_m$), it is important to factorize f, because in this case the stability test is separated into simpler ones. Both of these applications lie in the fields of image processing, geophysical and seismic data processing, biomedical digital systems, etc.

Finally there is an obvious importance of the m-D factorization subject from a pure mathematical point of view [9].

However, up to now, the general factorization problem, i. e. the factorization of each factorizable polynomial, has not been solved yet. For this reason, some more or less special types of m-D polynomial factorization have been studied [9, 10, 12, 15, 16, 17, 22].

In [22], the factorization in factors of one variable i. e. $f(z_1, \ldots, z_m) = f_1(z_1) \ldots \ldots f_m(z_m)$, and in factors with no common variables i. e. $f(z_1, \ldots, z_m) = f_1(\bar{z}_1) \ldots \ldots f_k(\bar{z}_k)$ ($\bar{z}_1, \ldots, \bar{z}_k$ are mutually disjoint groups of independent variables) has been studied. In [16], the factorization is succeeded by considering the given polynomial as $(1-D)$ polynomial with respect to z_j and applying the well known formulas from 1-D algebra. In [12], the factorization of the state-space model is investigated. In [10], the factorization of an m-D polynomial in factors where at least one factor contains no more than $m-1$ variables has been examined. In [17], the factorization is achieved by factorizing two other lower order polynomials. In [15], the factorization of an m-D polynomial in linear factors has been studied. Papers [10, 12, 15, 16, 17, 22] actually publish the material of [9] which is investigated exclusively by the author.

If one type of factorization does not hold though it is desirable, the (optimum) approximation of the original polynomial by a factorizable (of the considered type) one is attempted.

Work in approximate m-D polynomial factorization can be found in the papers [11, 13, 14]. In this paper, the results of [15] are briefly presented. Furthermore, the approximate factorization into linear but also stable factors is attempted. The problem is reduced into a non-linear minimization problem with constraints. The constraints are imposed by sufficient stability conditions. The constrained minimization problem can be solved by various numerical techniques.

2. EXACT FACTORIZATION INTO LINEAR m-D FACTORS

The problem of factorization of an m-D polynomial

$$f = f(z_1, \ldots, z_m) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_m=0}^{N_m} a(i_1, \ldots, i_m)\, z_1^{i_1} \ldots z_m^{i_m} \tag{1}$$

has been solved ([9] and [15]) in the case that the m-D polynomial can be written as a product of linear factors:

$$f(z_1, \ldots, z_m) = \prod_{i=1}^{N_1} (z_1 + a_{i,2} z_2 + \cdots + a_{i,m} z_m + c_i) \tag{2}$$

In [9] and [15], the sufficient and necessary conditions for such a factorization are given. More analytically, the following Theorems are proved. These Theorems provide the values of the unknown coefficients $a_{i,2}, \ldots, a_{i,m}, c_i$, as well as the necessary and sufficient conditions for the existence of such a factorization.

Theorem 1. For the polynomial given in (1) suppose that: $N_1 = \max(N_1, \ldots, N_m)$, $a(N_1, 0, \ldots, 0) = 1$. It is considered that no monomial of $f(z_1, \ldots, z_m)$ has degree greater than N_1. Iff (if and only if)

$$a(i_1, \ldots, i_m) = \sum \left(\prod^{i_2} a_{i,2}\right) \cdots \left(\prod^{i_m} a_{i,m}\right) \left(\prod^{N_1 - i_1 - \cdots - i_m} c_i\right) \tag{3}$$

where the product

$$\prod^{i_k} a_{i,k}, \quad k = 2, \ldots, m$$

involves only i_k coefficients of N_1 factors $a_{i,k}$, $i = 1, \ldots, N_1$; so the sum in the right-hand side of equality (3) involves all possible terms formed by splitting the integer N_1 into integers $i_1, \ldots, i_m, N_1 - i_1 - \cdots - i_m$, (therefore this sum has

$$\binom{N_1}{i_1, \ldots, i_m} = \frac{N_1!}{i_1! \cdots i_m!(N_1 - i_1 - \cdots - i_m)!}$$

terms), then the above polynomial can be factored as in (2) where $-c_i$ are the roots of the 1-D polynomial $f(z_1, 0, \ldots, 0)$ and

$$a_{i,k} = \left.\frac{df(z_1, \ldots, 0, z_k, 0, \ldots, 0)}{dz_k}\right|_{\substack{z_k=0 \\ z_1=-c_i}} \cdot \left.\frac{z_1 + c_i}{f(z_1, 0, \ldots, 0)}\right|_{z_1=-c_i}. \tag{4}$$

