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ONE METHOD FOR ROBUST CONTROL OF UNCERTAIN SYSTEMS: THEORY AND PRACTICE

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We present a controller design methodology for uncertain systems which is based on the constructive use of Lyapunov stability theory. The uncertainties, which are deterministic, are characterized by certain structural conditions and known as well as unknown bounds. As a consequence of the Lyapunov approach, the methodology is not restricted to linear or time-invariant systems. The robustness of these controllers in the presence of singular perturbations is considered. The situation in which the full state of the system is not available for measurement is also considered as are other generalizations. Applications of the proposed controller are noted, and examples of some resource management problems are discussed.

1. INTRODUCTION[1]

A fundamental feedback control problem is that of obtaining some specified desired behavior from a system about which there is incomplete or uncertain information. Here we consider systems whose uncertainties are characterized deterministically rather that stochastically or fuzzily; for a stochastic approach see [6], and for fuzzy one see [34].

Our model of an uncertain system is of the form

$$\dot{x}(t) = F(t, x(t), u(t), \omega) \tag{1}$$

where $t \in \mathbb{R}$ is the "time" variable, $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the control input. All the uncertainty in the system is represented by the lumped uncertain element $\omega \in \Omega$. It could be an element of $\mathbb{R}^q$ representing constant unknown parameters and inputs; it could also be a function from $\mathbb{R}$ into $\mathbb{R}^q$ representing unknown time varying parameters and inputs; it could also be a function from $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^q$ representing nonlinear elements which are difficult to characterize exactly; it could be merely an index. $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is known. The only information assumed about ω is the knowledge of a nonempty set Ω to which it belongs. A related characterization of uncertainties is via inclusions see [35].

[1] Throughout this paper, references are intended to be representative rather than exhaustive. For a more complete bibliography see [40], [41].

Discrete systems are usually modelled by a difference equation

$$x(k+1) = F(k,\, x(k),\, u(k)) \tag{2}$$

where $k \in \mathbb{Z}$ is the "time", $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control, and F is not known but rather belongs to a set $\mathcal{F}$, with $\mathcal{F}$ known.

2. CONTINUOUS SYSTEM CONTROL

For continuous systems modelled by ordinary differential equations of the form (1) we consider control to be given by a memoryless state feedback controller

$$u(t) = p(t,\, x(t)). \tag{3}$$

Ideally we wish to choose $p : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ so that the feedback controlled system

$$\dot{x}(t) = f(t,\, x(t),\, \omega), \tag{4}$$

where

$$f(t,\, x,\, \omega) := F(t,\, x,\, p(t,\, x),\, \omega), \tag{5}$$

has the property of g.u.a.s. (global uniform asymptotic stability) about the zero state for all $\omega \in \Omega$ and for all initial states in $\mathbb{R}^n$. However to assure g.u.a.s. of an uncertain system one sometimes has to resort to controllers which are discontinuous in the state; see [26]. To avoid such discontinuous controllers, we relax the problem to that of obtaining a family of controllers which assure that the behavior of (1) can be made arbitrarily close to g.u.a.s.; such a family is called a practically stabilizing family see [17], [20].

2.1. A specific class of uncertain continuous systems

An uncertain continuous system under consideration here is described by (1) and satisfies the following assumption.

Assumption C.1.[2] There exist a continuous function $B : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n,m}$, a candidate Lyapunov function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$, a class K function $\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, functions $\beta_1,\, \beta_2 : \mathbb{R} \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}_+$ and continuous functions $\kappa,\, \rho : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}_+$ such that

$$F(t,\, x,\, u,\, \omega) = f_s(t,\, x,\, \omega) + B(t,\, x)\, g(t,\, x,\, u,\, \omega) \tag{6}$$

for some functions f_s and g which satisfy:

1. For each $\omega \in \Omega$, $f_s(\cdot,\, \omega)$ is continuous and

$$\frac{\partial V}{\partial t}(t,\, x) + \frac{\partial V}{\partial x}(t,\, x)\, f_s(t,\, x,\, \omega) \leq -\, \gamma(\|x\|) \tag{7}$$

for all $t \in \mathbb{R},\ x \in \mathbb{R}^n$.

[2] For definition see [13], [20].

2. For each $\omega \in \Omega$, $g(\cdot, \omega)$ is continuous and

$$u^T g(t, x, u, \omega) \geq -\beta_1(t, x, \omega)\,||u|| + \beta_2(t, x, \omega)\,||u||^2 \tag{8}$$

where

$$\beta_1(t, x, \omega) \leq \beta_2(t, x, \omega)\,\rho(t, x) \tag{9}$$

$$\beta_1(t, x, \omega) \leq \kappa(t, x) \tag{10}$$

for all $t \in I\!R$, $x \in I\!R^n$, $u \in I\!R^m$.

2.2. Proposed controllers

Here we present some practically stabilizing controller sets for the system considered in the previous section. These controllers can be regarded as continuous approximations of those presented in [26].

