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PARALLEL ALGORITHMS FOR INITIAL AND BOUNDARY VALUE PROBLEMS FOR LINEAR ORDINARY DIFFERENTIAL EQUATIONS AND THEIR SYSTEMS

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New parallel algorithms for solving initial and boundary value problems for linear ODEs and their systems on large parallel MIMD computers are proposed.

The proposed algorithms are based on dividing a problem in similar so-called local problems, which can be solved independently and in parallel using any known (sequential or parallel) method. The solution is then built as a linear combination of the local solutions.

The recurrence relationships (for the case of non-homogeneous equations) and explicit expressions (for the case of homogeneous equations) for the coefficients of that linear combination are obtained.

Three elementary examples, illustrating the idea of the proposed approach, are given.

1. INTRODUCTION

The majority of parallel algorithms were developed for solving algebraic problems and boundary value problems for partial differential equations (PDEs). With the exception of the parallelization of methods of the Runge–Kutta type and their modifications, almost no attention was paid to the development of parallel algorithms for ordinary differential equations (ODEs), and the available literature reflects this state [1]–[5]. However, not every parallel algorithm for solving PDEs is applicable for solving ODEs.

Some new parallel algorithms for solving initial and boundary value problems for linear ODEs and their systems are described and illustrated in this paper.

The proposed approach is based on two main ideas.

1. The first idea is that, in fact, we always deal with finite intervals when we look for the numerical solution of any initial value problem. Even when the given interval (in the formulation of a problem) is infinite, we can obtain the numerical solution of a problem only for the finite subinterval of the given original infinite interval. So it seems to be natural to apply numerical methods directly to the finite (sub)interval of the researcher's interest.

Briefly: the first important idea is the finiteness of the considered interval for the numerical solution.

2. When one reads or hears the words "the initial value problem", only the sequentiality of the numerical procedure is automatically assumed. One starts at the beginning (say, at $x = 0$) and then goes step by step to the "future" (x runs to the end of the considered interval). It seems to be impossible to calculate something of the solution in "the far future" without a set of all those previous steps — the solution seems to be strongly dependent on its "past". Probably because of such traditional understanding, all known parallel methods for ODEs solve the common problem: how to make those sequential steps faster (but steps are still made sequentially).

However, for linear ODEs the rejection of such traditional understanding opens new ways. Instead of making sequential steps, we look for something like building elements (from the beginning of the considered interval to the end), and construct the solution using these building elements. It is very important that the building elements, corresponding to subintervals of the considered interval, can be found absolutely independently and in parallel.

Briefly: the second idea is to prepare (in parallel) a set of building elements for the whole considered interval and to use them for constructing the solution.

In this paper we introduce the parallel algorithms for linear ODEs and their systems. The proposed approach is illustrated with the help of three elementary examples. One of them (see Section 4.2) shows that such an approach can be used also for solving boundary value problems for LODEs.

Implementation, speed-up and related problems do not belong to the scope of this paper and will be discussed separately.

2. CAUCHY'S PROBLEM FOR NON-HOMOGENEOUS LINEAR ODE

Let us consider the following Cauchy problem:

$$L_m[y] = f(x), \quad x \in [a,b] \tag{1}$$

$$y^{(j-1)}(a) = y_{j-1}, \quad (j = 1, \ldots, m), \tag{2}$$

where $L_m[y] = \sum_{k=0}^{m} p_k(x) y^{(k)}(x)$; $p_k(x)$ $(k = 0, \ldots, m)$ and $f(x)$ are continuous in the closed interval $[a,b]$. We look for the solution of the problem (1)–(2) in the interval $[a,b]$.

Let us divide this interval in n subintervals:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

and look for the solution of the problem (1)–(2) in the form:

$$y(x) = \begin{cases} \sum_{k=1}^{m} a_{ik} u_{ik}(x) + v_i(x), \quad x \in I_i, \\ \qquad\qquad I_i = [x_{i-1}, x_i], \quad (i = 1, \ldots, n). \end{cases} \tag{3}$$

The function $u_{ik}(x)$ is equal to 0 outside I_i and it is equal to the solution of the problem

$$\begin{cases} L_m[u_{ik}] = 0, & x \in I_i \\ u_{ik}^{(j-1)}(x_{i-1}) = \delta_{j-1,k-1}, & j = 1, \dots, m, \quad (k = 1, \dots, m) \end{cases} \tag{4}$$

when $x \in I_i$ ($\delta_{j,k}$ is Kroneker's delta).