Proof. The proof can be found in [9] as well as in [15]. □

The number of the relations (3) is λ where $\lambda = ((N_1 + m)!/(N_1!m!)) - 1$, but one excludes the N_1 relations that in the left-hand side of (3) have $a(j, 0, \ldots, 0)$ with $j = 0, \ldots, N_1 - 1$ and the $N_1(m-1)$ relations that in the left-hand side of (3) have $a(j, 0, \ldots, 0, 1, 0, \ldots, 0)$ with $j = 0, \ldots, N_1 - 1$.

Theorem 1 can be applied in the case where all the roots of $f(z_1, 0, \ldots, 0)$ are simple. If there are i_1, i_2 such that $c_{i_1} = c_{i_2}$, the results of the theorem cannot be used. In that case, the next theorem is valid.

Theorem 2. For the polynomial given in (1) suppose that: $N_1 = \max(N_1, \ldots, N_m)$, $a(N_1, 0, \ldots, 0) = 1$. It is considered that no monomial of $f(z_1, \ldots, z_m)$ has degree greater than N_1. Iff

$$a(i_1, \ldots, i_m) = \sum \left(\prod^{i_2} a_{i,2}\right) \cdots \left(\prod^{i_m} a_{i,m}\right) \left(\prod^{N_1 - i_1 - \cdots - i_m} c_i\right) \tag{3}$$

where the sum $\sum$ and the product

$$\prod^{i_k} a_{i,k}, \quad k = 2, \ldots, m$$

are defined as in Theorem 1, then the above polynomial can be factored as in (2), where $-c_i$ are the roots of the 1-D polynomial $f(z_1, 0, \ldots, 0)$ and $a_{i,k}$ are found from

(4), if $-c_i$ is a simple root, and $a_{i_s,k}$ has p values given from the following system of equations, if $-c_i$ is a p-tuple root ($s = 1, \ldots, p$).

$$\left.\frac{f'_{z_k}|_{z_2=\ldots=z_m=0}}{(z_1+c_i)^{p-1}}\right|_{z_1=-c_i} = \left(a_{i_1,k} + \cdots + a_{i_p,k}\right) \cdot \left.\frac{f(z_1,0,\ldots,0)}{(z_1+c_i)^p}\right|_{z_1=-c_i} \tag{5.1}$$

$$\left.\frac{f''_{z_kz_k}|_{z_2=\ldots=z_m=0}}{(z_1+c_i)^{p-2}}\right|_{z_1=-c_i=q!} (a_{i_1,k}a_{i_2,k} + \cdots + a_{i_1,k}a_{i_p,k} + a_{i_2,k}a_{i_3,k}$$

$$+a_{i_2,k}a_{i_p,k} + \cdots + a_{i_{p-1},k}a_{i_p,k}) \cdot \left.\frac{f(z_1,0,\ldots,0)}{(z_1+c_i)^p}\right|_{z_1=-c_i} \tag{5.2}$$

$$\vdots$$

$$\left.f^{(p)}_{z_k\ldots z_k}\right|_{z_2=\ldots=z_m=0} = p!\left(a_{i_1,k}\cdots a_{i_p,k}\right) \cdot \left.\frac{f(z_1,0,\ldots,0)}{(z_1+c_i)^p}\right|_{z_1=-c_i} \tag{5.p}$$

Proof. The proof is presented in [9] as well as in [15]. □

The number of the relations (3) is λ, where $\lambda = \frac{(N_1+m)}{N_1!m!} - 1$ and one must exclude exactly the same relations as in Theorem 1.

The Theorems 1 and 2 can be stated in an algorithmic form:

Step 1: Read $a(i_1, \ldots, i_m)$.

Step 2: Arrange the variables $z_1, \ldots, z_m$ to get $N = N_1 = \max(N_1, \ldots, N_m)$.

