KYBERNETIKA — VOLUME 31 (1995), NUMBER 6, PAGES 530–540

PROPERTIES OF REACHABILITY AND ALMOST REACHABILITY SUBSPACES OF IMPLICIT SYSTEMS: THE EXTENSION PROBLEM

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A geometric characterisation of the reachability subspaces and almost reachability subspaces of implicit systems of the type $S(F,G) : F\dot{x} = Gx$ is given. Furthermore a classification of the almost reachability subspaces of such systems, based on the property that almost reachability spaces, or subspaces of such spaces can be extended to reachability spaces, is presented. In addition, necessary and sufficient conditions have been given for the above properties to hold true. The property of extension of a certain type of subspace to another type is integral part of the study of generalised dynamic cover problems of geometric theory.

1. INTRODUCTION

In this paper we are dealing with the concepts of reachability and almost reachability of implicit systems of the type $S(F,G) : F\dot{x} = G\boldsymbol{x}$ where $F, G \in R^{m\times n}$. The nature of properties examined, is based on the ability of a space of a certain type to be transformed by extension to another type of invariant space. Such a problem emerges in the study of generalised dynamic cover problems of geometric theory [11, 16], which are intimately connected with the problem of selection of inputs, outputs, on a given system and in particular with the problem of model projection [12]. The current approach is based on $S(F,G)$ type implicit descriptions. Clearly, such a system does not represent a dynamical system since the solutions which are due to some initial conditions, are not uniquely defined. In spite of this, the fact that a generalised system $S_e(E,A,B) : E\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ is strongly related to an autonomous system $S(NE, NA)$ where N is an annihilator of B, i.e. $NB = 0$, suggests that $S(F,G)$ defines a feedback free representation of a generalised system [15] and thus the family of solutions for a given initial condition corresponds to the set of all trajectories generated by the given initial condition and all possible control inputs or equivalently state feedbacks.

The notions of reachability and almost reachability concerning a dynamical system can be also introduced to the case of $S(F,G)$ systems. An attempt in this direction has been done in [9, 15] and this paper extends those introduced there. A summary of the background results is presented in Section 2, where in Section 3 a geometric characterisation of the concepts of the reachability, almost reachability, as well as their related spaces is given. In Section 4 we classify the almost reachability subspaces in two types by showing that there exists almost reachability subspaces that can be extended to reachability ones and others that cannot; the later family is called pure almost reachability spaces. The classification problem is linked to the following two problems: (i) given an almost reachability subspace, derive necessary and sufficient conditions which allow determining whether, or not the almost reachability space can be extended to a reachability space, (ii) given a pure reachability space derive necessary and sufficient conditions under which there exists a subspace of the pure reachability space that can be extended to a reachability space. Both the above two problems are linked to the property of an almost reachability space to be covered, or a subspace of it to be covered by a reachability space and they belong to the general family of cover problems considered in [16]; the tools however here are geometric rather than algebraic.

2. BACKGROUND RESULTS ON THE MATRIX PENCIL PROPERTIES OF REACHABILITY, ALMOST REACHABILITY SPACES

Consider a generalised autonomous system, or implicit form of the type

$$S(F,G) \quad F\,\dot{\boldsymbol{x}}(t) = G\,\boldsymbol{x}(t), \qquad F,\, G \in R^{m\times n}.$$

Clearly, such a system does not represent a dynamical system since the solution which is due to some initial condition is not always uniquely defined. However, it has been shown [13, 15] that a generalised system $S_e(E,A,B) : E\,\dot{\boldsymbol{x}} = A\,\boldsymbol{x} + B\,\boldsymbol{u}$ is strongly related to an autonomous system $S(NE,\ NA)$, where N is a left annihilator of B i.e., $NB = 0$ and thus $S(F,G)$ descriptions may simulate the feedback free representation of a generalised systems [15]. Inversely, $S(F,G)$ descriptions and related subspace problems are linked to the standard theory through the notion of invariant forced realisation [10], which allows the parameterisation of the family of solutions for a given initial condition in terms of control inputs, or feedbacks.

