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EXACT DECOMPOSITION OF LINEAR SINGULARLY PERTURBED H^∞-OPTIMAL CONTROL PROBLEM

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We consider the singularly perturbed H^∞-optimal control problem under perfect state measurements, for both finite and infinite horizons. We get the exact decomposition of the full-order Riccati equations to the reduced-order pure-slow and pure-fast equations. As a result, the H^∞-optimum performance and suboptimal controllers can be exactly determined from these reduced-order equations. The suggested decomposition allows the development of new effective algorithms of high-order accuracy.

1. INTRODUCTION

Consider the linear time-varying singularly perturbed system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u + D_1w, \quad \varepsilon\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u + D_2w, \quad x(0) = 0 \tag{1.1}$$

and the quadratic functional

$$J = x'(t_f)\,Fx(t_f) + \int_0^{t_f} [x'(t)\,Q(t)\,x(t) + u'(t)\,u(t)]\,\mathrm{d}t, \tag{1.2}$$

where $x = \mathrm{col}\{x_1, x_2\}$ is the state vector with $x_1(t) \in I\!R^{n_1}$ and $x_2(t) \in I\!R^{n_2}$, $u(t) \in I\!R^p$ is the control input, $w \in I\!R^q$ is the disturbance. The matrices $A_{ij} = A_{ij}(t)$, $B_i = B_i(t)$, $D_i = D_i(t)$ $(i = 1,2,\ j = 1,2)$ are continuously differentiable functions of $t \geq 0$, and ε is a small positive parameter. The symbol $(\cdot)'$ denotes the transpose of a matrix,

$$Q = Q' = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \geq 0, \quad F = F' = \begin{pmatrix} F_{11} & \varepsilon F_{12} \\ \varepsilon F_{21} & \varepsilon F_{22} \end{pmatrix} \geq 0.$$

Denote by $|\cdot|$ the Euclidean norm of a vector. Let $S_{ij} = B_iB_j' - \gamma^{-2}D_iD_j'$, $i = 1,2,\ j = 1,2,\ B_\varepsilon = \mathrm{col}\{B_1, \varepsilon^{-1}B_2\}, D_\varepsilon = \mathrm{col}\{D_1, \varepsilon^{-1}D_2\}$,

$$A_\varepsilon = \begin{pmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{pmatrix}, \quad S_\varepsilon = \begin{pmatrix} S_{11} & \varepsilon^{-1}S_{12} \\ \varepsilon^{-1}S_{21} & \varepsilon^{-2}S_{22} \end{pmatrix}.$$

With (1.1), (1.2) we associate the Riccati differential equation (RDE)

$$\dot{Z} + A_\varepsilon' Z + Z A_\varepsilon - Z S_\varepsilon Z + Q = 0; \quad Z(t_f) = F \tag{1.3}$$

for the matrix function

$$Z = Z' = Z(t,\varepsilon) = \begin{pmatrix} Z_{11}(t,\varepsilon) & \varepsilon Z_{12}(t,\varepsilon) \\ \varepsilon Z_{21}(t,\varepsilon) & \varepsilon Z_{22}(t,\varepsilon) \end{pmatrix}. \tag{1.4}$$

For each $\varepsilon > 0$ the H^∞-optimum performance $\gamma^*(\varepsilon)$ is computed by the formula [1], [10]

$$\gamma^*(\varepsilon) = \inf\{\gamma > 0 \,|\, (1.3) \text{ has a bounded solution on } [0; t_f]\}.$$

A controller that guarantees the performance level $\gamma > \gamma^*(\varepsilon)$ is determined by the relation

$$u(t) = -[B_1'; \; \varepsilon^{-1} B_2'] \, Z(t,\varepsilon) \, x(t), \quad t \in [0; t_f] \,, \tag{1.5}$$

where $Z(t,\varepsilon) = Z(t,\varepsilon,\gamma)$ is the solution of (1.3).