Consider any uncertain system described above and let $(B, V, \gamma, \rho, \kappa)$ be a quintuple which assures the satisfaction of Assumption C.1. Choose any continuous functions ρ_c, κ_c which satisfy

$$\rho_c(t, x) \geq \rho(t, x)\ ,\ \ \kappa_c(t, x) \geq \kappa(t, x) \tag{11}$$

and define

$$\alpha(t, x) := B(t, x)^T \frac{\partial V}{\partial x}(t, x)^T\,, \tag{12}$$

$$\eta(t, x) := \kappa_c(t, x)\,\alpha(t. x)\,. \tag{13}$$

A practically stabilizing family of controllers is the set

$$\mathcal{P} := \{p_\varepsilon \,|\, \varepsilon > 0\} \tag{14}$$

where p_ε is any continuous function which satisfies

$$||\alpha(t, x)||\,p_\varepsilon(t, x) = -||p_\varepsilon(t, x)\,||\,\alpha(t, x) \tag{15}$$

i.e., $p_\varepsilon(t, x)$ is opposite in direction to $\alpha(t, x)$, and

$$||\eta(t, x)|| > 0 \Longrightarrow ||p_\varepsilon(t, x)|| \geq \rho_c(t, x)\,[1 - ||\eta(t, x)||^{-1}\varepsilon]\,. \tag{16}$$

As an example of a function satisfying the above requirements on p_ε, consider

$$p_\varepsilon(t, x) := \begin{cases} -\frac{\eta(t,x)}{\varepsilon}\rho_c(t, x) & \text{if} \quad ||\eta(t, x)|| \leq \varepsilon \\ -\frac{\eta(t,x)}{||\eta(t,x)||}\rho_c(t, x) & \text{if} \quad ||\eta(t, x)|| > \varepsilon; \end{cases} \tag{17}$$

see [14].

As another example, consider

$$p_\varepsilon(t, x) := -\frac{\eta(t, x)}{||\eta(t, x)|| + \varepsilon}\,\rho_c(t, x)\,; \tag{18}$$

see [1].

Controllers of a discontinuous type as well as their continuous approximations, related to those proposed here, have been deduced by employing the theory of variable structure control; see [5]. Some early treatments of controller design for uncertain systems were based on "games against nature"; see [25]. Another class of controllers for systems of type (1) are deduced in [2].

2.3. Matching conditions

Given a system described by (1) the choice of B, f_s, g to assure satisfaction of Assumption C.1 (if possible) may not be obvious. This choice is usually easier if the uncertainties are *matched* in the sense that there exist functions f_0, B, g with $B(t, x) \in I\!R^{n,m}$ such that

$$F(t, x, u, \omega) = f_0(t, x) + B(t, x)\, g(t, x, u, \omega); \tag{19}$$

that is, the uncertainty ω and the control enter the system description via the same matrix $B(t, x)$.

Much of the literature concerns systems in which the uncertainties are matched. [4] and [11] consider systems with unmatched uncertainties; there the norm of the unmatched portion of the uncertain term must be smaller than a certain threshold value. In [52] linear systems are considered in which the uncertainty satisfies generalized matching conditions, that is, structural conditions which are less restrictive than the matching condition. In these cases, as in the matched case, the norm bounds of the uncertain terms can be arbitrarily large. Linear time-invariant systems with scalar control input are treated in [53], while Schmitendorf [47] requires the existence of a positive definite solution of a certain Riccati equation.

2.4. Other problems

While *global* uniform asymptotic stability or at least practical stability can be guaranteed provided the control is not constrained, only *local* stability can be assured if the available control is subject to constraints. One class of stabilization problems with control constraints is considered in [19], [51]. Controllers which assure not only practical stability but also exponential convergence at a prescribed rate are treated in [12], [13]. Corless and Leitmann [15] deal with systems in which the uncertainty bounds are not known exactly but depend on unknown constants; the controllers presented there are parameter adaptive controllers. Problems in which one wishes to keep the system state within or outside a prescribed region of the state space are considered in [21]. Systems with delay are considered in [55] and [38]. Corless and Leitmann [27] treat controllers which linearize a nominal system in addition to assuring stability of the actual one. Large scale uncertain systems with decentralized control are discussed in [9] and [49].

3. DISCRETE SYSTEMS CONTROL

The control of uncertain discrete systems modelled by difference equations of the form (2) has been treated in [22], [44] and [48]. Unlike in the continuous case reviewed in the previous section, arbitrarily large uncertainties cannot be tolerated, in general, and the region of ultimate attraction cannot be made arbitrarily small. Corless and Manela [22] consider the matched case, namely

$$x(k+1) = f(k,\, x(k)) + B(k,\, x(k))\, [u(k) + e(k,\, x(k),\, u(k))] \tag{20}$$

where $k \in \mathbb{Z}$, $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$. The functions $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ and $B : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^{n,m}$ are assumed known, with

$$\operatorname{rank}[B(k,\, x(k))] = m. \tag{21}$$

The function $e : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is not known; however, it is assumed that the class of functions $\mathcal{E}$ to which it belongs is known. We make the following two assumptions before stating a stabilization theorem.