By examining $S(F,G)$ independently from the links to an $S_e(E,A,B)$ system we may define the concepts of reachability and almost reachability using the standard dynamic notions introduced for regular systems [23, 24] as follows:

Definition 2.1. [9, 15] A subspace $\mathcal{R} \subset R^n$ will be called a *reachability subspace* of $S(F,G)$, if for all $x_1,\, x_2 \in \mathcal{R}$ there exists a family of smooth solutions $\mathcal{M} \in \mathcal{R}$ of $S(F,G)$ such that for any $\boldsymbol{x}(t) \in \mathcal{M}$ with $\boldsymbol{x}(0) = \boldsymbol{x}_1\ \exists\ T < \infty$ with $\boldsymbol{x}(T) = \boldsymbol{x}_2$ for all $t \geq 0$.

Definition 2.2. [9, 15] A subspace $\mathcal{V} \subset R^n$ is an *almost reachability subspace* of $S(F,G)$, if for all $\boldsymbol{x}_0,\, \boldsymbol{x}_1 \in \mathcal{V}$, some time t_0, and $\varepsilon > 0$, there is a smooth solution $\boldsymbol{x}(t)$ of $S(F,G)$ such that $\boldsymbol{x}_\varepsilon(0) = \boldsymbol{x}_0,\ \boldsymbol{x}_\varepsilon(t_0) = \boldsymbol{x}_1$ and $d(\boldsymbol{x}_\varepsilon(t),\, \mathcal{V}) < \varepsilon$ for all $t > 0$.

Remark 2.1. [9] An alternative characterisation of an almost reachability space of the $S(F,G)$ system is given using distributional solutions [22, 23] as follows: A subspace $\mathcal{V} \in R^n$ is an almost reachability subspace of $S(F,G)$ if for all $\boldsymbol{x}_0$, $\boldsymbol{x}_1 \in \mathcal{V}$, there is a distributional trajectory $\boldsymbol{x}(t)$ of $S(F,G)$ that joins $\boldsymbol{x}_0$, $\boldsymbol{x}_1$ and remains in $\mathcal{V}$.

Definition 2.3. We define as the maximal reachability subspace of $S(F,G)$, the space $\mathcal{R}^*$ with the property that if $\mathcal{R}^*$ is a reachability space and if $\mathcal{R}$ is any reachability subspace of $S(F,G)$ then $\mathcal{R} \subset \mathcal{R}^*$.

Similarly, we may define:

Definition 2.4. We define as the maximal almost reachability subspace of $S(F,G)$ the space $\mathcal{V}^*$ with the property that $\mathcal{V}^*$ is an almost reachability space and, if $\mathcal{V}$ is any almost reachability subspace of $S(F,G)$, then $\mathcal{V} \subset \mathcal{V}^*$.

The system $S(F,G)$ is said to be reachable, if $\mathcal{R}^* = R^n$ and almost reachable, if $\mathcal{V}^* = R^n$. Some of the basic results previously derived for such subsystems are given below [9, 11, 13, 15].

Theorem 2.1. Consider $S(F,G)$ and let $X(s) = [\boldsymbol{x}_1(s), \ldots, \boldsymbol{x}_p(s)]$ be any minimal basis matrix of $\mathcal{N}_r\{sF - G\}$. Then, the following properties hold true: [11]

(i) If $\boldsymbol{x}(s) \in R^n[s]$, $\boldsymbol{x}(s) = \boldsymbol{x}_0 + s\boldsymbol{x}_1 + \cdots + s^k \boldsymbol{x}_k$ and we denote by $\mathcal{R}(\boldsymbol{x}(s)) \triangleq \mathrm{sp}\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} = \mathcal{R}_{\boldsymbol{x}}$, then $\mathcal{R}_{\boldsymbol{x}}$ is a reachability space, if and only if $\boldsymbol{x}(s) \in \mathcal{N}_r\{sF - G\}$.

