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TIME–DISCRETIZATION FOR CONTROLLED MARKOV PROCESSES
Part I: General Approximation Results

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The method of time-discretization is investigated in order to approximate finite horizon cost functions for continuous-time stochastic control problems. The approximation method is based on approximating time-differential equations by one-step difference methods. In this paper general approximation results will be developed. An approximation lemma is presented. This lemma enables us to conclude orders of converge, which makes the method of computational interest. Also unbounded cost functions are allowed. We concentrate on approximations induced by discrete-time controlled Markov processes. The approximation can in principle be computed recursively by using discrete-time dynamic programming. In a subsequent second paper two applications will be studied in detail.

CONTENTS

1. INTRODUCTION

This paper is concerned with the approximation of continuous-time controlled Markov process by processes with discrete-time parameter $\{nh \mid n = 1, 2, \ldots\}$, where h denotes a step size. As functions of interest we focus on finite horizon cost functions.

Time discretization methods for controlled stochastic processes are studied extensively in the literature, cf. Kushner [13], Gihman and Skorohod [5], Haussmann [6], Bensoussan and Robin [1], Christopeit [2], Hordijk and Van der Duyn Schouten [7, 8, 9, 10], Van der Duyn Schouten [19], Van Dijk [20, 21], Plum [16] and Koole [11]. It is basically the *recursive structure of discrete-time systems* which makes time-discretization so interesting for computational purposes. The optimal cost functions as well as the corresponding optimal controls for discrete-time Markov processes can be computed recursively by using dynamic programming, whereas cost functions for continuous-time processes are usually given through implicit differential equations. The above-mentioned references focus on the convergence of discrete-time schemes as the step size h tends to 0, either from a theoretical point of view ([1, 5]), or to use time-discretization as a computational procedure ([6, 13]) or to prove the optimality of a limit control ([2, 7, 8, 9, 10, 11, 16, 19, 20, 21]). Particularly, Hordijk and Van der Duyn Schouten [7, 8, 9, 10] have been able to show the structure of the optimal control in several applications by using time-discretization. In these papers the controlled Markov process is a jump process with a deterministic drift between the jumps and permits both controls affecting jump rates together with jump sizes and impulsive control causing immediate transitions. Plum [16] combines the more general Markov processes of this paper with the more general controls (including impulsive control) of Hordijk and Van der Duyn Schouten [8, 9]. In Koole [11] (Chapter 5) the methods developed in this paper and in Van Dijk [21] are applied to stochastic scheduling models.

None of the above-mentioned references, however, is concerned with computational aspects of the approximation method such as: *orders of convergence* in some appropriate norm, the choice of such a norm and the choice of convenient discretizations. In this and a subsequent paper we attempt to make a first step in these directions.

In this paper we develop an approximation lemma for one-step difference methods. This lemma extends the Lax–Richtmeyer theorem (cf. [14]), in that it allows time inhomogeneous and *nonlinear* difference methods. The approximation lemma is applied to one-step difference methods induced by discrete-time controlled Markov processes. We derive conditions for the one-step transition probabilities which together with sufficient smoothness of the continuous-time functions imply the desired approximation results. Orders of convergence in the step-size h are given with respect to some appropriate chosen weightd supremum norm. The approximation conditions are not restricted to specifically chosen discretizations, but are applicable to a range of one-step difference methods, advanced numerical methods included. The possibility of choosing norms enables us to deal with unbounded cost rates and, in applications, unbounded infinitesimal characteristics.

In a subsequent paper, Part II, we will illustrate the approximation method and its consequences by applying it to a *controlled infinite server queue* and a *controlled* *investment* or *cash-balance* model. In both applications we will verify the necessary smoothness and approximation conditions, including, as a special result, the existence of a sufficiently smooth and bounded solution of the Bellman equation. Furthermore, the construction of ε-optimal controls by using time-discretization will be analyzed. In this paper we only give the discretizations used for these applications and some approximation results in order to illustrate the time-discretization method.

This paper is a shortened version of a technical report from 1985, we are grateful for this opportunity to publish our results.

2. CONTINUOUS–TIME CONTROLLED MARKOV PROCESS

2.1. Definitions and notation

In this section we give a formal definition of a continuous-time controlled Markov process and we illustrate it by two examples. First, we present a description of continuous-time controlled Markov processes by introducing the notion of a control object.

Notation 2.1.1. For S a separable and complete metric space and measurable $\mu : S \to \mathbb{R}$ such that $\mu(x) \geq \delta > 0$, for all $x \in S$ and some $\delta > 0$, it is easily verified that the space B^μ is a Banach space with norm $\|\cdot\|_\mu$, defined by

$$B^\mu = \left\{ f : S \to \mathbb{R} \,|\, f \text{ measurable and } \sup_{x \in S} |f(x)|/\mu(x) < \infty \right\}, \quad (2.1.1)$$

$$\|f\|_\mu = \sup_{x \in S} |f(x)|/\mu(x), \quad f \in B^\mu. \quad (2.1.2)$$

We call μ a bounding function.

Definition 2.1.2. A *control object* is a 7-tuple $(S, \Gamma, \mu, D_A, \{\boldsymbol{A}^\delta \,|\, \delta \in \Delta\}, L)$, where:

(i) S is a separable complete metric space with Borel-field β.

(ii) Γ is a separable complete metric space with Borel-field $\beta(\Gamma)$.

(iii) Δ is a subset of Borel-measurable functions $\delta : S \to \Gamma$.

(iv) μ denotes a bounding function (see Notation 2.1.1 above).

(v) D_A is a nonempty subset of B^μ (see Notation 2.1.1 above).

(vi) $\boldsymbol{A}^\delta$, for any $\delta \in \Delta$, is a linear operator from D_A into B^μ.

(vii) $\boldsymbol{L} : S \times \Gamma \to \mathbb{R}$ is a Borel measurable function.

The characteristics introduced above have the following interpretation:

(i) S is the state space of the controlled Markov process.

(ii) Γ is a set of decisions. At any time point a decision is taken from Γ.

(iii) Δ is a set of decision rules, i.e. if the current decision rule is $\delta \in \Delta$ and the actual state is x, then δ prescribes decision $\delta(x)$.

(iv) μ is a bounding function and determines the class B^{μ}.

(v) D_A is a domain on which the operator A^{δ} is defined for any $\delta \in \Delta$.