Similarly, $v_i(x)$ is equal to 0 outside I_i and it is equal to the solution of the problem

$$\begin{cases} L_m[v_i] = f(x), & x \in I_i \\ v_i^{(j-1)}(x_{i-1}) = 0, & j = 1, \dots, m, \end{cases} \tag{5}$$

when $x \in I_i$.

The constant a_{ik} must be determined by satisfying the initial conditions (2) and the condition of continuity of $y^{(j-1)}(x)$, $(j = 1, \dots, m)$ in $[a, b]$. The second condition leads to satisfying continuity of $y^{(j)}(x)$ at the points $x = x_i$, $(i = 1, \dots, n-1)$.

The above considerations along with the use of Picard's theorem lead to the following result:

Theorem 1. If $p_\nu(x), (\nu = 0, \dots, m)$ and $f(x)$ are continuous in $[a, b]$, then the solution of the Cauchy problem (1)–(2) for the interval $[a, b]$ can be obtained in the form (3), where functions $u_{ik}(x)$ and $v_i(x)$ are defined above and the coefficients a_{ik} are given by the recurrence relations:

$$a_{1k} = y_{k-1},$$
$$a_{ik} = a_{i-1,k}u_{i-1,k}(x_{i-1}) + v_{i-1}^{(k-1)}(x_{i-1}), \quad (i = 2, \dots, n; \quad k = 1, \dots, m). \tag{6}$$

3. CAUCHY'S PROBLEM FOR A SYSTEM OF NON-HOMOGENEOUS LINEAR ODES

As will be shown below, the use of the appropriate notation makes obtaining similar results for systems of linear ODEs possible. Since the problem for a system of linear ODE's of order higher than one can easily be reduced to a problem for systems of linear ODE's of the first order, we can consider only systems of the first-order equations without loss of generality.

Let us consider the following Cauchy problem:

$$\mathcal{Y}'(x) = \mathcal{C}(x)\mathcal{Y}(x) + F(x) \tag{7}$$
$$\mathcal{Y}(a) = \mathcal{Y}_0, \tag{8}$$

where

$$\mathcal{Y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_m(x) \end{pmatrix} \quad F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

$$\mathcal{C}(x) = \begin{pmatrix} c_{11}(x) & \cdots & c_{1m}(x) \\ c_{21}(x) & \cdots & c_{2m}(x) \\ \cdots & \cdots & \cdots \\ c_{m1}(x) & \cdots & c_{mm}(x) \end{pmatrix}$$

and $\mathcal{Y}(x)$, $F(x)$ and $\mathcal{C}(x)$ are continuous in the closed interval $[a,b]$. We will look for the solution of problem (9) – (10) in the interval $[a,b]$.

First, let us introduce the following notation:

$$U_{ik}(x) = \begin{pmatrix} u_{ik,1}(x) \\ u_{ik,2}(x) \\ \vdots \\ u_{ik,m}(x) \end{pmatrix} \quad V_i(x) = \begin{pmatrix} v_{i,1}(x) \\ v_{i,2}(x) \\ \vdots \\ v_{i,m}(x) \end{pmatrix} \quad A_i = \begin{pmatrix} a_{i,1} \\ a_{i,2} \\ \vdots \\ a_{i,m} \end{pmatrix}$$

$$e_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow (k) \;; \qquad \mathcal{U}_i(x) = (U_{i1}(x), U_{i2}(x), \ldots, U_{im}(x))$$

$U_{ik}(x)$ and $V_i(x)$ are column vector functions, A_i is a constant column vector, $\mathcal{U}_i(x)$ is a matrix consisting of the column vector functions $U_{i1}(x)$, $U_{i2}(x)$, …, $U_{im}(x)$.