(ii) The set of reachability spaces $\mathcal{R}_{\boldsymbol{x}_1}, \ldots, \mathcal{R}_{\boldsymbol{x}_p}$ are least dimension, $\dim \mathcal{R}_{\boldsymbol{x}_i} = \varepsilon_i + 1$ and form an independent set; ε_i, $i \in \underline{p}$ are the cmi of $sF - G$ and $\mathcal{R}^* = \mathcal{R}_{\boldsymbol{x}_i} \oplus \cdots \oplus \mathcal{R}_{\boldsymbol{x}_p}$ is the maximal reachability space of the system.

Before we proceed, we define some further notions needed for the subsequent developments.

Definition 2.5. [3, 11] i) Let $X(s) = [\boldsymbol{x}_1(s), \cdots, \boldsymbol{x}_p(s)]$ be a minimal basis for $\mathcal{N}_r\{sF - G\}$, where $\boldsymbol{x}_i(s) = \boldsymbol{x}_i^{d_i}\, s^{d_i} + \boldsymbol{x}_i^{d_i - 1}\, s^{d_i - 1} + \cdots + \boldsymbol{x}_i^0$, $\forall\, i \in \underline{p}$. Then, the space $\mathcal{P}^* = \mathrm{space}\{\boldsymbol{x}_1^{d_1}, \ldots, \boldsymbol{x}_p^{d_p}\}$ is called the *high coefficient space* of $X(s)$.

(ii) For the pair (F,G), a set of vectors $S(d; \boldsymbol{x}_1) = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d\}$ satisfying the conditions

$$F\boldsymbol{x}_1 = 0, \quad G\boldsymbol{x}_1 = F\boldsymbol{x}_2, \ldots, G\boldsymbol{x}_{k-1} = F\boldsymbol{x}_k, \quad \boldsymbol{x}_1 \neq 0$$

is said to define a *semicyclic chain* (sc) of length d with $\boldsymbol{x}_1$ generator, and $\mathcal{S}(d; \boldsymbol{x}_1) \triangleq \mathrm{sp}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d\}$ is the supporting space of the sc chain. A sc $S(d; \boldsymbol{x}_1)$ will be called *prime*, if $\dim \mathcal{S}(d; \boldsymbol{x}_1) = d$ and will be called *maximal*, if there is no prime $S(d'; \boldsymbol{x}_1)$ sc, $d' > d$ such that $S(d; \boldsymbol{x}_1)$ is a proper subset of $S'(d'; \boldsymbol{x}_1)$.

(iii) A set of sc chains $S(\mathcal{D},\mathcal{G}) = \{S(d_1;\boldsymbol{x}_1);\ldots;\{S(d_{\nu^i};\boldsymbol{x}_\nu)\}$, where $\mathcal{D} = \{d_1,\ldots$ $\ldots,d_\nu\}$ is the set of chain lengths and $\mathcal{G} = \{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_\nu\}$ is the set of generators, will be called *prime*, if the vectors is $S(\mathcal{D},\mathcal{G})$ are linearly independent, and will be called *complete* if each $S(d_i;\boldsymbol{x}_i)$ is maximal.

(iv) If $\mathcal{R}^*$ is the reachability space and $\mathcal{S}(\mathcal{D},\mathcal{G}) = \sum_{i=1}^{\nu} \mathcal{S}(d_i;\boldsymbol{x}_i)$ is the supporting space of $\mathcal{S}(\mathcal{D},\mathcal{G})$ then the set $\mathcal{S}(\mathcal{D},\mathcal{G})$ is called *pure* if $\mathcal{S}(\mathcal{D},\mathcal{G}) \cap \mathcal{R}^* = \{0\}$.

With these preliminary definitions we have the following result [4]

Theorem 2.2. For any (F,G) pair the following properties hold true:

(i) There is a pure set of sc chains $S(\mathcal{D},\mathcal{G})$, if and only if $sF - G$ has a set of infinite elementary divisors (ied).

