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ON ONE NP-COMPLETE PROBLEM

JIŘÍ DEMEL AND MARIE DEMLOVÁ

Let S be a finite set, and R be a set of three element subsets of S. An element r of R is interpreted as a production rule which enables to derive one of the elements of r from the others. A subset $X \subset S$ is conflicting if an element of S can be derived from X in two different ways. The problem of finding a largest non-conflicting subset is shown to be NP-complete.

Let S be a finite set; its elements will be called *constants.* Let R be a set of three element subsets of S. We interpret an element $r = \{a, b, c\} \in R$ as a *production rule*, which enables us to derive a value of any constant in r from the values of the remaining two constants.

Informally, we say that a subset of constants $X \subseteq S$ is conflicting if there is a constant which can be derived from X in two different ways. The problem treated here is to find, for a given set R of production rules, the largest non-conflicting set of constants. We show that this problem is NP-complete.

Let us point out that the problem is motivated by the study of models and useful constrains for qualitative physics. This is a new field of AI searching for an appropriate formalism supporting common sense reasoning, see [2] for a brief survey of this topic. The variables in the qualitative methodology are supposed to have only a fixed set of discrete values; mutual relations among variables are expressed by a limited set of dependencies (or constraints). The simplest constrains can be defined by the production rules mentioned above. The problem of existence of a non-conflicting set of a given size arises when trying to define a partially specified model for a given set of production rules, i.e. to find an evaluation of the set of variables corresponding to constraints given by production rules and the partial specification. The evaluation of a variable is called here a constant.

First, let us give some formal definitions. Let S be a non-empty finite set of constants, R be a set of production rules and X a non-empty subset of S. A *derivation D from X* is a finite sequence of ordered triples $\{(a_i, b_i, c_i)\}_{i=1}^{k}$ such that:

1. Members of each triple a_i, b_i, c_i form a production rule, i. e. $\{a_i, b_i, c_i\} \in R$. The third element, c_i, we consider to be derived from a_i, b_i.

2. Each of the first two members of any triple is either in X or has been derived earlier, i. e. $a_i, b_i \in X \cup \{c_j | 1 \leq j < i\}$.

The integer k is called the *length* of the derivation. An element $y \in S$ is *derived from* X by the derivation D if $y = c_i$ for some i. We say that all elements of X are derived from X by the empty derivation.

A *minimal derivation of an element* $y \in S$ *from* X is a derivation which derives y and it has no proper non-empty subderivation which derives y from X (i.e. we cannot omit any triples to get a smaller non-empty derivation of y from X). Every empty derivation is considered to be also a minimal one. Note that for every non-empty minimal derivation of y of the length k we have $y = c_k$, $k \leq |S|$ and $y \notin \{a_i, b_i, c_i | i < k\}$.

Two derivations are called *equivalent* if their sets of production rules are equal.

A set of constants X is called *conflicting* with respect to the set of rules R if there is an element of S which is derived by two non-equivalent minimal derivations from X.

Proposition 1. If there is an element $y \in X$ which is derived by a non-empty derivation from X then X is conflicting.

Proof. The proof is trivial; non-empty derivation of y contains a non-empty minimal one. The second minimal derivation is the empty one. □

Corollary 1. If there is a production rule $\{a, b, c\} \in R$ such that $\{a, b, c\} \subseteq X$ then X is conflicting with respect to R.

For a given set of constants $X \subset S$ and a set of production rules R the following simple polynomial algorithm decides whether X is conflicting with respect to R.

Algorithm 1.

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{Input: sets S, R and X as described above.}
{Auxiliary variables:}
{Z is the set of constants that has been derived so far.}
{D is a derivation which derives all elements of Z.}
{finished is a boolean variable indicating end of computation.}
{conflict is a boolean variable indicating discovery of a conflict.}
begin
  D := ∅; Z := X;
  finished := false; conflict := false;
  while not finished do
    begin
      finished := true;
      for all r ∈ R do
        if |r ∩ Z| = 3 and r is not in D then conflict := true;
        else if |r ∩ Z| = 2 then
          begin
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            denote a, b, c the elements of r such that {c} = r \ Z;
            append ordered triple (a, b, c) to D;
            Z := Z ∪ {c};
            finished := false;
          end;
    end;
  if conflict then write ("conflicting")
  else write ("non-conflicting");
end.
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Theorem 1. For a given set of constants $X \subseteq S$ and a set of production rules R the Algorithm 1 decides in polynomial time whether X is conflicting with respect to R.