Assumption D.1.[3] Given a positive definite $P \in \mathbb{R}^{n,n}$ there exist non-negative scalars ρ_0, ρ_1 and ρ_2 such that for all $e \in \mathcal{E}$

$$\|B(k,,\, x)\, e(k,\, x,\, u)\|_P \le \rho_0 + \rho_1 \|x\|_P + \rho_2 \|u\|_{R(k,x)} \tag{22}$$

for all $(k,\, x,\, u) \in \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^m$, where $R(k,\, x) := B(k,\, x)^T P B(k,\, x)$. Next we define

$$\psi(k,\, x) := \left[B(k,\, x)^T\, P B(k,\, x)\right]^{-1} B(k,\, x)^T P\,, \tag{23}$$

$$\phi(k,\, x) := B(k,\, x)\, \psi(k,\, x)\,, \tag{24}$$

$$\bar{f}(k,\, x) := \phi(k,\, x)\, f(k,\, x)\,, \tag{25}$$

and

$$\tilde{f}(k,\, x) := f(k,\, x) - \bar{f}(k,\, x)\,. \tag{26}$$

Assumption D.2. There exist a positive definite matrix $P \in \mathbb{R}^{n,n}$ and a non-negative scalar $\tilde{c} < 1$ such that

$$\|\tilde{f}(k,\, x)\|_P \le \tilde{c}\, \|x\|_P \tag{27}$$

for all $(k,\, x) \in \mathbb{Z} \times \mathbb{R}^n$. If $\rho_2 \ne 0$, then there also exist non-negative scalars c_0 and c_1 such that

$$\|\bar{f}(k,\, x)\|_P \le c_0 + c_1 \|x\|_P \tag{28}$$

for all $(k,\, x) \in \mathbb{Z} \times \mathbb{R}^n$.

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[3] Let $P \in \mathbb{R}^{n,n}$ be a positive definite matrix. We define the norm $\|\cdot\|_P : \mathbb{R}^n \to \mathbb{R}_+$ by $\|r\|_P := \sqrt{r^T P r}$.

3.1. Proposed controllers

Consider an uncertain discrete system (20) satisfying Assumptions D.1 – D.2 and subject to the control $u(k) = p(k, x(k))$ where $p(k, x(k))$ is defined as follows:

$$p(k, x(k)) := \begin{cases} 0 & \text{if} \quad \rho_2 \geq 1 \\ -\psi(k, x(k))\, f(k, x(k)) & \text{if} \quad \rho_2 < 1. \end{cases} \tag{29}$$

Suppose that

$$\tilde{c}^2 + (\rho_1 + c_1^*)^2 < 1 \tag{30}$$

where

$$c_1^* := \begin{cases} 0 & \text{if} \quad \rho_2 = 0 \\ \rho_2^* c_1 & \text{if} \quad \rho_2 \neq 0 \end{cases} \tag{31}$$

and

$$\rho_2^* := \min\{\rho_2, 1\}. \tag{32}$$

Then for all $e \in \mathcal{E}$, the feedback controlled system (20) is g.u.a.s. about the set

$$\mathcal{B}_P(d) := \{x \in I\!R^n \mid ||x||_P \leq d\} \tag{33}$$

where

$$d := \frac{\rho_0 + c_0^*}{\sqrt{1 - \tilde{c}^2 - (\rho_1 + c_1^*)}} \tag{34}$$

and

$$c_0^* := \begin{cases} 0 & \text{if} \quad \rho_2 = 0 \\ \rho_2^* c_0 & \text{if} \quad \rho_2 \neq 0. \end{cases} \tag{35}$$

4. ROBUSTNESS IN THE PRESENCE OF SINGULAR PERTURBATIONS

Consider an uncertain singularly perturbed system described by

$$\begin{aligned} \dot{x} &= F(t, x, y, u, \mu, \omega) \\ \mu\dot{y} &= G(t, x, y, u, \mu, \omega) \end{aligned} \tag{36}$$

where $(x, y) \in I\!R^n \times I\!R^l$ describe the state of the system. $\mu \in (0, \infty)$ is the singular perturbation parameter, and all the other variables are as described above. Here one wants to obtain memoryless feedback controllers (generating u) which assure that, for all $\omega \in \Omega$ and for all sufficiently small μ, the behavior of the feedback controlled system is close to that of g.u.a.s.

Assuming that, for each x, u, ω there exists a unique vector $H(x, u, \omega) \in I\!R^l$ such that

$$G(t, x, H(x, u, \omega), u, 0, \omega) = 0 \tag{37}$$

for all t, the reduced order system associated with (36) (let $\mu = 0$ in (36)) is given by

$$\dot{x} = \bar{F}(t, x, u, \omega) \tag{38}$$

where

$$\bar{F}(t,\ x,\ u,\ \omega) := F(t,\ x,\ H(x,\ u,\ \omega),\ u,\ 0,\ \omega). \tag{39}$$

For each $t,\ x,\ u,\ \omega$ the boundary layer system associated with (36) is given by

$$\frac{\mathrm{d}y}{\mathrm{d}\tau}(\tau) = G\,(t,\ x,\ y(\tau),\ u,\ 0,\ \omega). \tag{40}$$

[20] require that the boundary layer system satisfies g.u.a.s. about its equilibrium state $H(x,\ u,\ \omega)$ and present stabilizing controllers whose designs are based on the reduced order system. This situation occurs for systems with stable "neglected dynamics." In [24] the boundary layer system is not required to be stable. The "stabilizing" controllers presented there are composite controllers in the sense that they consist of two parts; one part is utilized to stabilize the boundary layer system and the other part is based on a nominal reduced order system.