(vi) $\boldsymbol{A}^{\delta}$ represents the infinitesimal operator of the controlled Markov process when using one and the same decision rule δ at any time $t \geq 0$.

(vii) $\boldsymbol{L}$ represents a cost-rate function; i.e. if during $[t, t + \Delta t]$ the state of the process is x and the decision is γ, then the costs incurred in that interval are $\Delta t \boldsymbol{L}(x, \gamma)$.

To illustrate the notion of a control object, let us give two examples:

Example 2.1.3. ($M|M|\infty$-queue with a controllable number of servers)
Customers arrive at a service facility according to a Poisson process with parameter λ. The number of servers is controlled continuously. A customer can only be served by a single server and the number of servers never exceeds the number of customers present. Each customer demands an amount of service according to an exponential distribution with parameter ν. The cost-rate function is bounded by a function polynomial in the number of customers and the number of servers. As will be pointed out in the subsequent Part II, the appropriate control object is given by:

$$\left.\begin{array}{l} S = \mathbb{N}; \quad \Gamma = \mathbb{N}; \quad \Delta = \{\delta : \mathbb{N} \to \mathbb{N} \mid \delta(i) \leq i, \ i \in \mathbb{N}\} \\ L(i,j) \leq C_0 + C_1[i]^{p_1} + C_2[j]^{p_2} \text{ for some } C_0,\, C_1,\, C_2,\, p_1,\, p_2 \end{array}\right\} \tag{2.1.3}$$

$$\left.\begin{array}{l} \text{With } p = \max(p_1, p_2): \\ \mu(i) = (1+i)^{p+2}, \quad i \in \mathbb{N}, \\ D_A = \left\{ f : \mathbb{N} \to \mathbb{R} \mid \sup_{i \in \mathbb{N}} |f(i)|/(1+i)^p < \infty \right\}. \end{array}\right\} \tag{2.1.4}$$

$$\boldsymbol{A}^{\delta} f(i) = \lambda[f(i+1) - f(i)] + \delta(i)\,\nu[f(i-1) - f(i)], \quad i \in \mathbb{N}. \tag{2.1.5}$$

Example 2.1.4. (Controlled investment model)
An investment of fixed amount is controlled by continuously allocating an investment opportunity (γ_1, γ_2) with γ_1 the rate of return and $[\gamma_2]^2$ the value of risk given by its variance per unit of time. The available opportunities (γ_1, γ_2) belong to a finite set B. The state x of the process denotes the current value of the investment. A cost-rate function is taken into account which depends on the value of the investment, the rate of return and the value of risk. This rate is bounded by a polynomial in x of order p. As will be pointed out in the subsequent Part II, the appropriate control object is given by:

$$\left.\begin{array}{l} S = I\!R; \quad \Gamma = B; \quad \Delta = \{\delta : I\!R \to B \,|\, \delta \text{ piecewise constant}\} \\ |\boldsymbol{L}(x,\gamma_1,\gamma_2)| \le C(1+|x|^p) \quad \text{for some constant } C, \\ \mu(x) = (1+|x|^p), \quad x \in I\!R. \end{array}\right\} \tag{2.1.6}$$

$$\left.\begin{array}{l} D_A = \Big\{ f : I\!R \to I\!R \,\Big|\, \frac{\mathrm{d}^k}{\mathrm{d}x^k} f(x) \text{ exists and is continuous for } k = 1,2,3 \\ \qquad \text{and } \left|\frac{\mathrm{d}^k}{\mathrm{d}x^k} f(x)\right| \le C(1+|x|^p), \; x \in I\!R, \text{ for } k = 0,1,2,3 \Big\} \end{array}\right\} \tag{2.1.7}$$

$$\boldsymbol{A}^\delta f(x) = [\gamma_1] \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{1}{2}[\gamma_2]^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) \quad \text{for } \delta(x) = (\gamma_1,\gamma_2), \; x \in I\!R. \tag{2.1.8}$$

Note that in the above examples an appropriate choice of the bounding function μ enables us to deal with polynomially bounded, hence unbounded, cost-rate functions as well as in the queueing model an unbounded infinitesimal characteristic (the jump rates $\delta(i)$, ν). We conclude this section with some notation.

Notation 2.1.5.

1. Z always represents a fixed but arbitrarily chosen finite time point, $t \le Z$ stands for: $t \in [0, Z]$, and if $h > 0$ is under consideration: ℓ denotes the entier of Zh^{-1}, i.e. $\ell = \lfloor Zh^{-1} \rfloor$, where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x.

2. A family $\{f_t \,|\, t \le Z\} \subset B^\mu$ is called μ-bounded if $\sup_{t \le Z} \|f_t\|_\mu < \infty$. A μ-bounded family is called integrable if for any $x \in S$: $f_t(x)$ is Lebesgue-integrable in t. We write

$$q_t = \int_t^Z f_s \,\mathrm{d}s \quad \text{if} \quad q_t(x) = \int_t^Z f_s(x)\,\mathrm{d}s, \qquad x \in S.$$

   Note that $\{q_t \,|\, t \le Z\}$ is a μ-bounded subset of B^μ.

3. For $\{q^\delta \,|\, \delta \in \Delta\}$ with $q^\delta : S \to I\!R$, $\delta \in \Delta$, let $\inf_{\delta\in\Delta}[q^\delta]$ denote the function $q^0 : S \to I\!R \cup \{-\infty\}$ given by $q^0(x) = \inf_{\delta\in\Delta}[q^\delta(x)]$, $x \in S$.

4. For an integral sign $\int$ without subscript S is the domain of integration.

5. For $\delta \in \Delta$: the function $\boldsymbol{L}^\delta : S \to I\!R$ is defined by: $\boldsymbol{L}^\delta(x) = \boldsymbol{L}(x, \delta(x))$, $x \in S$.

6. The function $\overline{0}$ denotes the function which is identically equal to zero.

In what follows we consider a fixed control object $(S, \Gamma, \Delta, \mu, D_A, \{\boldsymbol{A}^\delta \,|\, \delta \in \Delta\}, \boldsymbol{L})$ and a fixed but arbitrarily chosen finite time Z. The dependence on Z will not be mentioned explicitly in definitions and formulas.