In the infinite horizon case we take A_ε, B_ε, D_ε and $Q = C'C$ to be time invariant, $F = 0$ and assume:

A1. The triple $\{A_\varepsilon, B_\varepsilon, C\}$ is stabilizable and detectable for $\varepsilon \in (0, \varepsilon_0]$ ($\varepsilon_0 > 0$).

The H^∞-optimum performance is determined from the full-order generalized algebraic Riccati equation (ARE) of the form (1.3), where $\dot{Z} = 0$ as follows [1, 10]:

$$\begin{aligned} \gamma^*(\varepsilon) \; = \; & \inf\{\gamma > 0 \,|\, \text{the full} - \text{order ARE has a nonnegative definite solution such} \\ & \text{that the matrix } A_\varepsilon - S_\varepsilon Z \text{ is Hurwitz}\}. \end{aligned}$$

Computation of $\gamma^*(\varepsilon)$, and the corresponding suboptimal controller (1.5) for small values of $\varepsilon > 0$ presents serious difficulties due to high dimension and numerical stiffness, resulting from the interaction of slow and fast modes. In [10] an upper bound $\overline{\gamma}$ for $\gamma^*(\varepsilon)$ has been found on the basis of a slow and a fast control subproblems. For each $\gamma > \overline{\gamma}$ a composite controller has been designed that gives the zero-order approximation to the controller of (1.5) and achieves the performance γ for the full-order system for all small enough ε (see also [3] for a composite controller in the case $t_f = \infty$). In [7] and [9] the frequency domain decomposition of H^∞ control problems has been obtained, however the issue of optimal controller design has not been addressed.

The main objective of the paper is getting the exact decomposition of the problem.

2. MAIN RESULTS

We will develop the method of exact decomposition of the full-order Riccati equations initiated with the works [4, 12], to H^∞-optimal control problem. We begin with the finite horizon case. Consider the Hamiltonian system corresponding to (1.3) with the adjoint variables $y_1, \varepsilon y_2$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \varepsilon\dot{x}_2 \\ \varepsilon\dot{y}_2 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}, \quad R_{ij} = \begin{pmatrix} A_{ij} & -S_{ij} \\ -Q_{ij} & -A'_{ji} \end{pmatrix}, \tag{2.1a}$$

$$x_1(t_f) = x_1^0, \quad y_1(t_f) = F_{11}x_1^0 + \varepsilon F_{12}x_2^0, \quad x_2(t_f) = x_2^0, \quad y_2(t_f) = F_{21}x_1^0 + F_{22}x_2^0. \tag{2.1b}$$

Lemma 1. For each $\varepsilon > 0$, (1.3) has a bounded on $[0, t_f]$ solution iff there exists the matrix function of the form (1.4) such that for all $x_1^{(0)} \in I\!R^{n_1}$, $x_2^{(0)} \in I\!R^{n_2}$ a solution of (2.1) can be represented as follows:

$$\mathrm{col}\{y_1, \varepsilon y_2\} = Zx, \quad t \in [0, t_f]. \tag{2.2}$$

For proof of Lemma 1 and the other Lemmas of the paper see **Appendix.**

Let $C_2'C_2 = Q_{22}$. Consider the following ARE

$$A'_{22}M^{(0)} + M^{(0)}A_{22} + Q_{22} - M^{(0)}S_{22}M^{(0)} = 0, \quad t \in [0, t_f], \tag{2.3}$$

which corresponds, for each $t \in [0, t_f]$, to the fast infinite horizon subproblem. Assume

A2. The triple $\{A_{22}, B_2, C_2\}$ is stabilizable and detectable for all $t \in [0, t_f]$.