5. OUTPUT FEEDBACK

Heretofore it was assumed that the complete state is available for feedback. Consider now the more general situation in which the output $y(t) \in I\!R^s$ available for feedback is related to the state by

$$y(t) = c(t,\ x(t),\ \omega) \tag{41}$$

for some function $c : I\!R \times I\!R^n \times \Omega \rightarrow I\!R^s$.
Memoryless output feedback controllers are treated in [10], [23], [54]. The dynamic output feedback controllers in the literature utilize state estimators. The state is fed to a memoryless controller whose design is based on having the complete state available for feedback. Full order observers are utilized in [3], [56]. Breinl and Leitmann [7], [8] utilize reduced order observers. There the uncertain terms must satisfy certain structural conditions and the differential equation describing the evolution of the state estimation error is decoupled from the state equation.

6. APPLICATIONS

Controller designs based on a constructive use of Lyapunov stability theory or closely related methods have been applied to a variety of uncertain systems. In the realm of engineering these applications include tracking control for robotic manipulators including hybrid tracking and force control [46], suspension control for magnetically levitated vehicles [7], [8], control of seismically excited structures [33], of high speed rotors [57], and of nuclear power plants [45], as well as various aircraft and aerospace systems [42], [50], [53]. Experimental results may be found in [29], [32]. [43] concern applications in economics. Resource allocation in fisheries is discussed in [28], [30], [31]. Harvesting problems are treated in [16], [36]. Lee and Leitmann [37], [38] deal with pollution control in rivers and in [39] they treat a problem in pedagogy.

7. EXAMPLE: OPTIMAL LONG-TERM MANAGEMENT OF A DISTURBED ECOSYSTEM

Consider the following model of an exploited ecosystem

$$\dot{x}(t) = f(x(t)) - Hx(t), \qquad x(t_0) = x^0 \tag{42}$$

where

$$x(t) := (x_1(t),\ x_2(t),\ \dots,\ x_n(t))^T \in \mathcal{X} \subset \mathbb{R}^n$$

is the biomass vector with its ith component representing the biomass of the ith species with

$$\mathcal{X} := \{x \in \mathbb{R}^n \mid x_i > 0,\ i = 1, 2, \dots, n\},$$

and where

$$H := \operatorname{diag}(h_1, h_2, \dots, h_n)$$

is the constant harvest effort matrix; a constant harvest effort vector $h := (h_1, h_2, \dots \dots, h_n)^T$ is admissible if $h \in \mathcal{H}$, where $\mathcal{H} \subset \mathbb{R}^n$ is prescribed. The corresponding non-trivial solution of

$$f(x) - Hx = 0 \tag{43}$$

is assumed to be unique. Let h^* denote the harvest effort which maximizes $\beta^T Hx$ for all $h \in \mathcal{H}$ and subject to (43), and let x^* denote the corresponding equilibrium state of (42), where $\beta := (\beta_1, \beta_2, \dots, \beta_n)^T$ is a given constant price vector. Thus, under *optimal steady state harvesting*, the exploited ecosystem (42) becomes

$$\dot{x}(t) = f(x(t)) - H^* x(t), \quad x(t_0) = x^0 . \tag{44}$$

If the exploited ecosystem (44) is undisturbed, then the harvest rate $H^* x^*$ is indeed optimal for the long-term management of the ecosystem, that is, in the steady state. However, real ecosystems are continually disturbed by unpredictable events such as diseases, migrations, climatic changes, and others. To include such disturbances, we modify the model in the following way:

$$\dot{x}(t) = f(x(t)) - H^* x(t) + \Delta f(x(t),\ v(t)) + u(t) \tag{45}$$

where $\Delta f(\cdot) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a known continuous function, $v(t) \in \mathcal{R} \subset \mathbb{R}^p$ is a vector of *uncertain disturbances* with $\mathcal{R}$ a compact bounding set which may be known or unknown. To assure that the disturbed ecosystem can be *practically stabilized*, an additional harvest rate $u(t) \in U \subset \mathbb{R}^n$ is provided where U is a known or unknown depending on $\mathcal{R}$ (the bounding set of v).

Since we are concerned with $x_i > 0$, $i = 1, 2, \dots, n$, we consider transformed variables

$$z_i = \ln(x_i / x_i^*), \quad i = 1, 2, \dots, n \tag{46}$$

which is valid for x_i and $x_i^* > 0$. Under this transformation, eqn. (45) leads to

$$\dot{z}(t) = g(z(t)) + \Delta g(z(t),\ v(t)) + B(z(t))\, u(t), \quad z(t_0) = z^0 \tag{47}$$

where

$$\begin{aligned}
g(z) &:= E^{-z}X^{*-1}\left[f(X^*e^z) - H^*X^*e^z\right] \\
\Delta g(z, v) &:= B(z)\,\Delta f(X^*e^z, v) \\
B(z) &:= E^{-z}X^{*-1} \\
X^* &:= \operatorname{diag}(x_1^*, \ldots, x_n^*) \\
E^{-z} &:= \operatorname{diag}(e^{-z_1}, \ldots, e^{-z_n}) \\
e^z &:= (e^{z_1}, \ldots, e^{z_n}) \\
z^0 &:= (\ln(x_1^0/x_1^*), \ldots, \ln(x_n^0/x_n^*))\,.
\end{aligned}$$

The problem then is that of determining a *feedback* controller in the case of *known* uncertainty bound $\mathcal{R}$, or an *adaptive* (dynamic) controller in the case of *unknown* uncertainty bound $\mathcal{R}$, such that the corresponding harvest rate $u(\cdot)$ assures at least practical stability of the disturbed ecosystem model (47) regardless of the realization of disturbance $v(\cdot)$; in particular, such a controlled harvest guarantees that $z(t) \to 0$ (that is, $x(t) \to x^*$) arbitrarily closely within finite time. Furthermore, since then $z_i(t)$ remains bounded, it follows that $x_i(t) > 0,\ i = 1, 2, \ldots, n$.