Let us divide $[a,b]$ in n subintervals:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

and look for the solution of the problem (9) – (10) for the interval $[a,b]$ in the following form:

$$\mathcal{Y}(x) = \begin{cases} \mathcal{U}_i(x)\,A_i + V_i(x), & x \in I_i \\ & i = 1, \ldots, n, \end{cases} \tag{9}$$

where $U_{ik}(x)$ (the columns of $\mathcal{U}_i(x)$) and $V_i(x)$ are the solutions of the following problems:

$$\begin{cases} U'_{ik}(x) = \mathcal{C}(x)\,U_{ik}(x), & x \in I_i \\ U_{ik}(x_{i-1}) = e_k & (k = 1, \ldots, m) \end{cases} \tag{10}$$

$$\begin{cases} V'_i(x) = \mathcal{C}(x)\,V_i + F(x), & x \in I_i \\ V_i(x_{i-1}) = 0. \end{cases} \tag{11}$$

$U_{ik}(x)$ and $V_i(x)$ are equal to the zero vector outside their subintervals. We call $U_{ik}(x)$ and $V_i(x)$ the "local solutions".

The constant vector A_i must be determined by satisfying the initial condition (10) and the condition of continuity of $\mathcal{Y}(x)$ in $[a,b]$. The last condition reduces to satisfy continuity of $\mathcal{Y}(x)$ for $x = x_i$, $(i = 1, \ldots, n-1)$.

The above considerations along with the use of Picard's theorem complete the proof of the following theorem:

Theorem 2. If $\mathcal{C}(x)$ and $F(x)$ are continuous in $[a,b]$, then the solution of the Cauchy problem (9)–(10) for the interval $[a,b]$ can be obtained in the form (11), where $\mathcal{U}_i(x)$ and $V_i(x)$ are defined above and coefficients A_i are given by the recurrence relations:

$$A_1 = \mathcal{Y}_0; \quad A_i = \mathcal{U}_{i-1}(x_{i-1})\, A_{i-1} + V_{i-1}(x_{i-1}), \quad (i = 2, \ldots, n). \tag{12}$$

In the case of a homogeneous system ($F(x) \equiv 0$) the explicit formula for A_i can be obtained instead of the recurrence relations:

$$A_i = \mathcal{U}_{i-1}(x_{i-1})\,\mathcal{U}_{i-2}(x_{i-2}) \ldots \mathcal{U}_1(x_1)\,\mathcal{Y}_0 \tag{13}$$

and the solution can be expressed explicitly with the help of the local solutions:

Theorem 3. If $\mathcal{C}(x)$ is continuous in $[a,b]$, then the solution of Cauchy's problem (9)–(10) for $F(x) \equiv 0$ can be obtained in the form:

$$\mathcal{Y}(x) = \begin{cases} \mathcal{U}_i(x_i)\,\mathcal{U}_{i-1}(x_{i-1})\,\mathcal{U}_{i-2}(x_{i-2}) \ldots \mathcal{U}_1(x_1)\,\mathcal{Y}_0, & x \in I_i \\ & i = 1, \ldots, n. \end{cases}$$

4. EXAMPLES

To illustrate the proposed approach, we will consider three elementary examples, where local problems can be solved analytically as well as main problems. Moreover, for simplicity the division of a main interval into only two subintervals is used.

4.1. An initial value problem for a non-homogeneous equation

Let us consider the problem:

$$\begin{cases} y'(x) - y(x) = x \\ y(0) = 1 \end{cases} \tag{14}$$

(which has the obvious solution $y(x) = 2\exp(x) - x - 1$), to find its solution for the interval $[0,2]$ by the proposed method.

First, we divide this interval into two smaller ones: $[0,1]$ and $[1,2]$. Then we introduce functions $u_1(x)$, $u_2(x)$, $v_1(x)$ and $v_2(x)$, which are equal to the solutions of the independent local problems when x belongs to the corresponding subinterval and which are equal to 0 outside these subintervals. The local problems and the local solutions are listed below:

a) for the interval $[0,1]$:

$$\begin{cases} u_1'(x) - u_1(x) = 0; \qquad u_1(x) = \exp(x) \\ u_1(0) = 1 \end{cases} \tag{15}$$

$$\begin{cases} v_1'(x) - v_1(x) = x; & v_1(x) = \exp(x) - x - 1 \\ u_1(0) = 0. \end{cases} \tag{16}$$

b) for the interval $[1, 2]$:

$$\begin{cases} u_2'(x) - u_2(x) = 0; & u_2(x) = \exp(x - 1) \\ u_2(1) = 1 \end{cases} \tag{17}$$

$$\begin{cases} v_2'(x) - v_2(x) = x; & v_2(x) = 2\exp(x - 1) - x - 1 \\ v_2(1) = 0. \end{cases} \tag{18}$$