(ii) If $\mathcal{D}_\infty(F,G) = \{\hat{s}^{d_i}, i \in \underline{\nu}\}$ is the set of ied of $sF - G$, there exists complete and pure maximal sets of sc $S_\infty(\mathcal{D}_\infty;\mathcal{G})$, where $\mathcal{D}_\infty = \{d_1,\ldots,d_\nu\}$ and a supporting space $S_\infty = \sum_{i=1}^{\nu} \mathcal{S}(\mathrm{d}_i;\boldsymbol{x}_i)$ with $\dim S_\infty = \sum_{i=1}^{\nu} \mathcal{S}_{d_i}$.

(iii) If $\mathcal{R}^*$, $\mathcal{V}^*$ are the maximal reachability, almost reachability spaces respectively of (F,G), there exists always a maximal complete and pure set of sc $S_\infty(\mathcal{D}_\infty;\mathcal{G})$ with supporting space $\mathcal{S}_\infty$ such that

$$\mathcal{V}^* = \mathcal{S}_\infty \oplus \mathcal{R}^*.$$

The chains $S_\infty(\mathcal{D}_\infty;\mathcal{G})$ established by the above result are referred to as *normal* sc and characterise the set of ied of the pencil. The generation of such chains (structure of the set of generators $\mathcal{G}$) is discussed in [3,4].

Definition 2.6. The system $S(F,G)$ is said to be almost reachable if $\mathcal{V}^* = R^n$ and reachable if $\mathcal{R}^* = R^n$.

In [9] is has been shown that:

Theorem 2.3. The system $S(F,G)$ is almost reachable if and only if the pencil $sF - G$ has as possible Kronecker invariants infinite elementary divisors, column minimal indices and zero row minimal indices.

Theorem 2.4. The system $S(F,G)$ is reachable if and only if the pencil $sF - G$ has as possible Kronecker invariants column minimal indices and zero row minimal indices.

The above results are established by combining the notions of invariant forced realisation [10], the standard linear system characterisation of subspaces [23,24], and the matrix pencil characterisation of them [8]. The matrix pencil characterisation of an almost reachable, reachable system also leads to the following characterisation of subspaces [13,15]:

Proposition 2.1. Let $\mathcal{V} \subset R^n$ and V be a basis matrix for $\mathcal{V}$. Then: $\mathcal{V}$ is said to be an almost reachability subspace for the $S(F,G)$ system if the pencil $sFV - GV$ has a possible Kronecker invariants infinite elementary divisors, column minimal indices and zero row minimal indices.

Proposition 2.2. Let $\mathcal{V} \subset R^n$ and V be a basis matrix for $\mathcal{V}$. Then: $\mathcal{V}$ is said to be a reachability subspace, if the pencil $sFV - GV$ has as possible Kronecker invariants column minimal indices and zero row minimal indices. This is equivalent to that $\mathcal{V}$ is spanned by the vector coefficients of a polynomial vector $\boldsymbol{x}(s)$ for which $(sF - G)\,\boldsymbol{x}(s) = 0$.

The above results on the matrix pencil characterisation of subspaces provide the means for deriving some new geometric characterisation of them. A complete treatment of the Toeplitz based geometry of minimal bases which is related to reachability properties is given in [11]. Some further characterisation of the reachability and almost reachability spaces of an $S(F,G)$ system are summarised next. We first define some important families of vector space sequences for the pair (F,G) [13,18].