2.2. Finite horizon cost function: Fixed control

Definitions 2.2.1. A non-randomized Markov control is a function $\pi : [0, Z] \to \Delta$ (Notation: $\pi \in \Pi$). A Markov control $\pi \in \Pi$ is called *admissible* and *μ-bounded* (Notation $\pi \in \Pi(AB)$) if there exists a unique family of transition probabilities $\{\boldsymbol{P}^{\pi}_{s,t} \,|\, 0 \le s \le t\}$ and a constant M^{π} such that for all $t_1 \le t_2 \le t_3 \le Z$, $x \in S$ and $B \in \beta$:

$$\boldsymbol{P}^{\pi}_{t_1,t_3}(x; B) = \int \boldsymbol{P}^{\pi}_{t_2,t_3}(y; B)\, \boldsymbol{P}^{\pi}_{t_1,t_2}(x; \mathrm{d}y) \tag{2.2.1}$$

and

$$\left\| \int \mu(y)\, \boldsymbol{P}^{\pi}_{s,t}(\cdot\,; \mathrm{d}y) \right\|_{\mu} \le M^{\pi}, \quad 0 \le s \le t \le Z. \tag{2.2.2}$$

Remark 2.2.2. In applications, such as controlled jump and diffusion processes, the existence and uniqueness of the transition probabilities of the corresponding Markov processes has to be guaranteed by smoothness conditions on the infinitesimal characteristics and the control π.

Consider a fixed Markov control $\pi \in \Pi(AB)$. Then, by virtue of (2.2.2), we can define for all s, $t \le Z$, an operator $\boldsymbol{T}^{\pi}_{s,t} : B^{\mu} \to B^{\mu}$ by

$$\boldsymbol{T}^{\pi}_{s,t}\, f(x) = \int f(y)\, \boldsymbol{P}^{\pi}_{s,t}(x; \mathrm{d}y). \tag{2.2.3}$$

Moreover, from (2.2.1) and (2.2.2) it easily follows that for all $f \in B^{\mu}$:

$$\boldsymbol{T}^{\pi}_{s,t}\, f = \boldsymbol{T}^{\pi}_{s,\tau}(\boldsymbol{T}^{\pi}_{s,\tau} f), \qquad s \le \tau \le t \le Z, \tag{2.2.4}$$

$$\left\|\boldsymbol{T}^{\pi}_{s,t}\, f\right\|_{\mu} \le \|f\|_{\mu}\, M^{\pi}, \qquad s \le t \le Z. \tag{2.2.5}$$

Finite horizon cost function. We are now able to define the finite horizon cost function $\{\boldsymbol{V}^{\pi}_{t} \,|\, t \le Z\}$. Herein $\boldsymbol{L}^{\pi}(s)$ denotes the function $S \to I\!R$ defined by $\boldsymbol{L}^{\pi(s)}(x) = \boldsymbol{L}(x, \delta(x))$ for $\delta = \pi(s)$ while Assumption 2.2.5 below guarantees that the integral involved is well-defined:

$$\boldsymbol{V}^{\pi}_{t} = \int_{t}^{Z} \boldsymbol{T}^{\pi}_{t,s}\, \boldsymbol{L}^{\pi(s)}\, \mathrm{d}s, \quad (t \le Z), \quad \boldsymbol{V}^{\pi}_{Z} = \bar{0}. \tag{2.2.6}$$

The value $\boldsymbol{V}^{\pi}_{t}(x)$ represents the expected total costs from time t up to time Z under policy π, given that the state at time t is x. In order to let $\boldsymbol{V}^{\pi}_{t}$ be well-defined as well as for purposes that we need later on, the following assumption is made.

Assumption 2.2.3.

(i) $\{\boldsymbol{L}^{\pi(s)} \,|\, s \le Z\}$ is μ-bounded.

(ii) $\{\boldsymbol{T}^{\pi}_{t,s}\, \boldsymbol{L}^{\pi(s)} \,|\, s \le Z\}$ is integrable.

(iii) $\{\boldsymbol{V}^{\pi}_{t} \,|\, t \le Z\}$ is a μ-bounded family $\subset D_A$.

Let $h > 0$. Then by virtue of the semigroup property (2.2.4), the fact that $\boldsymbol{T}^{\pi}_{t,t+h}$ is a linear and bounded operator on B^{μ} (see relation (2.2.5)), and condition (i) of Assumption 2.2.3, for any t, $t+h \leq Z$ we can write:

$$\boldsymbol{V}^{\pi}_{t} = \int_{t}^{t+h} \boldsymbol{T}^{\pi}_{t,s}\, \boldsymbol{L}^{\pi(s)}\, \mathrm{d}s + \boldsymbol{T}^{\pi}_{t,t+h}(\boldsymbol{V}^{\pi}_{t+h}). \tag{2.2.7}$$

This relation can also be written as:

$$\left.\begin{aligned} &\boldsymbol{V}^{\pi}_{t} - \boldsymbol{V}^{\pi}_{t+h} = h\left[\boldsymbol{L}^{\pi(t)} + \boldsymbol{A}^{\pi(t)}\,\boldsymbol{V}^{\pi}_{t+h}\right] + R^{\pi}_{t}(\boldsymbol{V},h), \quad \text{where} \\ &R^{\pi}_{t}(\boldsymbol{V},h) = \left[\int_{t}^{t+h} \boldsymbol{T}^{\pi}_{t,s}\,\boldsymbol{L}^{\pi(s)}\,\mathrm{d}s - h\boldsymbol{L}^{\pi(t)}\right] + \left(\left[\boldsymbol{T}^{\pi}_{t,t+h} - I\right] - h\boldsymbol{A}^{\pi(t)}\right)\boldsymbol{V}^{\pi}_{t+h}. \end{aligned}\right\} \tag{2.2.8}$$

2.3. Finite horizon optimal cost function

Assumption 2.3.1. There exists an operator $\boldsymbol{J} : D_A \to B^{\mu}$ such that

$$\boldsymbol{J}\, f(x) = \inf_{\delta\in\Delta}\left[\boldsymbol{L}^{\delta}(x) + \boldsymbol{A}^{\delta}\, f(x)\right], \quad f \in D_A,\ x \in S. \tag{2.3.1}$$

Note that this assumption requires that the right-hand side of (2.3.1) exists and is μ-bounded for any $f \in D_A$. If Assumption 2.3.1 holds we consider

As umption 2.3.2. There exists a unique family $\{\boldsymbol{\Phi}_t \mid t \leq Z\} \subset D_A$ with $\{\boldsymbol{J}(\boldsymbol{\Phi}_t) \mid t \leq Z\}$ a μ-bounded and integrable family satisfying the continuous-time optimality equation:

$$\boldsymbol{\Phi}_t = \int_{t}^{Z} \boldsymbol{J}(\boldsymbol{\Phi}_s)\,\mathrm{d}s, \quad (t \leq Z), \qquad \boldsymbol{\Phi}_Z = \bar{0}. \tag{2.3.2}$$

Remark 2.3.3. It is well-known that for jump- and diffusion-type applications the value $\boldsymbol{\Phi}_t(x)$ represents the optimal (minimal) expected costs from time t up to time Z given that the state at time t is x, and where the minimum is taken over all Markov controls (cf. [4]).