Let $\gamma_f^t = \inf\{\gamma' \,|\, \text{ARE (2.3) has a solution } M^{(0)} \geq 0 \text{ such that } \Lambda_0 = A_{22} - S_{22}M^{(0)}$ is Hurwitz$\}$. We choose $\gamma_f = \sup_{t\in[0,t_f]} \gamma_f^t$. Under A2 $\gamma_f < \infty$ [10]. We shall further consider only $\gamma \geq \gamma_f + \delta$ with $\delta > 0$ fixed. From [2, Lemma 4] and from the continuous dependence of R_{22} on $t \in [0, t_f]$ and $1/\gamma \in [0, (\gamma_f + \delta)^{-1}]$ it follows that for all $\gamma \geq \gamma_f + \delta$ and $t \in [0, t_f]$ the matrix R_{22} has n_2 stable eigenvalues λ, $\mathrm{Re}\lambda < -\alpha < 0$ (corresponding to Λ_0) and n_2 unstable ones, $\mathrm{Re}\lambda > \alpha$. This implies [11] the existence of $\varepsilon_\gamma > 0$ such that for each $\gamma \geq \gamma_f + \delta$ and $\varepsilon \in [0, \varepsilon_\gamma)$ there are the matrix functions $H = -R_{22}^{-1}R_{21} + \varepsilon\bar{H}(t, \varepsilon)$, $P = R_{12}R_{22}^{-1} + \varepsilon\bar{P}(t, \varepsilon)$, $M = M^{(0)} + \varepsilon\bar{M}(t, \varepsilon)$ and $L = L^{(0)} + \varepsilon\bar{L}(t, \varepsilon)$ that satisfy the equations

$$\varepsilon\dot{H} + \varepsilon H(R_{11} + R_{12}H) = R_{21} + R_{22}H, \tag{2.4a}$$

$$\varepsilon\dot{P} + P(R_{22} - \varepsilon HR_{12}) = \varepsilon(R_{11} + R_{12}H)P + R_{12}, \tag{2.4b}$$

$$\varepsilon\dot{M} + M[A_{22} + \varepsilon K_1 + (\varepsilon K_2 - S_{22})M] = -Q_{22} + \varepsilon K_3 + (-A'_{22} + \varepsilon K_4)M, \tag{2.4c}$$

$$\varepsilon\dot{L} - L[A'_{22} - \varepsilon K_4 + M(\varepsilon K_2 - S_{22})] = [A_{22} + \varepsilon K_1 + (\varepsilon K_2 - S_{22})M]L + \varepsilon K_2 - S_{22}, \tag{2.4d}$$

where

$$\begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} = -HR_{12}, \quad H = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}, \quad P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}. \tag{2.5}$$

The matrix $M^{(0)}$ is a solution of (2.3) and $L^{(0)}$ satisfies the Lyapunov equation, that results from (2.4d) by setting $\varepsilon = 0$. If the coefficients of (1.1) and (1.2) are smooth, the functions H, P, M and L can be easily found in the form of asymptotic expansions. The terms of these expansions can be determined from linear algebraic equations [11]. In the time-invariant case, H, P, M and L can be also computed numerically [6].

For $\gamma \geq \gamma_f + \delta$ and $\varepsilon \in [0, \varepsilon_\gamma)$ the nonsingular transformation [11]

$$\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} I & 0 & \varepsilon G_1 & \varepsilon G_2 \\ 0 & I & \varepsilon G_3 & \varepsilon G_4 \\ H_1 & H_2 & E_1 & E_2 \\ H_3 & H_4 & E_3 & E_4 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}, \tag{2.6}$$

where

$$\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} = (I + \varepsilon HP) \begin{pmatrix} I & L \\ M & I + ML \end{pmatrix}, \quad \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} = P \begin{pmatrix} I & L \\ M & I + ML \end{pmatrix},$$

decomposes (2.1) into the slow system for $u_1 \in I\!R^{n_1}$ and $v_1 \in I\!R^{n_1}$

$$\begin{pmatrix} \dot u_1 \\ \dot v_1 \end{pmatrix} = W \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad W = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} = R_{11} + R_{12}H, \tag{2.7a}$$

and the two fast decoupled equations for $u_2 \in I\!R^{n_2}$ and $v_2 \in I\!R^{n_2}$

$$\varepsilon \dot u_2 = (A_{22} + \varepsilon K_1 + (-S_{22} + \varepsilon K_2)M)\, u_2, \quad \varepsilon \dot v_2 = (-A_{22}' + \varepsilon K_4 + M(S_{22} - \varepsilon K_2))\, v_2. \tag{2.7b}$$