Step 3: Check, if $a(N_1, 0, \ldots, 0) \neq 0$ and $a(i_1, \ldots, i_m) = 0$ for $i_1 + \cdots + i_m > N_1$. If this is not the case, the method is not applied. → END.

Step 4: Let $a(i_1, \ldots, i_m) := \frac{a(i_1,\ldots,i_m)}{a(N_1,0,\ldots,0)}$.

Step 5: Find (numerically) the roots $-c_i$ of the polynomial $f(z_1, 0, \ldots, 0)$.

Step 6: Check if $-c_i$ is a simple or a multiple root then *Find* $a_{i,k}$ from (4). If $-c_i$ is a $p-tuple$ root then *Find* $a_{i_1,k}, \ldots a_{i_p,k}$ from (5.1)–(5.p). *Solve* this system using the Vieta polynomial.

Step 7: Check the validity of (3). → END.

3. NECESSARY AND SUFFICIENT CONDITIONS OF STABILITY

It is remembered that a system is said to be Bounded-Input-Bounded-Output (BIBO) stable whenever a bounded input always produces a corresponding bounded output.

Consider an m-D system with transfer function:

$$G(z_1,\ldots,z_m)=\frac{g(z_1,\ldots,z_m)}{f(z_1,\ldots,z_m)}=\frac{\sum_{i_1=0}^{K_1}\cdots\sum_{i_m=0}^{K_m} g(i_1,\ldots,i_m)\,z_1^{i_1}\ldots z_m^{i_m}}{\sum_{i_1=0}^{N_1}\cdots\sum_{i_m=0}^{N_m} f(i_1,\ldots,i_m)\,z_1^{i_1}\ldots z_m^{i_m}} \tag{6}$$

where $g(z_1,\ldots,z_m)$, $f(z_1,\ldots,z_m)$ are coprime polynomials.

In the case that $g(z_1,\ldots,z_m)=1$, several BIBO stability criteria and tests have been formulated recently. Some of them are the following.

Test 1 (Shanks, Treital, Justice 1972) [21].
The system described by Equation (6) with $g(z_1,\ldots,z_m)\equiv 1$ is BIBO stable iff $f(z_1,\ldots,z_m)\neq 0$, for $|z_1|\leq 1,\ldots,|z_m|\leq 1$.

Test 2 (DeCarlo, Murray, Saeks) [3, 4, 19].
The system described by Equation (6), with $g(z_1,\ldots,z_m)\equiv 1$, is BIBO stable iff $f(z,\ldots,z)\neq 0$ for $|z|\leq 1$, and $f(z_1,\ldots,z_m)\neq 0$ for$|z_1|=\ldots=|z_m|=1$.

Using the Test 1 (Shanks, Treital, Justice), one can derive the following sufficient stability condition for the polynomial given in (1)

$$\sum_{\substack{i_1=0\\(i_1,\ldots,i_n)\neq(0,\ldots,0)}}^{N_1}\cdots\sum_{i_m=0}^{N_m} a^2(i_1,\ldots,i_m)<\frac{a_{0,\ldots,0}^2}{(N_1+1)\cdots(N_m+1)-1}. \tag{7}$$

Definition 1. A polynomial $f(z_1,\ldots,z_m)$ is said to be stable polynomial iff the system described by the transfer function $G(z_1,\ldots,z_m)=\frac{1}{f(z_1,\ldots,z_m)}$ is stable.

4. PROBLEM FORMULATION AND SOLUTION

Consider the polynomial $f=f(z_1,\ldots,z_m)$ as in (1). Suppose that, using the previous algorithm, this is not factorized into a product of linear m-D factors. In attempt to "factorize" f approximately an unknown factorizable polynomial $\tilde{f}(z_1,\ldots,z_m)$ is considered:

$$\begin{aligned}\tilde{f}(z_1,\ldots,z_m) &= \sum_{i_1=0}^{N_1}\cdots\sum_{i_m=0}^{N_m}\tilde{a}(i_1,\ldots,i_m)\,z_1^{i_1}\ldots z_m^{i_m}\\ &= \prod_{i=1}^{N_1}(z_1+\tilde{a}_{i,2}z_2+\cdots+\tilde{a}_{i,m}z_m+\tilde{c}_i)\end{aligned}$$