For t, $t+h \leq Z$ relation (2.3.2) can be written as:

$$\left.\begin{aligned} &\boldsymbol{\Phi}_t - \boldsymbol{\Phi}_{t+h} = h\boldsymbol{J}(\boldsymbol{\Phi}_{t+h}) + R_t(\boldsymbol{\Phi},h), \quad \text{where} \\ &R_t(\boldsymbol{\Phi},t) = \textstyle\int_t^{t+h} \boldsymbol{J}(\boldsymbol{\Phi}_s)\,\mathrm{d}s - h\boldsymbol{J}(\boldsymbol{\Phi}_{t+h}). \end{aligned}\right\} \tag{2.3.3}$$

3. DISCRETE–TIME CONTROLLED MARKOV PROCESSES

3.1. Definitions and notation

This section focuses on controlled Markov processes in which the state and decision can only change at discrete-time points $\{nh \mid n = 0, 1, \ldots\}$ for some $h > 0$. In analogy with the continuous-time controlled Markov processes we will present such a process by introducing the notion of an h-control object.

Definition 3.1.1. An *h-control object* is a 7-tuple $(S,\ \Gamma,\ \Delta,\ \mu,\ h,\ \{\boldsymbol{P}_h^\delta \mid \delta \in \Delta\},\ \boldsymbol{L})$, where

(i) S, Γ, Δ and μ are defined as in Section 2.1.

(ii) $h > 0$ denotes the step size of the time parameter.

(iii) For any $\delta \in \Delta : \boldsymbol{P}_h^\delta : S \times \beta \to I\!R$ is a transition probability.

It has the following interpretation: If at time nh the current decision rule is δ and the actual state is x, then $\boldsymbol{P}_h^\delta(x; B)$ is the probability that at time $nh + h$ the state is contained in B.

(iv) $\boldsymbol{L} : S \times \Gamma \to I\!R$ is the measurable function defined in Section 2.1. In this section, $h\boldsymbol{L}$ represents the one-step cost function, i.e.: If at time nh the actual state and decision are x and γ then $h\boldsymbol{L}(x,\gamma)$ are the expected costs incurred during $[nh,\ nh+h]$.

Definition 3.1.2. For $\delta \in \Delta$ with $\left\| \int \mu(y)\, \boldsymbol{P}_h^\delta(\cdot;\, \mathrm{d}y) \right\|_\mu < \infty$, the operators $\boldsymbol{T}_h^\delta$ and $\boldsymbol{A}_h^\delta : B^\mu \to B^\mu$ are defined by:

$$\left.\begin{aligned} \boldsymbol{T}_h^\delta f(x) &= \int f(y)\, \boldsymbol{P}_h^\delta(x;\, \mathrm{d}y); \quad x \in S,\ f \in B^\mu, \\ \boldsymbol{A}_h^\delta &= [\boldsymbol{T}_h^\delta - I]\, h^{-1}, \quad \text{where } If = f,\ f \in B^\mu. \end{aligned}\right\} \tag{3.1.1}$$

3.2. Finite horizon cost function; fixed control

Definition 3.2.1. A non-randomized h-Markov control is a function $\pi : \{nh \mid n = 0, 1, \ldots, \ell\} \to \Delta$ (Notation: $\pi \in \Pi^h$). A control $\pi \in \Pi^h$ will also be denoted by $(\pi(0),\ \pi(1h), \ldots, \pi(\ell h))$.

Construction 3.2.2. For any $\pi \in \Pi^h$ one can construct a unique family of transition probabilities $\{\boldsymbol{P}_{j,n}^h \mid j,\ n \le \ell\}$ such that for all $j \le \ell$, $x \in S$ and $B \in \beta$:

$$\left.\begin{aligned} \boldsymbol{P}_{j,j}^h(x; B) &= 1_B(x), \\ \boldsymbol{P}_{j,n+1}^h(x; B) &= \int \boldsymbol{P}_h^{\pi(nh)}(y; B)\, \boldsymbol{P}_{j,n}^h(x;\, \mathrm{d}y), \quad j < n < \ell. \end{aligned}\right\} \tag{3.2.1}$$

Consequently, in contrast with the continuous-time model given in Section 2.2, *any* h-Markov control $\pi \in \Pi^h$ can be called *admissible*. In addition, an h-Markov control $\pi \in \Pi^h$ is also called *μ-bounding*, (Notation: $\pi \in \Pi^h(AB)$) if for some constant M^h:

$$\left\| \int \mu(y)\, \boldsymbol{P}_{j,n}^h(\cdot;\, \mathrm{d}y) \right\|_\mu \le M^h, \quad j \le n \le \ell. \tag{3.2.2}$$

Consider a fixed h-Markov control $\pi \in \Pi^h(AB)$ and let $\{\boldsymbol{P}_{j,n}^h \mid j,\ n \le \ell\}$ be given by (3.2.1). Then, by virtue of (3.2.2), we can define for all $j,\ n \le \ell$ an operator

$\boldsymbol{T}^h_{j,n} : B^\mu \to B^\mu$ as per (3.2.3) below and the relations (3.2.4) and (3.2.5) follow directly from (3.1.1) and (3.2.2)

$$\boldsymbol{T}^h_{j,n} f(x) = \int f(y)\, \boldsymbol{P}^h_{j,n}(x;\, \mathrm{d}y), \qquad j < n \le \ell, \tag{3.2.3}$$

$$\boldsymbol{T}^h_{j,n} f = \boldsymbol{T}^{\pi(j,h)}_h (\boldsymbol{T}^h_{j+1,n} f), \qquad j < n \le \ell, \tag{3.2.4}$$

$$\left\| \boldsymbol{T}^h_{j,n} f \right\|_h \le \|f\|_\mu M^h, \qquad j \le n \le \ell. \tag{3.2.5}$$

Next, in contrast with Assumption 2.2.3 for the continuous-time model, we make the following assumption.