Then we look for the solution in the form:

$$y(x) = \begin{cases} a_1 u_1(x) + v_1(x), & x \in [0, 1] \\ a_2 u_2(x) + v_2(x), & x \in [1, 2] \end{cases} \tag{19}$$

(which is the particular case of (3)).

Satisfying the initial condition ($y(0) = 1$), we obtain $a_1 = 1$. The continuity of $y(x)$ gives $a_2 = 2(e - 1)$. Therefore,

$$y(x) = \begin{cases} u_1(x) + v_1(x), & x \in [0, 1] \\ 2(e - 1)\, u_2(x) + v_2(x), & x \in [1, 2]. \end{cases} \tag{20}$$

It is easy to check that the derivative $y'(x)$ is continuous too, and that expression (22) gives for $x \in [0, 2]$ exactly $2\exp(x) - x - 1$, which is the classical form of the solution of problem (16).

4.2. A boundary value problem for the ODE of the second order

The proposed approach can also be used for solving boundary value problems for ODEs.

Let us consider the following problem:

$$\begin{cases} y''(x) - y(x) = 0 \\ y(0) = 1 \\ y(2) = \exp(2) \end{cases} \tag{21}$$

which has the obvious solution $y(x) = \exp(x)$. To use the proposed method, we divide the interval $[0, 2]$ into two subintervals and introduce, as above, the local solutions, which are equal to the solutions of the independent local problems when x belongs to the corresponding subinterval and which are equal to 0 outside these subintervals. The local problems and the corresponding local solutions are listed below:

a) for the interval $[0,1]$:

$$\begin{cases} u''_{10}(x) - u_{10}(x) = 0; \qquad u_{10}(x) = \cosh(x) \\ u_{10}(0) = 1 \\ u'_{10}(0) = 0 \end{cases} \tag{22}$$

$$\begin{cases} u''_{11}(x) - u_{11}(x) = 0; \qquad u_{11}(x) = \sinh(x) \\ u_{11}(0) = 0 \\ u'_{11}(0) = 1. \end{cases} \tag{23}$$

b) for the interval $[1,2]$:

$$\begin{cases} u''_{20}(x) - u_{20}(x) = 0; \qquad u_{20}(x) = \cosh(x-2) \\ u_{20}(2) = 1 \\ u'_{20}(2) = 0 \end{cases} \tag{24}$$

$$\begin{cases} u''_{21}(x) - u_{21}(x) = 0; \qquad u_{21}(x) = \sinh(x-2) \\ u_{21}(2) = 0 \\ u'_{21}(2) = 1. \end{cases} \tag{25}$$

Then we look for the solution in the following form:

$$y(x) = \begin{cases} a_{10}u_{10}(x) + a_{11}u_{11}(x), & x \in [0,1] \\ a_{20}u_{20}(x) + a_{21}u_{21}(x), & x \in [1,2]. \end{cases} \tag{26}$$

Satisfying the boundary conditions, we find: $a_{10} = 1$, $a_{20} = \exp(2)$. The continuity of $y(x)$ and $y'(x)$ gives: $a_{11} = 1$, $a_{21} = \exp(2)$.
Therefore,

$$y(x) = \begin{cases} u_{10}(x) + u_{11}(x), & x \in [0,1] \\ \exp(2)\,(u_{20}(x) + u_{21}(x)), & x \in [1,2]. \end{cases} \tag{27}$$

It is easy to check that $y''(x)$ is continuous too, and that expression (29) gives for $x \in [0,2]$ exactly $\exp(x)$, which is the classical form of the solution of problem (23).