In all previous derivations ε_γ can be chosen independent of γ. Really, the matrix functions H, P, M, L define integral manifold of (2.1) and some auxiliary singularly perturbed systems [11]. Due to the inequality $\mathrm{Re}\lambda < -\alpha$ for the eigenvalues of Λ_0 and since the coefficients of (2.1) are uniformly bounded on $\gamma^{-1} \in [0, (\gamma_f + \delta)^{-1}]$, these integral manifolds exist for all small enough ε and $\gamma \geq \gamma_f + \delta$. Thus we get:

Proposition. There is $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0]$ and $\gamma \geq \gamma_f + \delta$ the transformation (2.14) exists and decomposes (2.1) into the systems of (2.7).

Substituting (2.6) into the terminal conditions of (2.1) and further eliminating x_1^0 and x_2^0, we obtain the following terminal conditions for u_1, v_1, u_2, v_2:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\Big|_{t=t_f} = \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix}, \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\Big|_{t=t_f} = \begin{pmatrix} U_{11} & \varepsilon U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix}, \tag{2.8}$$

where

$$\begin{pmatrix} U_{11} & \varepsilon U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} Y_2 \\ Y_4 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_3 \end{pmatrix}^{-1}\Bigg|_{t=t_f}, \quad \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \begin{pmatrix} \Phi_1 & \Phi_2 & -\varepsilon P_1 & -\varepsilon P_2 \\ \Phi_3 & \Phi_4 & -\varepsilon P_3 & -\varepsilon P_4 \\ \Psi_1 & \Psi_2 & \Xi_1 & \Xi_2 \\ \Psi_3 & \Psi_4 & \Xi_3 & \Xi_4 \end{pmatrix} \begin{pmatrix} I & 0 \\ F_{11} & \varepsilon F_{12} \\ 0 & I \\ F_{21} & F_{22} \end{pmatrix} \tag{2.9}$$

$$\begin{pmatrix}\Phi_1 & \Phi_2\\ \Phi_3 & \Phi_4\end{pmatrix} = I+\varepsilon PH, \quad \begin{pmatrix}\Xi_1 & \Xi_2\\ \Xi_3 & \Xi_4\end{pmatrix}=\begin{pmatrix}I+LM & -L\\ -M & I\end{pmatrix}, \quad \begin{pmatrix}\Psi_1 & \Psi_2\\ \Psi_3 & \Psi_4\end{pmatrix} = -\begin{pmatrix}\Xi_1 & \Xi_2\\ \Xi_3 & \Xi_4\end{pmatrix} H.$$

By straightforward computations we get

$$\begin{pmatrix}Y_1\\ Y_3\end{pmatrix} = \begin{pmatrix}I & 0\\ \dots & I+L^{(0)}(M^{(0)}-F_{22})\end{pmatrix} + O(\varepsilon). \tag{2.10}$$

To assure the existence of the inverse matrix in (2.9) we assume

A3. The matrix $I+L^{(0)}(M^{(0)}-F_{22})$ is invertible at $t=t_f$ for all $\gamma \geq \gamma_f+\delta$.

Consider the pure-slow RDE for the $n_1 \times n_1$-matrix function $N=N(t,\varepsilon)$

$$\dot N + N(W_1+W_2N) = W_3+W_4N, \quad N(t_f)=U_{11}, \tag{2.10}$$

and the pure-fast linear equations for the $n_i \times n_j$-matrix functions $N_{ij}=N_{ij}(t,\varepsilon)$:

$$\varepsilon\dot N_{12} = -N_{12}(\Lambda+\varepsilon(K_1+K_2M+W_2))+\varepsilon W_4N_{12}, \qquad N_{12}(t_f)=U_{12}, \tag{2.11}$$

$$\varepsilon\dot N_{21} = -(\Lambda'-\varepsilon(K_4-MK_2))\,N_{21}-\varepsilon N_{21}(W_1+W_2N), \quad N_{21}(t_f)=U_{21}, \tag{2.12}$$