Assumption 3.2.3. $\{\boldsymbol{L}^{\pi(nh)} \mid n = 0, 1, \ldots, \ell\}$ is μ-bounded.

Then, for any $j \le \ell$, Assumption 3.2.3 together with (3.2.5) justifies the following definition of the finite horizon cost function $\boldsymbol{V}^h_j$:

$$\boldsymbol{V}^h_j = \sum_{n=j}^{\ell-1} \boldsymbol{T}^h_{j,n}(\boldsymbol{L}^{\pi(nh)})\, h. \tag{3.2.6}$$

Further, by virtue of the semigroup-property (3.2.4), the fact that $\boldsymbol{T}^h_{j,j+1}$ is a linear and bounded operator on B^μ (see relation (3.2.5)) and Assumption 3.2.5, it is easily verified that $\boldsymbol{V}^h_j$, $j \le \ell$, can be solved recursively by:

$$\boldsymbol{V}^h_j = h\, \boldsymbol{L}^{\pi(jh)} + \boldsymbol{T}^{\pi(jh)}_h (\boldsymbol{V}^h_{j+1}), \quad j < \ell, \qquad \boldsymbol{V}^h_\ell = \bar{0}. \tag{3.2.7}$$

3.3. Finite horizon optimal cost function

Assumption 3.3.1.

(i) Relation (3.1.1) is satisfied for all $\delta \in \Delta$.

(ii) There exists a subset $F \subset B^\mu$ with $\bar{0} \in F$ and

$$\inf_{\delta \in \Delta} [h\, \boldsymbol{L}^\delta + \boldsymbol{T}^\delta_h f] \in F, \quad \text{for all } f \in F. \tag{3.3.1}$$

If Assumption 3.3.1 is satisfied, then there exists a unique family $\{\boldsymbol{\Phi}^h_j \mid j \le \ell\}$ satisfying the discrete-time optimality equation:

$$\boldsymbol{\Phi}^h_j = \inf_{\delta \in \Delta} \left[h\, \boldsymbol{L}^\delta + \boldsymbol{T}^\delta_h\, \boldsymbol{\Phi}^h_{j+1} \right], \quad (j < \ell), \qquad \boldsymbol{\Phi}^n_\ell = \bar{0}. \tag{3.3.2}$$

It is well-known in dynamic programming that the value $\boldsymbol{\Phi}^h_j(x)$ represents the minimal expected total costs from time jh to ℓh, given that the state at time jh is x. Further, note that (3.3.2) is a recursive system of equations.

4. APPROXIMATION LEMMA

This section contains a general approximation lemma which is the key lemma for our approximations results. In view of the time-difference equations (2.2.8) and (2.3.3) and the time-recursive equations (3.2.7) and (3.3.2) corresponding to the continuous- respectively discrete-time cost functions, this lemma is concerned with systems of backwards time-evaluation equations of the form

$$\left.\begin{aligned} \boldsymbol{U}_{jh} &= \boldsymbol{C}_{jh}(\boldsymbol{U}_{jh+h}), \quad j < \ell, \text{ and} \\ \boldsymbol{U}_j^h &= \boldsymbol{C}_j^h(\boldsymbol{U}_{j+1}^h), \quad j < \ell, \end{aligned}\right\} \tag{4.1}$$

where

$$\left.\begin{aligned} &\{\boldsymbol{U}_{jh} \mid j \le \ell\} \text{ and } \{\boldsymbol{U}_j^h \mid j \le \ell\} \text{ are families within a Banach space } (B, \|\cdot\|) \\ &\{\boldsymbol{C}_{jh} \mid j \le \ell\} \text{ and } \{\boldsymbol{C}_j^h \mid j \le \ell\} \text{ are families of operators from } B \text{ into } B. \end{aligned}\right\}$$

Lemma 4.1. Suppose that for constants ε, $K > 0$:

$$\left\|\boldsymbol{C}_j^h(\boldsymbol{U}_{jh+h}) - \boldsymbol{C}_{jh}(\boldsymbol{U}_{jh+h})\right\| h^{-1} \le \varepsilon, \qquad j < \ell \tag{4.2}$$

$$\left\|\boldsymbol{C}_j^h(c_1) - \boldsymbol{C}_j^h(c_2)\right\| \le (1 + hK)\, \|c_1 - c_2\|, \quad j < \ell,\ c_1, c_2 \in B. \tag{4.3}$$

Then for any $n < m < \ell$:

$$\left\|\boldsymbol{U}_n^h - \boldsymbol{U}_{nh}\right\| < \varepsilon \left[e^{K(mh-nh)} - 1\right] K^{-1} + e^{K(mh-nh)} \left\|\boldsymbol{U}_m^h - \boldsymbol{U}_{mh}\right\|. \tag{4.4}$$

Proof. Write

$$\left.\begin{aligned} \delta_j^h &= \boldsymbol{U}_j^h - \boldsymbol{U}_{jh}, & j < \ell, \\ \varepsilon_j^h &= \boldsymbol{C}_j^h(\boldsymbol{U}_{jh+h}) - \boldsymbol{C}_{jh}(\boldsymbol{U}_{jh+h}), & j < \ell. \end{aligned}\right\} \tag{4.5}$$

Then from (4.1):

$$\delta_j^h = \boldsymbol{C}_j^h(\boldsymbol{U}_{j+1}^h) - \boldsymbol{C}_j^h(\boldsymbol{U}_{jh+h}) + \varepsilon_j^h. \tag{4.6}$$

Consequently, (4.2) and (4.3) yield:

$$\|\delta_j^h\| \le (1 + hK)\, \|\delta_{j+1}^h\| + \varepsilon h. \tag{4.7}$$

So that by iterating (4.7) for $j = n, \ldots, m-1$:

$$\|\delta_n^h\| \le (1 + hK)^{(m-n)}\, \|\delta_m^h\| + \varepsilon \sum_{j=n}^{m-1} (1 + hK)^{(j-n)} h. \tag{4.8}$$

Since $(1 + hK) \le e^{hk}$ and $(e^{hK})^p \le e^{tK}$ for $t \in [ph, ph + h)$, relation (4.8) and some simple calculus imply relation (4.4). □

Remarks 4.2.