and the norm $\|f-\tilde{f}\|_2$ is minimized (subject to the stability constraints), where

$$
\begin{aligned}
\|f-\tilde{f}\|_2^2 &= \|f(z_1,\dots,z_m)-\tilde{f}(z_1,\dots,z_m)\|_2^2 \\
&= \left\| f(z_1,\dots,z_m) - \prod_{i=1}^{N_1}(z_1+\tilde{a}_{i,2}z_2+\dots+\tilde{a}_{i,m}z_m+\tilde{c}_i) \right\|_2^2 \\
&\hat{=} \sum_{\substack{i_1=0 \\ (i_1,\dots,i_n)\neq(0,\dots,0)}}^{N_1} \cdots \sum_{i_m=0}^{N_m} \left(a(i_1,\dots,i_m)-\tilde{a}(i_1,\dots,i_m) \right)^2
\end{aligned} \tag{8}
$$

where the symbol $\tilde{}$ is used for the corresponding quantities of the unknown factorizable polynomial $\tilde{f}(z_1,\dots,z_m)$.

The stability constraints are derived by (7)

$$
1+\sum_{k=2}^{m}\tilde{a}_{i,k}^2 < \frac{\tilde{c}_i^2}{2^m-1}, \quad i=1,\dots,N_1.
$$

In attempt to make the inequality "weak" inequality (and proper for the constraint minimization problem), a small positive number ε is considered, for example $\varepsilon = .95$

$$
1+\sum_{k=2}^{m}\tilde{a}_{i,k}^2 \le \varepsilon\cdot\frac{\tilde{c}_i^2}{2^m-1}, \quad i=1,\dots,N_1. \tag{9}
$$

This minimization problem is solved by several numerical techniques. In the present paper, the minimization is achieved using the Penalty method [18]. To this end, we write the constraints

$$
1+\sum_{k=2}^{m}\tilde{a}_{i,k}^2 - \varepsilon\cdot\frac{\tilde{c}_i^2}{2^m-1}+y_i^2=0, \quad i=1,\dots,N_1.
$$

We set

$$
\begin{aligned}
F &= \|f-\tilde{f}\|_2^2 = \|f(z_1,\dots,z_m)-\tilde{f}(z_1,\dots,z_m)\|_2^2 \\
&= \|f(z_1,\dots,z_m) - \prod_{i=1}^{N_1}(z_1+\tilde{a}_{i,2}z_2+\dots+\tilde{a}_{i,m}z_m+\tilde{c}_i)\|_2^2 \\
&\hat{=} \sum_{\substack{i_1=0 \\ (i_1,\dots,i_n)\neq(0,\dots,0)}}^{N_1} \cdots \sum_{i_m=0}^{N_m} \left(a(i_1,\dots,i_m)-\tilde{a}(i_1,\dots,i_m) \right)^2 .
\end{aligned}
$$

Following the penalty method, we seek for the following minimum for the increasing values of μ, $\mu = 10, 10^2, 10^3, \dots, 10^\nu, \dots$

$$
\min\left(F+\mu\left(\sum_{i=1}^{N_1}\left[\max\left(0, 1+\sum_{k=0}^{m}\tilde{a}_{i,k}^2-\varepsilon\cdot\frac{\tilde{c}_i^2}{2^m-1}+y_i^2\right)\right]^2\right)\right).
$$

A sequence of minima is obtained, the limit of which is the final minimum. Therefore $\tilde{a}_{i,k}$, $\tilde{c}_i$, $i = 1, \dots, N_1$, $k = 2, \dots, m$ can be found.

Another approach to the above problem is to transform the constrained minimization problem into an unconstrained minimization problem, [20]. For example, for the inequality $1 + x^2 + y^2 \le a^2 z^2$, where a is a constant, an adequate transformation of the variables is: $x = \sinh t_1 \cdot \cos t_2$, $y = \sinh t_1 \cdot \sin t_2$ and $z = (\cosh t_1)/(a \cos t_3)$.

5. EXAMPLE

The Example refers to a 2-D polynomial which can be, for example, the characteristic polynomial of a 2-D filter.

$$f(z_1, z_2) = z_1^2 + 1.2z_2^2 + 2.3z_1z_2 + 4.2z_1 + 3.6z_2 + 2.7.$$

This polynomial is an unstable 2-D polynomial (Test 1), since for $z_2 = 0$ and $z_1 = -0.7923$ we have $f(-0.7923, 0) = 0$.