A detailed discussion the controllers mentioned above can be found in [36] for known $\mathcal{R}$ and in [16] for unknown $\mathcal{R}$. Here we present a simple example of a single species harvested population

$$\dot{x}(t) = \frac{r}{K}\,x(t)\,[K - x(t)] - hx(t), \qquad x(0) = x^0 \tag{48}$$

whence the maximum harvest rate at equilibrium is

$$h = h^* = r/2$$

with corresponding equilibrium population

$$x = x^* = K/2\,.$$

Now consider that the growth is subject to unpredictable disturbances of the form $v(t)\,x(t)$, where $v(\cdot) : I\!R \to \mathcal{R}$, $\mathcal{R} := \{v \mid |v| \le \alpha = \text{const.} > 0\}$. Thus, we have

$$\dot{x}(t) = \frac{r}{K}\,x(t)\,[K - x(t)] - h^*x(t) + v(t)\,x(t) + u(t) \tag{49}$$

where the control $u(t)$ corresponds to adjusting the total harvest rate $-h^*x(t)+u(t)$. Of course, if $u(t) > h^*x(t)$, harvesting is replaced by stocking (replenishing) which is not ruled out here. If stocking is not allowed, a constraint $u(t) \le h^*x(t)$ must be imposed; see, for instance, [19], [51].

On employing transformation (46), eqn. (49) leads to

$$\dot{z}(t) = \frac{r}{2}(1 - e^{z(t)}) + v(t) + \frac{2}{K}e^{-z(t)}\,u(t)$$

corresponding to eqn. (47). For this system, the result of Sec. 2.2 yields a stabilizing feedback control. After retransformation from z to x, one such control $u(t) = p(x(t))$

is one with

$$p(x) = \begin{cases} \left[-\operatorname{sgn} \alpha\left(\frac{x}{x^*} - 1\right)\right] \frac{Kx}{2x^*} & \text{if } \left|\alpha\left(\frac{x}{x^*} - 1\right)\right| > \varepsilon \\ -\frac{\alpha\left(\frac{x}{x^*} - 1\right)}{\varepsilon} \frac{Kx}{2x^*} & \text{if } \left|\alpha\left(\frac{x}{x^*} - 1\right)\right| \le \varepsilon \end{cases} \tag{50}$$

where $\varepsilon > 0$ is at our disposal; however, the smaller ε, the smaller is the assured value of $|x(t) - x^*|$ for all $t \ge T$ for some computable T.

For simulation purposes we use $K = 1.5$, $r = 0.25$, $\alpha = 0.1$, and the uncertain disturbance realization $v = -0.1 \cos t$. Figure 1 shows the biomass evolution of the uncontrolled ($u(t) \equiv 0$) and undisturbed ($v(t) \equiv 0$) ecosystem (49) for two initial values. Figure 2 portrays the biomass evolution for the disturbed but uncontrolled system, while Figures 3(a) and 3(b) present the biomass evolution of the disturbed and controlled system for two values of the design parameter ε. Finally, Figures 4(a) and 4(b) show the accumulative yield

$$Y(t) := \int_0^t [h^* x(\tau) - u(\tau)]\, d\tau ,$$

corresponding to the biomass history of Figure 3(b), as a function to time t. Clearly, in the long run the yield of the controlled system exceeds that of the uncontrolled one.

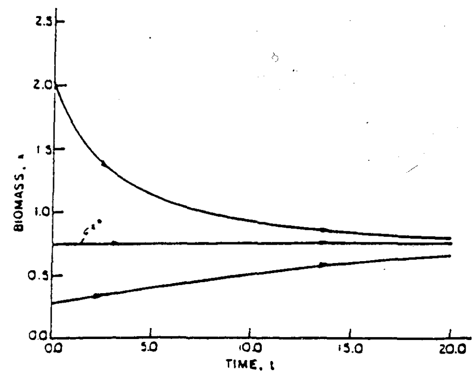

Fig. 1. Harvested one species system, undisturbed and uncontrolled.

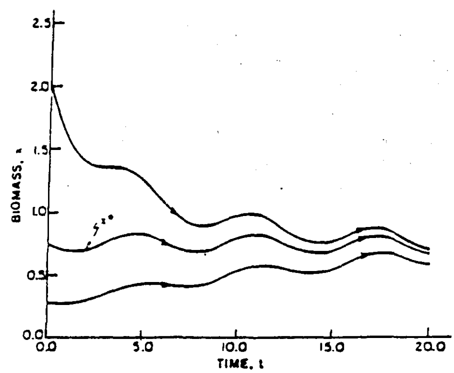

Fig. 2. Harvested one species system, disturbed and uncontrolled.