Definition 2.7. Let $F, G \in R^{m\times n}$. We may define the following sequences of subspaces of R^n:

$$\mathcal{Q}(F,G) \triangleq \{\mathcal{K}_0 = \{0\},\ \mathcal{K}_{k+1} = F^{-1}(G\mathcal{K}_k),\ k \geq 0\} \tag{2.1}$$

$$\mathcal{P}(G,F) \triangleq \{\mathcal{T}_0 = R^n,\ \mathcal{T}_{k+1} = G^{-1}(F\mathcal{T}_k),\ k \geq 0\} \tag{2.2}$$

Note that the above sequences have been studied in [1,2,13,17] and they are the pencil forms of the sequences characterising standard geometric properties [23,24]. Using the above sequences we have:

Theorem 2.5. (i) The maximal almost reachability space $\mathcal{V}^*$ in $S(F,G)$ is the limit of the sequence $\{\mathcal{K}_k\}_{k\in N}$ defined by (2.1), that is $\mathcal{V}^* = \mathcal{K}^*$.

(ii) For the maximal reachability space of the S(F,G) system $\mathcal{R}^*$, we have that

$$\mathcal{R}^* = \mathcal{T}^* \cap \mathcal{V}^*,$$

where $\mathcal{V}^* = \mathcal{K}^*$ is the limit of the $\{\mathcal{K}_k\}_{k\in N}$ sequence defined as in (2.1) and $\mathcal{T}^*$ is the limit of the sequence $\{\mathcal{T}_k\}_{k\in N}$ defined as by (2.2).

Clearly, the above characterisations of the maximal almost reachable maximal reachable subspace of an $S(F,G)$ system are very similar to those in [20] concerning those of the corresponding spaces of $S_e(E,A,B)$ system.

Theorem 2.6. A subspace $\mathcal{V} \subset R^n$ is an almost reachability subspace for the $S(F,G)$ system if and only if

$$\mathcal{V} = \lim \mathcal{V}_k \quad \text{where} \quad \mathcal{V}_k = \mathcal{V} \cap F^{-1} G \mathcal{V}_k, \quad \mathcal{V}_0 = \{0\}. \tag{2.3}$$

Proof. Let V be a basis matrix for $\mathcal{V}$. Then, the proof follows by the fact that the limit space of the (2.3) sequence is the sum of the spaces spanned by the vector coefficients of the polynomial belonging to a minimal basis for $sFV - GV$ and of the chains characterising the set of the i. e. d. of this pencil (see [14, 18]). □

Theorem 2.7. A subspace $\mathcal{R} \subset R^n$ is a reachability subspace for the $S(F,G)$ system if and only if

$$\mathcal{R} = \lim R_k \cap \lim \tilde{\mathcal{R}}_k$$

where

$$\mathcal{R}_k = \mathcal{R} \cap F^{-1} G \mathcal{R}_{k-1}, \quad \mathcal{R}_0 = \{R\}, \quad \text{and} \quad \tilde{\mathcal{R}}_k = \lim \mathcal{R} \cap G^{-1} F \tilde{\mathcal{R}}_{k-1}, \quad \tilde{\mathcal{R}}_0 = \mathcal{R}.$$

Proof. Let R be any basis matrix for $\mathcal{R}$. Then, the intersection of the limit spaces of the above sequences can be expressed as a sum of the subspaces formed by the vector coefficients of the polynomials belonging to a minimal basis of the right null space of $sFR - G\mathcal{R}$ pencil (for a proof of the above fact see [6, 18]). This fact combined with the characterisation of a reachability space, as in Proposition 2.2 establishes the result. □

3. CLASSIFICATION OF THE ALMOST REACHABILITY SUBSPACES OF THE $S(F,G)$ SYSTEM, BASED ON THE EXTENSION

Some further properties of the almost reachability spaces related to their classification are considered next. The present classification is based on the property of whether the space can be extended to a reachability space, or not, which is equivalent to a cover type of classification. In fact, it has been shown [12] that one of the fundamental family of problems involved in the selection of systems of inputs, outputs on a system are those referred to as Model Projection Problems (MPP); such problems involve the reduction of the potential sets of inputs, outputs to smaller sets, referred to as effective sets. The study of MPPs is equivalent to the problem of covering a given invariant space with another one of certain invariant type; an integral part of this study is the extension of a certain invariants subspace type of another type, which is considered here. We first define:

Definition 3.1. A subspace $\mathcal{V} \subset R^n$ is said to be a *single generated almost reachability space* (s.g.a.r.s.) for $S(F,G)$ system if

i) $\mathcal{V} = \lim \mathcal{V}_k$, $\mathcal{V}_k = \mathcal{V} \cap F^{-1} G \mathcal{V}_{k-1}$, $\mathcal{V}_0 = \{0\}$,

ii) $\dim(\mathcal{V} \cap \mathcal{N}_r F) = 1$.