After the calculations, it is seen that the necessary and sufficient conditions for factorization into linear 2-D factors are not satisfied. So, the approximation of $f(z_1, z_2)$ by the factorizable polynomial $\tilde{f}(z_1, z_2)$ is attempted:

$$\tilde{f}(z_1, z_2) = \prod_{i=1}^{2} (z_1 + \tilde{a}_{i,2} z_2 + \tilde{c}_i)$$

or in a simpler notation

$$\begin{aligned} \tilde{f}(z_1, z_2) &= (z_1 + qz_2 + p)(z_1 + uz_2 + r) = z_1^2 + quz_2^2 + (u + q)\, z_1 z_2 \\ &\quad + (p + r)\, z_1 + (qr + up)\, z_2 + pr \end{aligned}$$

where

$$p = \tilde{c}_1, \quad q = \tilde{a}_{1,2}, \quad r = \tilde{c}_2, \quad u = \tilde{a}_{2,2}.$$

First the unconstrained minimization of $||f - \tilde{f}||_2$ or $||f - \tilde{f}||_2^2$ is attempted, where:

$$||f - \tilde{f}||_2^2 = (1.2 - qu)^2 + (2.3 - u - q)^2 + (4.2 - p - r)^2 + (3.6 - qr - up)^2 + (2.7 - pr)^2.$$

Using a numerical routine (for example Levenberg–Marquardt routine [1, 2, 6, 7], the following solution is obtained: $p = 0.791725$, $q = 0.68437$, $r = 3.39725$, $u = 1.64481$ and $||f - \tilde{f}||_2^2 = 0.00734$. So, we can write: $f \cong \tilde{f}$ i.e.

$$\begin{aligned} f(z_1, z_2) &\cong \tilde{f}(z_1, z_2) = (z_1 + 0.68437z_2 + 0.791725)(z_1 + 1.64481z_2 + 3.39752) \\ &= z_1^2 + 1.12566z_2^2 + 2.32918z_1z_2 + 4.18924z_1 + 3.6274z_2 + 2.6899. \end{aligned}$$

One can see that the new factorizable polynomial $\tilde{f}(z_1, z_2)$ that approximates $f(z_1, z_2)$ is also unstable, since $\tilde{f}(-.791725, 0) = 0$. The instability can also be checked by various tests that have been published in the relevant literature [5], [23]. Therefore, we ought to seek for a new stable factorizable polynomial $\hat{f}(z_1, z_2)$. For this reason the minimization of $||f - \tilde{f}||_2^2$ is attempted subject to the constraints:

$$1 + q^2 < p^2/3, \quad 1 + u^2 < r^2/3$$

or

$$1+q^2 \leq p^2/3.1, \quad 1+u^2 \leq r^2/3.1.$$

A succesful transformation of the variables is

$$q=\sinh t_1,$$

$$p=(\sqrt{3.1}\ \cosh t_1)/\cos t_2, \quad u=\sinh t_3, \quad r=(\sqrt{3.1}\ \cosh t_1)/\cos t_4.$$

Therefore the problem is transformed into an unconstrained optimization problem, the solution of which is:

$$p=2.11356, \quad q=0.664088,$$

$$r=2.11356, \quad u=0.664088 \quad \text{and} \quad \|f-\hat{f}\|_2^2=5.27258.$$

Finally, the factorizable and stable polynomial is:

$$\begin{aligned} f(z_1,z_2) &\cong \hat{f}(z_1,z_2)=(z_1+0.664088z_2+2.11356)^2 \\ &= z_1^2+0.441012z_2^2+1.32818z_1z_2+4.22712z_1+2.80718z_2+4.46714. \end{aligned}$$

In Figure 1, the amplitude of the transfer function $B(z_1,z_2)=f$ is sketched when $z_1=e^{j\omega_1}$, $z_2=e^{j\omega_2}$ and $\omega_1 \in [0,2\pi]$, $\omega_2 \in [0,2\pi]$. In Figures 2 and 3, the amplitude of the transfer functions $B(z_1,z_2)=\tilde{f}$ and $B(z_1,z_2)=\hat{f}$ are also sketched with $z_1=e^{j\omega_1}$, $z_2=e^{j\omega_2}$, and $\omega_1 \in [0,2\pi]$, $\omega_2 \in [0,2\pi]$.