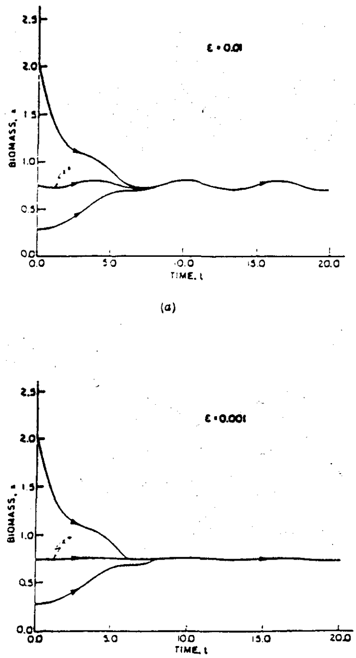

Fig. 3. Harvested one species system, disturbed and controlled. (a) $\varepsilon = 0.01$, (b) $\varepsilon = 0.001$.

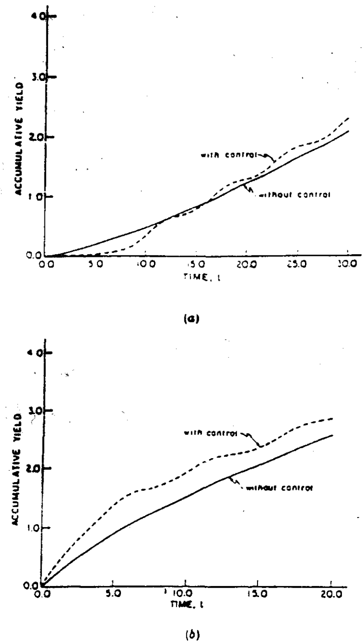

Fig. 4. Harvested and disturbed one species system. (a) $x(0) = 0.276$, (b) $x(0) = 2.039$.

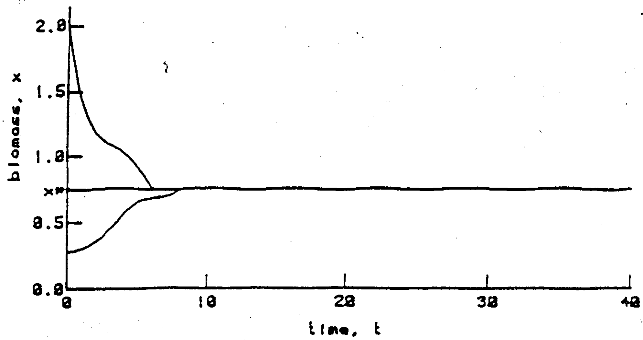

Fig. 5. Biomass responses for uncertain model with non-adaptive controller.

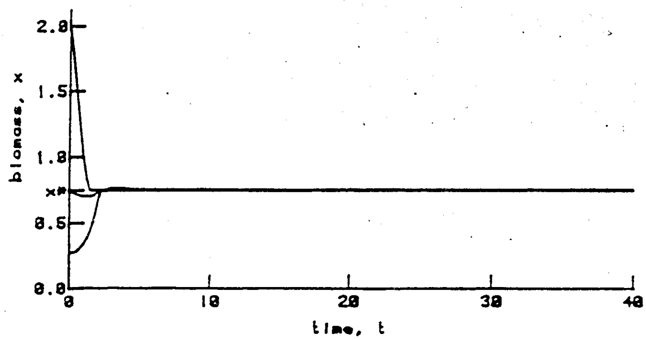

Fig. 6. Biomass responses for uncertain model with adaptive controller.

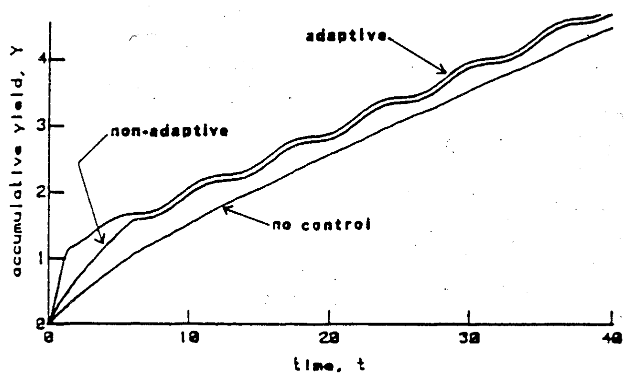

Fig. 7. Accumulative yields for $x(0) = 2.039$.

As mentioned above, if the bound of the disturbance, here α, is *not known*, we may employ a dynamic controller. As shown in [16], for the example treated here such a controller is

$$u(t) = \tilde{s}(t)\,\hat{\alpha}(t)\,x(t) \tag{51}$$

where

$$\tilde{s}(t) := \begin{cases} -\text{sgn}[x(t) - x^*] & \text{if} \quad |\hat{\alpha}(t)\,[x(t) - x^*]\,| > \varepsilon(t) \\ \frac{-\hat{\alpha}(t)\,[x(t)-x^*]}{\varepsilon(t)} & \text{if} \quad |\hat{\alpha}(t)\,[x(t) - x^*]\,| \le \varepsilon(t) \end{cases} \tag{52}$$

and

$$\dot{\hat{\alpha}}(t) = L\,|\,x(t) - x^*\,|\,, \qquad \hat{\alpha}(0) > 0 \tag{53}$$
$$\dot{\varepsilon}(t) = -l\,\varepsilon(t)\,, \qquad \varepsilon(0) > 0$$

for L, l = const. > 0.