1. Clearly, Lemma 4.1 can be extended by relaxing (4.2) and (4.3) to time dependent bounds, i.e. ε and K replaced by ε_j and K_j. Since, however, the above form appears to be sufficient for our purposes we prefer to avoid further notational complexity.
2. Informally speaking, in numerical analysis a relation of the form (4.2) with ε converging to 0 and h tends to 0 is known as *consistency* of the approximation scheme, while relation (4.3) as stability of the finite difference method (cf. [17]). The standard Lax–Richtmeyer theorem (cf. [15] or [17]) states that a consistent and stable difference-method is convergent.
3. The approximation lemma presented above differs from the standard Lax–Richtmeyer theorem in numerical analysis in that the one-step difference operators $\boldsymbol{C}_j^h$ are time inhomogeneous and may be *nonlinear*. The nonlinearity will be essential for dealing with optimality equations.

5. DISCRETE–TIME APPROXIMATIONS

5.1. Introduction

By applying the approximation lemma of Section 4, in this section we show that the continuous-time finite horizon cost functions defined by (2.2.6) and (2.3.2), can be approximated by their discrete-time analogues from (3.2.4) and (3.3.2) respectively. Roughly speaking, the discrete-time approximation is guaranteed if the following three conditions are satisfied:

(i) The continuous-time function is (piecewise) sufficiently smooth with respect to the time parameter (*smoothness*).

(ii) The discrete-time one-step generators approximate the infinitesimal operators (*consistency*).

(iii) The discrete-time one-step transition probabilities are sufficiently bounded (*stability*).

In this section we consider a fixed control object

$$(S,\,\Gamma,\,\Delta,\,\mu,\,D_A,\,\{\boldsymbol{A}^\delta\,|\,\delta\in\Delta\},\,\boldsymbol{L})\text{ and fixed }h\text{-control objects}$$
$$(S,\,\Gamma,\,\Delta,\,\mu,\,h,\,\{\boldsymbol{P}_h^\delta\,|\,\delta\in\Delta\},\,\boldsymbol{L})\text{ for all }h\le h_0\text{ and some }h_0\ge 0.$$

Further, we note that we do not make any of the assumptions of the preceding section a priori, but that we collect all necessary assumptions for finite horizon cost functions under fixed control in Theorem 5.2.1 and for finite horizon optimal cost functions in Theorem 5.3.1. The notation of the preceding sections will be adopted, where the use of $\boldsymbol{T}_h^\delta$ and $\boldsymbol{A}_h^\delta$ (see Definition 3.1.2) is justified by either (5.2.3) or (5.3.3).

5.2. Finite horizon cost function; fixed control

Let $\pi \in \Pi(AB)$ and define $\pi^h \in \Pi^h$ by: $\pi^h(nh)$, $n = 0, 1, \ldots, \ell$. First, we remark that the construction (3.2.1) together with the boundedness relation (5.2.3) given below yield for all $j \leq n \leq \ell$:

$$\left\| \int \mu(y)\, \boldsymbol{P}^h_{j,n}(\cdot; \mathrm{d}y) \right\|_\mu \leq (1 + hK^\pi)^{(n-j)} \leq e^{ZK^\pi}.$$

Hence, relation (5.2.3) implies: $\pi^h \in \Pi^h(AB)$. Further, recall the expression (2.2.6) for $\boldsymbol{V}^\pi_t$, (2.2.8) for $R^\pi_t(\boldsymbol{V}, h)$, (3.2.6) for $\boldsymbol{V}^h_j$ and (3.1.1) for $\boldsymbol{A}^{\pi(jh)}_h$, $j \leq \ell$.

Theorem 5.2.1. Suppose that

(i) Assumptions 2.2.3 and 3.2.3 hold and that

(ii) For some constants ε^h_1, ε^h_2 and K^π, and all $j \leq \ell$:

$$\left\| R^\pi_{jh}(\boldsymbol{V}, h)\, h^{-1} \right\|_\mu \leq \varepsilon^h_1, \tag{5.2.1}$$

$$\left\| (\boldsymbol{A}^{\pi(jh)}_h - \boldsymbol{A}^{\pi(jh)})\, \boldsymbol{V}^\pi_{jh+h} \right\|_\mu \leq \varepsilon^h_2, \tag{5.2.2}$$

$$\left\| \int \mu(y)\, \boldsymbol{P}^{\pi(jh)}_h(\cdot; \mathrm{d}y) \right\|_\mu \leq (1 + hK^\pi). \tag{5.2.3}$$

Then, for some constant C^π_L and all $n \leq \ell$:

$$\left\| \boldsymbol{V}^h_n - \boldsymbol{V}^\pi_{nh} \right\|_\mu \leq [\varepsilon^h_1 + \varepsilon^h_2]\, [e^{ZK^\pi} - 1]/K^\pi + h\, e^{ZK^\pi}\, C^\pi_L. \tag{5.2.4}$$

Proof. By virtue of (2.2.8) and (3.2.7) system (4.1) holds with:

$$\left.\begin{aligned} &B = B^\mu; \quad \boldsymbol{U}_{jh} = \boldsymbol{V}^\pi_{jh}; \quad \boldsymbol{U}^h_j = \boldsymbol{V}^h_j; \\ &\boldsymbol{C}_{jh}(f) = h\, \boldsymbol{L}^{\pi(jh)} + \left[I + h\, \boldsymbol{A}^{\pi(jh)}\right](f) + R^\pi_{jh}(\boldsymbol{V}, h); \\ &\boldsymbol{C}^h_j(f) = h\, \boldsymbol{L}^{\pi(jh)} + \boldsymbol{T}^{\pi(jh)}_h(f) = h\, \boldsymbol{L}^{\pi(jh)} + \left[I + h\, \boldsymbol{A}^{\pi(jh)}_h\right](f). \end{aligned}\right\} \tag{5.2.5}$$