It is rather straightforward to show that:

Theorem 3.1. A subspace $\mathcal{V}$ of dimension k is a s.g.a.r.s. for the $S(F,G)$ system if and only if there exists vectors $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\}$ such that they form a basis for $\mathcal{V}$ and satisfy the following conditions

$$F\boldsymbol{x}_1 = \mathbf{0}, \quad G\boldsymbol{x}_1 = F\boldsymbol{x}_2, \ldots, F\boldsymbol{x}_{k-1} = G\boldsymbol{x}_k \tag{3.1}$$

and

$$\mathrm{sp}\{\boldsymbol{x}_1\} = \mathcal{V} \cap \mathcal{N}_r\{F\}.$$

Definition 3.2. We say that an almost reachability subspace $\mathcal{V} \subset R^n$ can be extended to a reachability one, if there exists a reachability subspace $\mathcal{K}$, $\mathcal{K} \subset R^n$ such that $\mathcal{V} \subseteq \mathcal{K}$.

The next theorem describes the necessary and sufficient conditions which must be satisfied by an almost reachability subspace in order that it can be extended to a reachability one.

Theorem 3.2. An almost reachability subspace $\mathcal{V} \subset R^n$ can be extended to a reachability space, if and only if $\mathcal{V} \subset \mathcal{R}^*$.

Proof. Let $\mathcal{V} \subset R^n$ be such that it can be extend to a reachability subspace. Then, there exists a subspace $\mathcal{W} \subset R^n$ such that $\mathcal{V}+\mathcal{W}$ is a reachability space. Since $\mathcal{V}+\mathcal{W} \subset \mathcal{R}^*$ the proof of the only if part is obvious. Assume now that $\mathcal{V} \subset \mathcal{R}^*$ and let $\{\boldsymbol{x}_i,\ i = 1, \ldots, \mu\}$ be a basis matrix for $\mathcal{V}$. Then for $\forall\, \boldsymbol{x}_i$ there exists a reachability subspace say $\mathcal{R}_i$ containing $\boldsymbol{x}_i$ (for the proof of this fact see [14]). Consider now the space $\sum_{i=1}^{\mu} \mathcal{R}_i$; then, by taking into account the statement of Theorem 2.7 it is not hard to show that this is a reachability subspace; furthermore, $\mathcal{V} \subset \sum_{i=1}^{\mu} \mathcal{R}_i$ and this completes the proof. □

It is well known [7] that the i.e.d. of a right singular pencil $sF-G$ (i.e. $\mathcal{N}_r\{(sF-G)\} \neq \{0\}$) are related to chains of linearly independent vectors of (3.1) type with the addition property that $\mathrm{span}\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k\} \cap \mathcal{R}^* = \{0\}$; furthermore by Theorem 2.7 it is not hard to see that the space spanned by the vectors belonging to such a chain is an almost reachability space. Then, Theorem 3.2 ensures that such a space cannot be extended to a reachability space. Thus, it is convenient to define:

Definition 3.3. An almost reachability space will be called *pure*, if it cannot be extended to a reachability subspace.

The following theorem describes a sufficient condition under which a single generated almost reachability subspace can be characterized as pure.

Theorem 3.3. If a s.g.a.r.s. for the $S(F,G)$ system is generated by a vector not belonging to the high coefficient space of $\mathcal{N}_r\{sF-G\}$, then it is pure.