$$\varepsilon\dot N_{22} = -N_{22}(\Lambda+\varepsilon(K_1+K_2M))-(\Lambda'-\varepsilon(K_4-MK_2))\,N_{22}, \quad N_{22}(t_f)=U_{22}, \tag{2.13}$$

where $\Lambda = A_{22}-S_{22}M$. Similarly to Lemma 1, equations (2.10)–(2.13) have bounded solutions on $[0,t_f]$ iff a solution of (2.7) can be represented in the form

$$v_1 = Nu_1+\varepsilon N_{12}u_2, \quad v_2 = N_{21}u_1+N_{22}u_2, \quad t\in[0,t_f] \tag{2.14}$$

for every $u_1^0 \in I\!R^{n_1}$, $u_2^0 \in I\!R^{n_2}$. Finally, substituting (2.14), (2.6) into (2.2), and equating separately terms with u_1 and u_2, we get

$$Z\begin{pmatrix}I+\varepsilon G_2N_{21} & \varepsilon G_1+\varepsilon G_2N_{22}\\ H_1+H_2N+E_2N_{21} & E_1+E_2N_{22}+\varepsilon H_2N_{12}\end{pmatrix} = \begin{pmatrix}N+\varepsilon G_4N_{21} & \varepsilon N_{12}+\varepsilon G_3+\varepsilon G_4N_{22}\\ \varepsilon(H_3+H_4N+E_4N_{21}) & \varepsilon E_3+\varepsilon E_4N_{22}+\varepsilon^2H_4N_{12}\end{pmatrix}. \tag{2.15}$$

If for $\gamma \geq \gamma_f+\delta$ and small ε RDE (2.10) has a uniformly bounded solution on $[0,t_f]$ then the linear equations (2.11)–(2.13) have solutions, exponentially decaying on $[0,t_f]$:

$$|N_{ij}(t,\varepsilon)| \leq Ke^{\alpha(t-t_f)/\varepsilon}, \quad t\in[0,t_f], \quad K>0. \tag{2.16}$$

Lemma 2. Under A2 and A3 for any $\delta>0$ there exists $\varepsilon_\delta>0$ such that for all $0<\varepsilon\leq\varepsilon_\delta$ and $\gamma\geq\gamma_f+\delta$ the following holds:

(i) The full-order RDE (1.3) has a bounded solution on $[0,t_f]$ iff the slow RDE (2.10) has a bounded solution on $[0,t_f]$;

(ii) If (1.3) has a bounded solution on $[0,t_f]$, then this solution can be uniquely defined from the equations (2.4), the decoupled pure-slow and pure-fast differential equations (2.10)–(2.13) and the linear algebraic equation (2.15).

From Lemma 2 it follows immediately:

Theorem 1 (finite horizon case). Under A2 and A3 the following holds:

i) For a prechosen $\delta > 0$ and all small enough ε, the suboptimal controller (1.5), that guarantees a $\gamma > \max\{\gamma^*(\varepsilon), \gamma_f + \delta\}$ performance level, can be determined from (2.4), the decoupled reduced-order pure-slow and pure-fast differential equations (2.10)–(2.13), and the linear algebraic equation (2.15) instead of (1.3);

(ii) If $\gamma^*(\varepsilon) \geq \gamma_f + \delta_0$ for $0 < \varepsilon < \varepsilon_0$, then for all small enough ε, the value of $\gamma^*(\varepsilon)$ can be found from (2.4a) and the slow RDE (2.10) by the formula:

$$\gamma^*(\varepsilon) = \inf\{\gamma > 0 \mid \text{RDE (2.10) has a bounded on } [0, t_f] \text{ solution}\}. \tag{2.17}$$

In the infinite-horizon case we take A, B, D, Q to be constant and $F = 0$. In this case (2.4) are algebraic equations and H, P, M and L are constant.