4.3. An initial value problem for a system of linear ODEs

Let us consider the following problem:

$$\begin{cases} \mathcal{Y}'(x) = \mathcal{C}(x)\,\mathcal{Y}(x) \\ \mathcal{Y}(0) = e_2, \end{cases} \tag{28}$$

where

$$\mathcal{Y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \quad \mathcal{C}(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This problem has the obvious solution:

$$\mathcal{Y}(x) = \begin{pmatrix} \sinh(x) \\ \cosh(x) \end{pmatrix}. \tag{29}$$

To use the proposed method, we divide the interval $[0, 2]$ into two subintervals and introduce the local solutions, which are equal to the solutions of the independent local problems when x belongs to the corresponding subinterval and which are equal to 0 outside these subintervals. The local problems and the corresponding local solutions are listed below:

a) for the interval $[0, 1]$:

$$\begin{cases} U'_{11}(x) = \mathcal{C}(x)\, U_{11}(x) \\ U_{11}(0) = e_1 \end{cases} \qquad U_{11}(x) = \begin{pmatrix} \cosh(x) \\ \sinh(x) \end{pmatrix} \tag{30}$$

$$\begin{cases} U'_{12}(x) = \mathcal{C}(x)\, U_{12}(x) \\ U_{12}(0) = e_2 \end{cases} \qquad U_{12}(x) = \begin{pmatrix} \sinh(x) \\ \cosh(x) \end{pmatrix}. \tag{31}$$

b) for the interval $[1, 2]$:

$$\begin{cases} U'_{21}(x) = \mathcal{C}(x)\, U_{21}(x) \\ U_{21}(1) = e_1 \end{cases} \qquad U_{21}(x) = \begin{pmatrix} \cosh(x-1) \\ \sinh(x-1) \end{pmatrix} \tag{32}$$

$$\begin{cases} U'_{22}(x) = \mathcal{C}(x)\, U_{22}(x) \\ U_{22}(1) = e_2 \end{cases} \qquad U_{22}(x) = \begin{pmatrix} \sinh(x-1) \\ \cosh(x-1) \end{pmatrix}. \tag{33}$$

Then we look for the solution in the following form:

$$\mathcal{Y}(x) = \begin{cases} \mathcal{U}_1(x)\, A_1, & x \in [0, 1] \\ \mathcal{U}_2(x)\, A_2, & x \in [1, 2]. \end{cases} \tag{34}$$

Satisfying the initial condition, we find $A_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The continuity of $\mathcal{Y}(x)$ leads to the equation $\mathcal{U}_1(1)\, A_1 = \mathcal{U}_2(1)\, A_2$ from which we obtain $A_2 = \begin{pmatrix} \sinh(1) \\ \cosh(1) \end{pmatrix}$.

Therefore, the solution of problem (30) is:

$$\mathcal{Y}(x) = \begin{cases} U_{12}(x), & x \in [0, 1] \\ \sinh(1)\, U_{21}(x) + \cosh(1)\, U_{22}(x), & x \in [1, 2]. \end{cases} \tag{35}$$

It is easy to see that the derivative $\mathcal{Y}'(x)$ is continuous too, and that expression (37) gives for $x \in [0,2]$ exactly $\begin{pmatrix} \sinh(x) \\ \cosh(x) \end{pmatrix}$, which is the classical form of the solution of the considered problem (30).

5. CONCLUDING REMARKS

The proposed approach can be successfully used for solving initial and boundary value problems for linear ordinary differential equations especially on large parallel MIMD computers.

The most natural parallel computer architecture for the proposed method is a simple tree structure, where the local problems are solved independently and in parallel (using any known sequential or parallel numerical method) at leaves and their outputs are after that gathered at the root, where the recurrence relations (Theorems 1 – 4) are realized.

The accuracy of the main solution is determined by the accuracy of the obtained local solutions.

If every coefficient of the considered equations is given by the same analytical expression in the considered main interval, then the proposed approach could be easily vectorized. In such a case, number and lengths of steps in all subintervals must be the same. Under this condition, all computations necessary for obtaining local solutions can be performed simultaneously using vector computing facilities provided by hardware (vector processor) or software (such as MATLAB).

The proposed method requires almost no communication between processors (or the local problem solvers). This makes even the use of cheap computer networks (with slow data transfer) possible instead of expansive parallel computers.

However, the proposed method in its present form is not applicable for solving non-linear problems.

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