Consequently, relations (5.2.1) and (5.2.2) imply (4.2) with $\varepsilon = \varepsilon^h_1 + \varepsilon^h_2$. Furthermore, relation (4.3) holds for $K = K^\pi$ since (5.2.3) implies:

$$\left\| \boldsymbol{C}^h_j(f_1) - \boldsymbol{C}^h_j(f_2) \right\|_\mu = \left\| \boldsymbol{T}^{\pi(jh)}_h(f_1 - f_2) \right\|_\mu \leq (1 + hK^\pi)\, \|f_1 - f_2\|_\mu. \tag{5.2.6}$$

Finally, with $\ell = \lfloor Zh^{-1} \rfloor$, we obtain, according to (5.2.5), (2.2.6) and (3.2.7), that for some constant C^π_L (see condition (i) of Assumption 2.2.3):

$$\left\| \boldsymbol{U}^h_\ell - \boldsymbol{U}_{\ell h} \right\|_\mu = \|\boldsymbol{V}^\pi_{\ell h}\|_\mu \leq h \sup_{t \leq Z} \|L^\pi_t\|_\mu \leq h\, C^\pi_L. \tag{5.2.7}$$

Substituting (5.2.7) in (4.4) with $m = \ell$ and applying Lemma 4.1 yields (5.2.4). □

Theorem 5.2.2. Under the conditions from Theorem 5.2.1 with $\varepsilon_1^h \to 0$ and $\varepsilon_2^h \to 0$ as $h \to 0$, and with $n = \lfloor th^{-1} \rfloor$:

$$\left\| \boldsymbol{V}_n^h - \boldsymbol{V}_t^\pi \right\|_\mu \to 0 \quad \text{as } h \to 0, \text{ uniformly in } t \leq Z. \tag{5.2.8}$$

Proof. A direct result of Theorem 5.2.1 and the inequalities:

$$\left\| \boldsymbol{V}_n^h - \boldsymbol{V}_t^\pi \right\|_\mu \leq \left\| \boldsymbol{V}_n^h - \boldsymbol{V}_{nh}^\pi \right\|_\mu + \|\boldsymbol{V}_{nh}^\pi - \boldsymbol{V}_t^\pi\|_\mu , \tag{5.2.9}$$

$$\|\boldsymbol{V}_{nh}^\pi - \boldsymbol{V}_t^\pi\|_\mu \leq h \sup_{t \leq Z} \|\boldsymbol{L}_t^\pi\|_\mu \leq h\, C_L^\pi. \tag{5.2.10}$$

□

Remarks 5.2.3.

1. Clearly, from Theorem 5.2.1 and inequality (5.2.10) it follows that if ε_1^h and ε_2^h are convergent of order $O(h^p)$ (order of consistency), then also the convergence in (5.2.8) is of order $O(h^p)$ for $p \leq 1$ and of order $O(h)$ for $p > 1$ (order of convergence).
2. Note that Theorem 5.2.1 allows us to deal with controls π which are only piecewise smooth on intervals $[nh, nh + h]$, (e.g. piecewise constant (step) controls). In this respect also the following remark is of interest.
3. By virtue of the symmetry in system (4.1) it is easily seen that Lemma 4.1 remains valid if we interchange $(\boldsymbol{U}_{jh}, \boldsymbol{C}_{jh})$ and $(\boldsymbol{U}_j^h, \boldsymbol{C}_j^h)$. Consequently, Theorem 5.2.1 remains valid if we replace

$$\left.\begin{array}{lll} \boldsymbol{V}_{jh+h}^\pi & \text{by} \quad \boldsymbol{V}_{j+1}^h & \text{in (5.2.2), and} \\ \boldsymbol{P}^{\pi(jh)} & \text{by} \quad \boldsymbol{P}_{jh,jh+h}^\pi & \text{in (5.2.3).} \end{array}\right\} \tag{5.2.11}$$

4. In Theorem 5.2.1 the discrete-time controls $\pi^h \in \Pi^h$ are the projections on $\{nh \,|\, n = 0, 1, \ldots, \ell\}$ of the continuous-time control π. Obviously, however, Theorems 5.2.1 and 5.2.2 remain valid if we consider any sequence of discrete-time controls $\pi^h \in \Pi^h$ and substitute

$$\left.\begin{array}{lll} \boldsymbol{A}_j^{\pi^h(jh)} & \text{for} \quad \boldsymbol{A}_h^{\pi(jh)} & \text{in (5.2.2), and} \\ \boldsymbol{P}_j^{\pi^h(jh)} & \text{for} \quad \boldsymbol{P}_h^{\pi(jh)} & \text{in (5.2.3).} \end{array}\right\} \tag{5.2.12}$$

Of particular interest is the case where the controls π^h are (ε-)optimal for the h-discrete-time models. If there exists a continuous-time (limit) control π such that the conditions of the Theorem 5.2.2 are satisfied, then π is optimal in a wide class of controls (cf. [8,9]).

Since, however, in our applications (see Part II) we focus on either discrete-time projections of a continuous-time control or a continuous-time embedding of a discrete-time control, we prefer to present the approximation results, more specifically the conditions (5.2.2) and (5.2.3), in the simpler form.

5.3. Finite horizon optimal cost functions

In this section we derive an approximation theorem for the optimal cost function. Recall the defining relations (2.3.2) for $\boldsymbol{\Phi}_t$, (2.3.3) for $R_t(\boldsymbol{\Phi}, t)$ and (3.3.2) for $\boldsymbol{\Phi}^h_j$.