Lemma 3. Under A1 and A2 for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that for all $0 < \varepsilon \leq \varepsilon_\delta$ and $\gamma \geq \gamma_f + \delta$ the full-order ARE of (1.3), where $\dot{Z} = 0$, has a unique solution Z, such that the matrix $A_\varepsilon - S_\varepsilon Z$ is Hurwitz, iff the slow ARE of (2.10), where $\dot{N} = 0$, has a unique solution such that $\Delta_1 = W_1 + W_2 N$ is Hurwitz. The solutions of ARE (1.3) and of ARE (2.10) are related by formula:

$$Z = \begin{pmatrix} N & \varepsilon G_3 \\ \varepsilon(H_3 + H_4 N) & \varepsilon E_3 \end{pmatrix} \begin{pmatrix} I & \varepsilon G_1 \\ H_1 + H_2 N & E_1 \end{pmatrix}^{-1}, \tag{2.18}$$

where the inverse matrix exists.

Note that A1, imposed on the full-order problem (1.1), (1.2) can be decomposed into corresponding conditions for the slow and fast subproblems [8]. From Lemma 3 it follows

Theorem 2 (infinite horizon case). Under A1 and A2 the following holds:

(i) For a pechosen $\delta > 0$ and all small enough ε, the suboptimal controller, that guarantees a $\gamma > \max\{\gamma^*(\varepsilon), \gamma_f + \delta\}$ performance level, can be determined from (2.4), (1.5) and (2.18), where N is the solution of ARE (2.10) with the Hurwitz matrix Δ_1 and $Z \geq 0$;

(ii) If $\gamma^*(\varepsilon) \geq \gamma_f + \delta_0$ for $0 < \varepsilon < \varepsilon_0$, then for all small enough ε

$$\gamma^*(\varepsilon) = \inf\{\gamma > 0 \mid \text{ARE (2.10) has a solution such that } \Delta_1 \text{ is Hurwitz and } Z, \text{ defined by (2.18), is nonnegative definite}\}.$$

3. CONCLUSIONS

Solutions to the ε-dependent reduced-order equations (2.10)–(2.13) can be found without difficulty by standard numerical and asymptotic methods. This would lead to effective reduced-order algorithms for H^∞-Riccati equations. For a nonlinear counterpart of the infinite horizon results see [5], where an asymptotic approximation to the suboptimal controller is constructed on the basis of exact decomposition, and it is shown that the high-order accuracy controller improves the performance.

APPENDIX

Proof of Lemma 1. Let RDE (1.3) has a bounded solution on $[0, t_f]$. Consider the equation

$$\dot{x} = (A_\varepsilon + B_\varepsilon Z)\, x, \quad t \in [0, t_f]. \tag{A.1}$$

Let $x(t)$ be a solution of (A.1) with $x(t_f) = x^0$, and $y_1(t)$, $y_2(t)$ be defined by (2.2). Then $y_1(t_f)$, $y_2(t_f)$ satisfy the terminal condition of (2.1). Differentiating (2.2) and applying (1.3) and (A.1) we shall see that the functions $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ satisfy (2.1).

Conversely, let there exists $Z(t)$, satisfying (2.2), where $\{x_1(t), x_2(t), y_1(t), y_2(t)\}$ is a solution of (2.1). Then $x(t)$ satisfies (A.1). Let (t_0, x_0), $t_0 \in [0, t_f]$ be an arbitrary initial value for (A.1). Then (A.1) has a unique solution $x(t)$ on $[0, t_f]$, satisfying $x(t_0) = x_0$. Differentiating (2.2) on t, at $t = t_0$, we shall get (1.3) multiplied by x_0. This implies (1.3) since t_0 and x_0 are arbitrary. □

Proof of Lemma 2. Let (1.3) has a bounded on $[0, t_f]$ solution. Since Lemma 1 for any x_1^0, x_2^0 the Hamiltonian system (2.1) has a solution, represented in the form (2.2). Consider the system of (2.7), (2.8) with arbitrary terminal values u_1^0 and u_2^0. This system has a solution represented in the form of (2.14) iff the following algebraic system, that is obtained by substituting (2.6) into (2.2),

$$\begin{pmatrix} v_1 + \varepsilon G_3 u_2 + \varepsilon G_4 v_2 \\ H_3 u_1 + H_4 v_1 + E_3 u_2 + E_4 v_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & \varepsilon Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} u_1 + \varepsilon G_1 u_2 + \varepsilon G_2 v_2 \\ H_1 u_1 + H_2 v_1 + E_1 u_2 + E_2 v_2 \end{pmatrix} \tag{A.2}$$

is solvable with respect to v_1 and v_2.