In Figures 4, 5, 6 the amplitude of the transfer functions $B(z_1,z_2)=\frac{1}{f}$, $B(z_1,z_2)=\frac{1}{\tilde{f}}$, and $B(z_1,z_2)=\frac{1}{\hat{f}}$ are also sketched when z_1 and z_2 belong to the same domains.

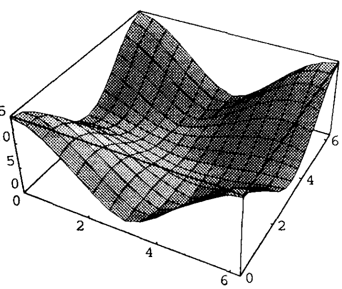

Fig. 1. Abs[f] versus ω_1, ω_2 : $\omega_1 \in [0,2\pi], \omega_2 \in [0,2\pi]$.

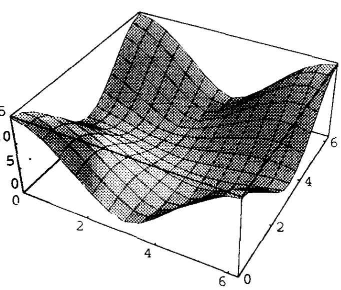

Fig. 2. Abs[$\tilde{f}$] versus ω_1, ω_2 : $\omega_1 \in [0,2\pi], \omega_2 \in [0,2\pi]$.

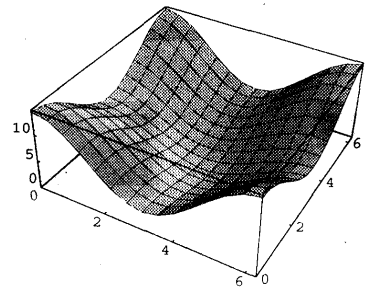

Fig. 3. Abs[$\hat{f}$] versus ω_1, $\omega_2 : \omega_1 \in [0, 2\pi], \omega_2 \in [0, 2\pi]$.

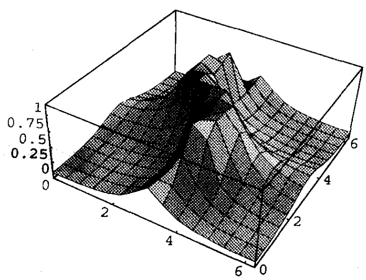

Fig. 4. Abs[$1/f$] versus ω_1, $\omega_2 : \omega_1 \in [0, 2\pi], \omega_2 \in [0, 2\pi]$.

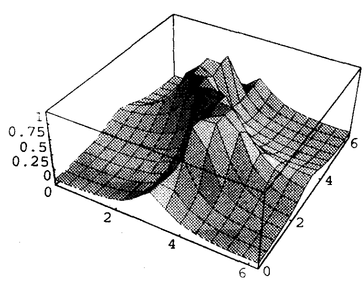

Fig. 5. Abs[$1/\tilde{f}$] versus ω_1, $\omega_2 : \omega_1 \in [0, 2\pi], \omega_2 \in [0, 2\pi]$.

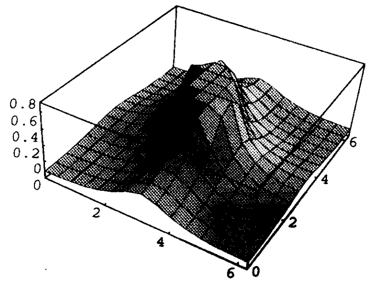

Fig. 6. Abs$[1/\hat{f}]$ versus $\omega_1,\ \omega_2 : \omega_1 \in [0, 2\pi], \omega_2 \in [0, 2\pi]$.

In Figures 7, 8, 9, 10, 11, 12 the impulse response of the 2-dimensional systems described by the transfer functions $B(z_1, z_2) = f,\ B(z_1, z_2) = \tilde{f}, B(z_1, z_2) = \hat{f},\ B(z_1, z_2) = \frac{1}{f},\ B(z_1, z_2) = \frac{1}{\tilde{f}}, B(z_1, z_2) = \frac{1}{\hat{f}}$ are also sketched respectively.