The use of adaptive control (cf. [15]) in this problem is illustrated in Figures 5–8. For comparison with the feedback control (50) we present simulation results with the same parameter values K and r, and for the same disturbance realization of $v(\cdot)$, as those used in the preceding simulations. As can be seen, the adaptive control results in improved behavior, that is, a more rapidly convergent biomass and increased long-term yield (albeit, for "small" initial biomass, replenishing is required at the outset).

For a detailed discussion and a multi-species example see [16], [36].

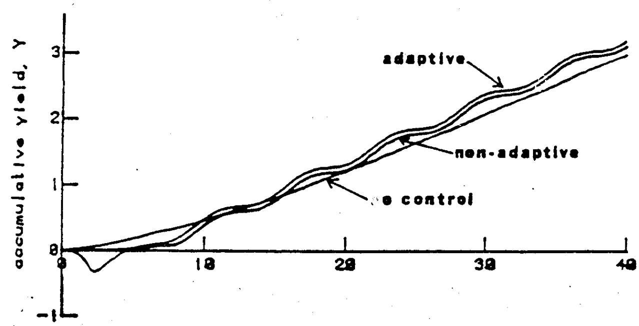

Fig. 8. Accumulative yields for $x(0) = 0.276$.

8. EXAMPLE: STABILIZING EMPLOYMENT IN A FLUCTUATING RESOURCE ECONOMY

Now we consider a resource management problem in which the management objective is the suppression of fluctuations in resource economics. In particular, we desire to stabilize the employment level of fishermen in an open-access common-property fishery subject to uncertain fluctuations in the resource level and in the value of the resource. This is to be accomplished by using a subsidy/tax policy based on current resource and employment levels.

Unlike in the previous example, here we adopt a discrete model. We consider an uncertain density-dependent resource described by a difference equation

$$N(k+1) = F[N(k),\, v(k,\, N(k),\, l(k))] - H[N(k),\, l(k)] \tag{54}$$

where $k \in \{\ldots, -1, 0, 1, \ldots\}$ is the time, $N(k)$ is aggregate stock size at time k, $F(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is the resource growth function, and $l(k)$ is the fishing effort during period $[k, k+1)$. The uncertainty in the system is modelled by $v(\cdot)$ which depends on time, stock level and fishing effort; $H(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is the catch during period $[k, k+1)$.

We assume again that the growth function is of the logistic type

$$F[N(k), v(k, N(k), l(k))] = N(k)\left[1 + (r + v(k, N(k), l(k)))\left(1 - \frac{N(k)}{K}\right)\right] \tag{55}$$

where r and K are positive constants, while the harvest rate is of the form

$$H[N(k), l(k)] = \frac{l(k)\, N(k)}{a + l(k)} \tag{56}$$

for $l(k) \geq 0$, and a is a positive constant. Finally, we take the growth uncertainty to be bounded by a known constant $\bar{v} < r$, that is,

$$|v(k, N(k), l(k))\,| \leq \bar{v}\,. \tag{57}$$

Next we investigate the possible range of stock levels for the fishery model (54) – (57). In particular, it is readily shown that

$$N(k) \in (0, K] \Longrightarrow N(k+l) \in (0, K] \tag{58}$$

for non-negative fishing effort, $l(k) \geq 0$, provided

$$r + \bar{v} \leq 1\,. \tag{59}$$

Since one major goal of regulation is the avoidance of species extinction, we restrict the subsequent treatment to the case for which this can be assured; namely $N(0) \in (0,\ K]$ and condition (59).

Next we postulate a regulatory agency which can employ a subsidy/tax policy with the aim of supressing employment fluctuations ascribable to varying economic conditions in an open-access fishery. The model of the entry/exit behavior of fishermen is based on the assumption that the change in the number of employed fishermen depends on the short-term revenues which they can receive. Let π denote the value of unit resource and c the cost of unit fishing effort. With fishing effort $l(k)$ equivalent to the number of fishermen employed (labor-intensive fishing), the entry/exit behavior is modelled by

$$l(k+1) = l(k) + t\, l(k)\left[\frac{\pi N(k)}{a + l(k)} - c\right] \tag{60}$$

where $t > 0$ is an entry coefficient. Thus, the number of employed fishermen increases as long as the enterprise is profitable; it decreases when cost exceeds profit.
We assume that

$$tc \leq 1 \tag{61}$$

so that, since $N(k) \geq 0$,

$$l(k) \geq 0 \Longrightarrow l(k+1) \geq 0\,; \tag{62}$$

that is, the fishing effort (employment level) remains non-negative.

Now we consider bounded fluctuations in the value (price) of the resource. The entry/exit equation (66) is modified to reflect this price uncertainty:

$$l(k+1) = l(k) + t\,l(k)\left[(\pi + w(k))\frac{N(k)}{a + l(k)} - c\right] \tag{63}$$

with

$$|w(k)| \leq \bar{w} \leq \pi \tag{64}$$

where $\bar{w}$ is a known (assumed) bound.