The linear algebraic system (A.2) is solvable with respect to v_1, v_2 iff the equations (2.10),–(2.13) have bounded on $[0, t_f]$ solutions. The uniqueness off the solutions of (2.10)–(2.13) implies that the linear algebraic system (A.2) can possess only one solution. It means that the latter system has the unique solution (2.14) and N obtained is the bounded on $[0, t_f]$ solution of (2.10).

Conversely, let (2.10) and, hence, (2.11)–(2.13) have bounded on $[0, t_f]$ solutions. Then the terminal value problem of (2.1) has a solution related in the form of (2.2) iff the linear algebraic equation (2.15) is solvable with respect to components of Z or iff (1.3) has a bounded on $[0, t_f]$ solution. The uniqueness of the solution of (1.3) implies the existence and the uniqueness of solution of (2.15) and, therefore, the existence of the bounded on $[0, t_f]$ solution of (1.3). This completes the proof of (i) and (ii). □

Proof of Lemma 3. Let ARE of (1.3) has a solution Z, such that the matrix $A_\varepsilon - S_\varepsilon Z$ is Hurwitz. It means [2], that the set

$$X^- = \{(x_1, x_2, y_1, y_2) \mid (2.2) \text{ is valid}\} \tag{A.3}$$

is the stable eigenspace of the matrix Ham_γ of the Hamiltonian system (2.1). Moreover, Ham_γ has $n_1 + n_2$ stable and $n_1 + n_2$ unstable eigenvalues and such Z is unique. Applying to X^- the nonsingular transformation of (2.6), we get the stable eigenspace M^- of the matrix V of the system of (2.7). The latter stable manifold can be represented in the form

$$M^- = \{(u_1, v_1, u_2, v_2) \mid (2.14) \text{ is valid}\} \tag{A.4}$$

iff (A.2) is solvable with respect to v_1, v_2. Eigenvalues of the matrix V coincide with those of Ham_γ. Therefore the matrices N, N_{12}, N_{21}, N_{22} in (A.4) are uniquely defined. This implies the existence and the uniqueness of the solution (2.14) of (A.2) and, hence, the existence of M^- given as (A.4). The matrices N, N_{12}, N_{21}, N_{22} in (A.4) satisfy ARE of (2.10) and algebraic equations of (2.11) – (2.13), where $N_{ij} = 0$. The linear homogeneous algebraic equations (2.11) and (2.13) have the unique solutions $N_{i2} = 0, i = 1, 2$ due to the nonsingularity of Λ_0.Then the equation $v_1 = Nu_1$ defines the stable eigenspace of the matrix W, that has no eigenvalues on the imaginary axis, and Δ_1 is Hurwitz. The uniqueness of the solution of ARE (2.10) with the Hurwitz matrix Δ_1 follows from the uniqueness of the stable eigenspace of W. Note, that $N_{21} = 0$ since it is the solution of the linear homogeneous algebraic equation (2.12), the matrix of which is nonsingular.

Conversely, let there exist a unique N satisfying (2.10) and such that Δ_1 is Hurwitz. Then the system of (2.7) has the unique stable manifold given as (A.4) with the zero matrices N_{12}, N_{21} and N_{22}. By means of the inverse to (2.6) transformation this stable eigenspace of the matrix V is mapped to the eigenspace of Ham_γ. The latter manifold can be represented as (A.3) iff the linear algebraic equation (2.15) has a unique solution. Due to the uniqueness of the stable manifold of X^-, the linear algebraic equation (2.15) has a unique solution of the form (2.18). This implies existence and uniqueness of the function Z satisfying ARE of (1.3) and such that $A_\varepsilon - B_\varepsilon Z$ is Hurwitz. □

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