Remark 1. One must note that our factorization problem is applied for the systems having $B(z_1, z_2) = f$ and $B(z_1, z_2) = \frac{1}{\tilde{f}}$. In the case of $B(z_1, z_2) = f$, we have to solve an unconstrained minimization problem, while in the case of $B(z_1, z_2) = \frac{1}{\tilde{f}}$, we have to solve the corresponding constrained one. So, the system with $B(z_1, z_2) = \hat{f}$ has no (practical) meaning and therefore the impulse response of Figure 9 is provided only for the reason of completeness.

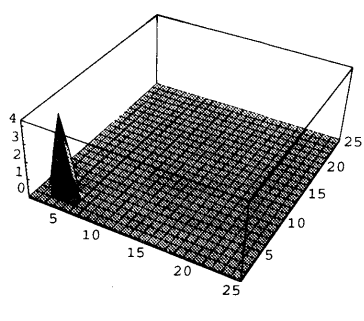

Fig. 7. Impulse Response of a 2-D system with $B(z_1, z_2) = f$.

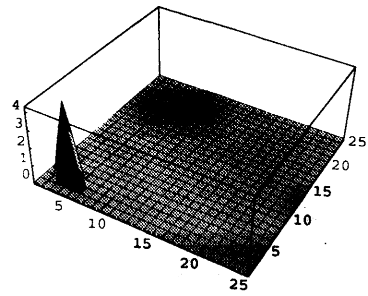

Fig. 8. Impulse Response of a 2-D system with $B(z_1, z_2) = \tilde{f}$.

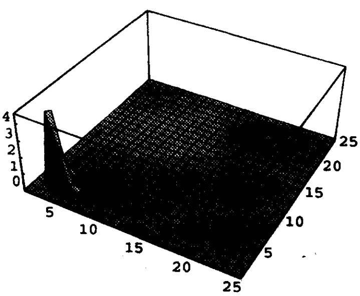

Fig. 9. Impulse Response of a 2-D system with $B(z_1, z_2) = \hat{f}$.

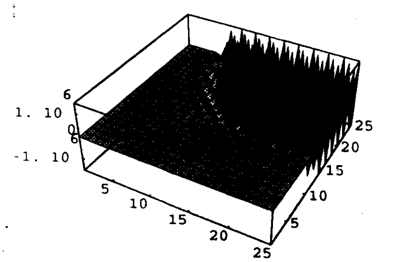

Fig. 10. Impulse Response of a 2-D system with $B(z_1, z_2) = 1/f$.

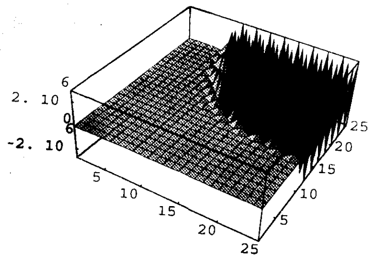

Fig. 11. Impulse Response of a 2-D system with $B(z_1, z_2) = 1/\tilde{f}$.

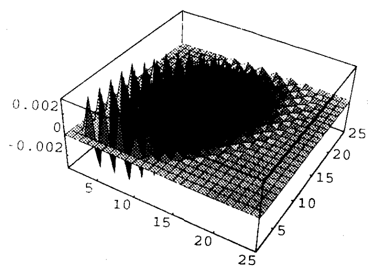

Fig. 12. Impulse Response of a 2-D system with $B(z_1, z_2) = 1/\hat{f}$.

6. CONCLUSION

An m-D polynomial which is not exactly factorized into linear m-D polynomial factors is considered. This polynomial can be approximately factorized into linear m-D factors in the sense of the least square approach. This simple technique can be proved very useful in m-D filters, m-D networks and DPS (Distributed Parameter Systems) design since, in the most cases, the exact m-D factorization is impossible. Furthermore, the "stability" of the new polynomial is desirable. So, a constrained optimization problem results. A 2-D numerical example is given in which, finally, we compare a transfer function having the original (unfactorizable) polynomial as numerator or denominator and two other transfer functions having the corresponding factorizable/factorizable and stable polynomial as numerator or denominator respectively.

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