Proof. Let $\mathcal{W} = \mathrm{span}(\boldsymbol{y}_1, \dots, \boldsymbol{y}_k)$ be a single generated almost reachability space for which $\boldsymbol{y}_1 \notin \mathcal{P}^*$. Then, we claim that $\mathcal{W} \cap \mathcal{R}^* = \{0\}$. To prove this let $\boldsymbol{z}_1 \in \mathcal{R}^* \cap \mathcal{W}$. Then, $\boldsymbol{z}_1$ may be expressed as

$$\boldsymbol{z}_1 = \boldsymbol{a}_1 \boldsymbol{y}_1 + \dots + \boldsymbol{a}_k \boldsymbol{y}_k. \tag{3.2}$$

Consider the vectors $\boldsymbol{z}_\nu$, $\nu = 1, \dots, k$ defined by

$$\boldsymbol{z}_\nu = \sum_{j=\nu}^{k} a_j \boldsymbol{y}_{j-\nu+1}, \quad \nu = 1, \dots, k. \tag{3.3}$$

Note that they satisfy the following conditions

$$F \boldsymbol{z}_1 = G \boldsymbol{z}_2, \dots, F \boldsymbol{z}_{k-1} = G \boldsymbol{z}_k, \quad F \boldsymbol{z}_k = \mathbf{0}. \tag{3.4}$$

□

Since for $\boldsymbol{z}_1$ it has been assumed that $\boldsymbol{z}_1 \in \mathcal{R}^*$ we have that there exists polynomial vector $\boldsymbol{x}(s)$ of $\mathcal{N}_r\{sF - G\}$ having $\boldsymbol{z}_1$ as one of its nonzero vector coefficients (for the proof of this fact see [11]).

Let $\boldsymbol{x}(s) = \boldsymbol{x}_\nu s^\nu + \dots + \boldsymbol{x}_{\mu+1} s^{\mu+1} + \boldsymbol{z}_1 s^\mu + \boldsymbol{x}_{\mu-1} s^{\mu-1} + \dots + \boldsymbol{x}_1 s + \boldsymbol{x}_0$ be such a vector. Then, by the way that the vector coefficients of $\boldsymbol{x}(s)$ are related and by the fact that the $\boldsymbol{y}_1, \dots, \boldsymbol{y}_k$ satisfy relations of (3.1) type it follows that

$$G \boldsymbol{x}_0 = \mathbf{0},\ F \boldsymbol{x}_0 = G \boldsymbol{x}_1 \cdots F \boldsymbol{x}_\mu = G \boldsymbol{z}_1,\ F \boldsymbol{z}_1 = G \boldsymbol{z}_2 \cdots F \boldsymbol{z}_{k-1} = G \boldsymbol{z}_k,\ F \boldsymbol{z}_k = \mathbf{0}. \tag{3.5}$$

The above according to the properties of the high coefficient space $\mathcal{P}^*$ see [9, 14] implies $\boldsymbol{z}_k = a_k \boldsymbol{y}_1 \in \mathcal{P}^*$; however, it is given that $\boldsymbol{y}_1 \notin \mathcal{P}^*$ and therefore we conclude $a_k = 0$. By substituting $a_k = 0$ in the expression for $\boldsymbol{z}_1, \dots, \boldsymbol{z}_{k-1}$ given by (3.3), (3.4) yields once more that $a_{k-1} = 0$. By proceeding along the same lines we get $a_2 = \dots = a_k = 0$. Then, (3.2) show that $\mathcal{R}^* \cap \mathcal{W} = \mathrm{span}\{\boldsymbol{y}_1\} \cap \mathcal{P}^*$; however by our hypothesis span $\{\boldsymbol{y}_1\} \cap \mathcal{P}^* = \{0\}$ and therefore, $\mathcal{R}^* \cap \mathcal{W} = \{0\}$. □

The generalisation of Theorem 3.3 is given below:

Theorem 3.4. Let $\mathcal{V}$ be an almost reachability space for which $\mathcal{V} \cap \mathcal{N}_r F \cap \mathcal{P}^* = \{0\}$. Then $\mathcal{V}$ is a pure reachability space.