Thus, both the resource growth rate and the value of the resource are subject to unknown but bounded variations with known bounds. And while we suppose that $N(k)$ and $l(k)$ are known, due to uncertainties $v(\cdot)$ and $w(\cdot)$, the resource manager cannot predict their future values. However, as stated above, he can control the entry/exit behavior so as to drive it to and maintain it "near" a target level, l_s, by supporting (subsidy) or penalizing (tax) the fishermen. Thus, the model reflecting such social control becomes

$$l(k+1) = l(k) + t\,l(k)\left[(\pi + w(k))\frac{N(k)}{a + l(k)} - c + u(k)\right] \tag{65}$$

where $u(k) > 0$ denotes a subsidy and $u(k) < 0$ a tax.

On letting

$$x(k) := l(k) - l_s$$

equation (65) becomes

$$x(k+1) = f(k,\, x(k)) + C(k,\, x(k))\,[u(k) + e(k,\, x(k),\, u(k))] \tag{66}$$

where

$$\begin{aligned} f(k,\, x) &:= x + t(x + l_s)\left[\tfrac{N(k)}{a+x+l_s} - c\right] \\ C(k,\, x) &:= t(x + l_s) \\ e(k,\, x,\, u) &:= \tfrac{w(k)N(k)}{a+x+l_s}\,. \end{aligned}$$

On applying the results given in Sec. 3.1, the stabilizing control is

$$u(k) = -f(k,\, x(k)\,/\,C(k,\, x(k))\,. \tag{67}$$

Now one can show that use of control (67) assures the non-negativity of fishing effort, that is,

$$l(k) \geq 0 \quad \text{for all } k > 0 \tag{68}$$

for all possible realizations of the uncertainties $v(\cdot)$ and $w(\cdot)$, if

$$r + \bar{v} \leq 1\,, \qquad t\,\bar{w}\,K < l_s\,, \tag{69}$$

provided

$$N(0) \in (0, K], \quad l(0) \geq 0. \tag{70}$$

To illustrate the efficacy of control (67), we present simulation results for the system with parameter values

$$r = 0.25, \quad K = 1000, \quad a = 500, \quad \pi = 1, \quad c = 1, \quad t = 0.7,$$

uncertainty bounds

$$\bar{v} = 0.1, \quad \bar{w} = 0.1$$

and uncertainty realizations

$$v(k, N(k), l(k)) = \bar{v} \sin 0.2\,k$$

$$w(k) = \bar{w} \sin 0.5\,k$$

with initial values

$$N(0) = 700, \quad l(0) = 50.$$

In the absence of disturbances, employment stabilizes at $l = 61.6$ with corresponding stock level $N = 562$. The resource manager prefers a higher employment level, namely, $l_s = 80$; the corresponding steady-state stock level of the undisturbed controlled system is $N = 448$.

Figures 9 and 10 show the behavior of the stock level and of the employment level in the presence of the assumed uncertainty realizations for the system without and with control (67). Clearly, the use of the proposed subsidy/tax policy serves to suppress the fluctuations in employment, albeit with increased intensity of resource utilization.

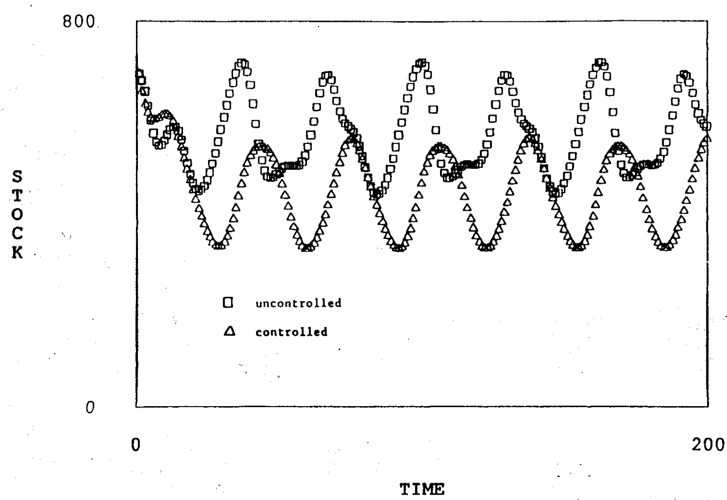

Fig. 9. Stock level.

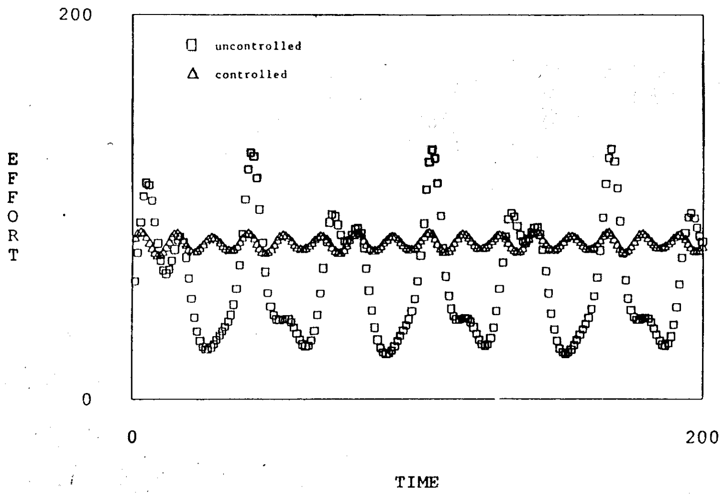

Fig. 10. Employment level.

Detailed derivations as well as further discussions can be found in [30]. More complex systems allowing for labor as well as capital intensive fishery sectors are treated in [28], [31].

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