Theorem 5.3.1. Suppose that

(i) Assumptions 2.3.1, 2.3.2 and 3.3.1 hold, and that

(ii) for some constants ε^h_1, ε^h_2 and K_Δ and all $j \le \ell$:

$$\left\| R_{jh}(\boldsymbol{\Phi}, h)\, h^{-1} \right\|_\mu \le \varepsilon^h_1, \tag{5.3.1}$$

$$\sup_{\delta \in \Delta} \left\| (\boldsymbol{A}^\delta_h - \boldsymbol{A}^\delta)\, \boldsymbol{\Phi}_{jh} \right\|_\mu \le \varepsilon^h_2, \tag{5.3.2}$$

$$\sup_{\delta \in \Delta} \left\| \int \mu(y)\, \boldsymbol{P}^\delta_h(\cdot;\, \mathrm{d}y) \right\|_\mu \le (1 + h\, K_\Delta). \tag{5.3.3}$$

Then for some constant C_Φ and all $n \le \ell$:

$$\left\| \boldsymbol{\Phi}^h_n - \boldsymbol{\Phi}_{nh} \right\|_\mu \le \left[\varepsilon^h_1 + \varepsilon^h_2\right] \left[e^{ZK_\Delta} - 1\right] / K_\Delta + h\, e^{ZK_\Delta}\, C_\Phi. \tag{5.3.4}$$

P r o o f. By virtue of (2.3.2), (2.3.3), (3.1.1) and (3.3.2), system (4.1) holds with:

$$\left.\begin{aligned} &B = B^\mu; \quad \boldsymbol{U}_{jh} = \boldsymbol{\Phi}_{jh}; \quad \boldsymbol{U}^u_j = \boldsymbol{\Phi}^h_j; \\ &\boldsymbol{C}_{jh}(f) = \inf_{\delta\in\Delta} \left[h\, \boldsymbol{L}^\delta + [I + h\, \boldsymbol{A}^\delta](f)\right] + R_{jh}(\boldsymbol{\Phi}, h) \\ &\boldsymbol{C}^h_j(f) = \inf_{\delta\in\Delta} \left[h\, \boldsymbol{L}^\delta + \boldsymbol{T}^\delta_h\, h\right] = \inf_{\delta\in\Delta} \left[h\, \boldsymbol{L}^\delta + [I + h\, \boldsymbol{A}^\delta_h](f)\right]. \end{aligned}\right\} \tag{5.3.5}$$

Hence,

$$\left.\begin{aligned} &\left\| \boldsymbol{C}_{jh}(f) - \boldsymbol{C}^h_j(f) \right\|_\mu \\ &\le \|R_{jh}(\boldsymbol{\Phi}, h)\|_\mu + \left\| \inf_{\delta\in\Delta}[h\, \boldsymbol{L}^\delta + h\, \boldsymbol{A}^\delta f] - \inf_{\delta\in\Delta}[h\, \boldsymbol{L}^\delta + h\, \boldsymbol{A}^\delta_h f] \right\|_\mu \\ &\le \sup_{\delta\in\Delta} \left\| (\boldsymbol{A}^\delta_h - \boldsymbol{A}^\delta)\, f \right\|_\mu h + \|R_{jh}(\Phi, h)\|_\mu . \end{aligned}\right\} \tag{5.3.6}$$

The relations (5.3.1), (5.3.2) and (5.3.6) imply (4.2) with $\varepsilon = \varepsilon^h_1 + \varepsilon^h_2$. Furthermore, relation (4.3) holds for $K = K_\Delta$, since (5.3.3) implies:

$$\left\| \boldsymbol{C}^h_j(f_1) - \boldsymbol{C}^h_j(f_2) \right\|_\mu \le \sup_{\delta\in\Delta} \left\| \boldsymbol{T}^\delta_h(f_1 - f_2) \right\|_\mu \le (1 + h\, K_\Delta)\, \|f_1 - f_2\|_\mu . \tag{5.3.7}$$

According to Assumption 2.3.2: $\{\boldsymbol{J}(\boldsymbol{\Phi}_t) \,|\, t \le Z\}$ is μ-bounded. Hence, from $\ell = \lfloor Zh^{-1} \rfloor$, (5.3.5), (2.3.2) and (3.3.2) we obtain for some constant C_Φ:

$$\left\| \boldsymbol{U}^h_\ell - \boldsymbol{U}_{\ell h} \right\|_\mu = \|\boldsymbol{\Phi}_{\ell h}\|_\mu \le h \sup_{t\le Z} \|\boldsymbol{J}(\boldsymbol{\Phi}_t)\|_\mu \le h\, C_\Phi. \tag{5.3.8}$$

Substituting (5.3.8) in (4.4) with $m = \ell$ and applying Lemma 4.1 yields (5.3.4). □

Theorem 5.3.2. Suppose that for all $h \le h_0$ the conditions (i), (ii) of Theorem 5.3.1 are satisfied with $\varepsilon_1^h \to 0$ and $\varepsilon_2^h \to 0$ as $h \to 0$. Then, with $n = \lfloor th^{-1} \rfloor$:

$$\left\| \boldsymbol{\Phi}_n^h - \boldsymbol{\Phi}_t \right\|_\mu \to 0 \quad \text{as} \quad h \to 0, \text{ uniformly in } t \le Z. \tag{5.3.9}$$

P r o o f. Since $\{\boldsymbol{J}(\boldsymbol{\Phi}_t) \mid t \le Z\}$ is μ-bounded the proof is similar to the proof of Theorem 5.2.2. □

Remarks 5.3.3.

1. As pointed out in 1 of Remark 5.2.3 an order of convergence in (5.3.9) can be obtained if we have an order of consistency, i. e., an order of convergence for ε_1^h and ε_2^h.
2. Similarly to 4 of Remark 5.2.3 it can be argued that Theorem 5.3.2 remains valid if we allow a finite set of time-points as exclusion set.

5.4. Remark on terminal cost functions

It is easily seen that the approximation results of Subsections 5.2 and 5.3 remain valid if the cost function $\boldsymbol{V}^\pi$ and $\boldsymbol{\Phi}$ also take into account a terminal cost depending on the actual state at time Z, say $q \in B^\mu$ is the terminal cost function. For including a terminal cost we only need to *add*:

$$\begin{cases} \text{Add } \boldsymbol{T}_{t,Z}^\pi\, q \text{ and } \boldsymbol{T}_{j,\ell}^h\, q \text{ to the right-hand side of (2.2.6) resp. (3.2.6).} \\ \text{Replace } \overline{0} \text{ by } q \text{ in the relations (2.3.2), (3.2.7), (3.3.1) and (3.3.2).} \end{cases}$$

By taking $\boldsymbol{L}^\delta = \overline{0}$ for any $\delta \in \Delta$ and using only suitably chosen terminal cost functions q, one can show weak convergence of the discrete-time transition probabilities $\boldsymbol{P}_{j,\ell}^h$ to $\boldsymbol{P}_{t,z}^\pi$ for any fixed control π. By varying t and z, and applying arguments for weak convergence of processes on appropriate sample path spaces (see the Appendix of [20]) we can also conclude weak convergence of the underlying processes.

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