The proof of the above result follows by generalising the previous arguments. Some further properties related to the cover of subspaces of pure almost s.g.a.r.s. are examined below.

Proposition 3.1. Let $\mathcal{V}$ be a pure s.g.a.r.s. for the $S(F, G)$ system. Then, there exists a subspace $\mathcal{W}$ of $\mathcal{V}$ with the property that $\mathcal{W}$ can be extended to a reachability subspace, if and only if $\mathcal{V}$ has as a generator a vector belonging to $\mathcal{P}^*$.

Proof. Consider $\mathcal{V} = \mathrm{span}\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k\}$ such that $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ are linearly independent and satisfy conditions of the (3.1) type. Clearly, if $\boldsymbol{x}_1 \in \mathcal{P}^*$ then we have that $\mathcal{V} \cap \mathcal{R}^* \neq \{0\}$. Let $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_\lambda\}$ be a basis for the $\mathcal{R}^* \cap \mathcal{V}$ space. Since for $\forall\, i = 1, \ldots, \lambda$ there exists a reachability space containing $\boldsymbol{x}_i$ as it has been discussed in the proof of Theorem 3.1. The proof of the sufficiency is rather obvious.

To prove the necessity of the results let $\mathcal{V} = \mathcal{W} \oplus \tilde{\mathcal{W}}$ be such that there exists $\mathcal{K}$ with $\mathcal{W} \subset \mathcal{K}$ and $\mathcal{K}$ being a reachability subspace. Furthermore, assume that $\mathcal{K}$ is of minimal dimension. Then, for $\mathcal{K}$ we will have that $\dim(\mathcal{K} \cap \mathcal{N}_r\, F) = 1$ (see [3, 14]); by the fact that the vectors $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$ satisfy condition of (3.1) type we have that $(\mathcal{V} \cap \mathcal{N}_r\, F) = (W \cap \mathcal{N}_r\, F) = \mathrm{span}\{\boldsymbol{x}_1\}$ and thus $\mathcal{K} \cap \mathcal{N}_r\, F = \mathrm{span}\{\boldsymbol{x}_1\}$. Since $\mathcal{K}$ is a reachability space then $\boldsymbol{x}_1 \in \mathcal{P}^*$ (see [14]) and this completes the proof. □

By generalising the above arguments we have:

Theorem 3.5. Let $\mathcal{V}$ be a pure almost reachability space. Then there exists a subspace $\mathcal{W}$ of $\mathcal{V}$ with the property that $\mathcal{W}$ can be extended to a reachability subspace, if and only if $\mathcal{V} \cap \mathcal{N}_r\, F \cap \mathcal{P}^* \neq \{0\}$.

The results here provide a geometric framework for the study of generalised dynamic cover problems, in the special case where the space to be covered is an almost reachability space; the approach allows the construction of minimal covers for a large family of spaces to be covered by reducing the overall problem to a problem of extending semicyclic chains, bases.

4. CONCLUSIONS

We have shown that the reachability and almost reachability subspaces of a implicit system description $S(F, G)$ can be characterised in geometric terms in a way very similar to that of the corresponding spaces of a generalised $S_e(E, A, B)$ system, as it has been given in [20]. We have proved that there exists almost reachability subspaces that can be extended to reachability ones and others that cannot. Furthermore, we described the condition which allow us to determine whether or not a given almost reachability subspace can be extended to a reachability one. In addition, we have given the necessary and sufficient conditions under which there exists a subspace of a pure reachability space that can be extended to a reachability space.

The classification of the almost reachability spaces according to their property to be extended, or not to a reachability space, is intimately related to the study of minimal covers for subspaces of the almost reachability type. Such problems are important in the study of many geometric theory problems and are central in the study of selection of effective sets of inputs, outputs from larger sets associated with progenitor type models [12].

(Received February 13, 1995.)

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