Proof. The time bound follows from the fact that the while-loop is repeated at most $(|S \setminus X| + 1)$-times.

Let us prove the correctness.

a) Assume that the algorithm answered "conflicting". Let $r = \{a, b, c\}$ be the rule for which the variable *conflict* changed its value from false to true, i. e. $|r \cap Z| = 3$, so $r \subseteq Z$.

If $r \subseteq X$ then X is conflicting by Corollary 1.

Let $r \nsubseteq X$. Then, without loss of generality we can assume that $c \in Z \setminus X$ and each of a, b either belongs to X or was derived by D earlier than c. Denote $t = (p, q, c)$ the triple of D which derives c. Since $r \notin D$ it must hold $\{p, q\} \neq \{a, b\}$. Denote by D_1 the minimal non-empty derivation of c obtained from D by omitting some triples. Note that t is the last triple in D_1. The second minimal derivation D_2 of c we obtain from D by replacing t by (a, b, c) and then omitting unnecessary triples. Derivations D_1, D_2 are non-equivalent, hence X is conflicting.

b) Now, assume that the algorithm answered "non-conflicting". Then all constants which have a derivation from X are derived by D and all rules which can be used in any derivation from X are used in D. Let us prove that X is not conflicting in this case.

Assume for contrary that X is conflicting. Then there is a constant y with two non-equivalent minimal derivations D_1, D_2 from X. Without loss of generality we can assume that the sum of lengths of D_1, D_2 is minimal. Denote by B the set of all rules used in at least one of D_1, D_2.

Each constant which is contained in a rule from B is either in X or it is contained in at least two different rules of B. Indeed, for y it follows from the minimality of the sum of lengths: the last rules of D_1 and D_2 must be different. For other constants it follows from the minimality of derivations D_1, D_2: a constant $x \notin X$, $x \neq y$ is derived by a rule from B and (since $x \neq y$ and D_1, D_2 are minimal) is used by at least one other rule from B.

Let (a, b, c) be the last triple in D which is a use of a rule from B. Each of the constants a, b, c either is in X or it appeared in some earlier triple of D. So, the algorithm instead of appending (a, b, c) to D had to discover a conflict, a contradiction. □

Problem 1. Given a set of constants S, a set of rules R and an integer K. Decide whether there exists a non-conflicting set $X \subseteq S$ with respect to R with $|X| \geq K$.

Theorem 2. The Problem 1 is NP-complete.

Proof. First, the problem belongs to the class NP: One can non-deterministically guess a set X with at least K elements and use the above algorithm to verify (in a polynomial time) that X is non-conflicting with respect to R.

To prove that Problem 1 is NP-complete we show that the following well-known NP-complete problem [1] can be polynomially reduced to Problem 1.

> The Independent Set Problem: For a given undirected graph G and a given integer K, does there exist an independent set X of vertices with $|X| \geq K$. (A set of vertices is independent if it contains no two adjacent vertices.)

Let us have an undirected graph G and an integer K, we shall construct an instance of the Problem 1.

First, the Independent Set Problem can be easily reduced to a slightly restricted version in which the graph has no isolated vertices and $K \geq 3$. (Each isolated vertex can be replaced by a pair of adjacent vertices.)

Hence, let $G = (V, E)$, where V is the set of vertices, E is the set of undirected edges. Take three new elements $p, q, r \notin V$ and define a set of constants S and a set of production rules R as follows:

$$\begin{aligned} S &= V \cup \{p, q, r\} \\ R &= \{\{v, w, t\} | \{v, w\} \in E \text{ and } t \in \{p, q, r\}\} \\ &\quad \cup \{\{p, q, r\}\} \end{aligned}$$

To prove the Theorem it suffices to show that for every $X \subseteq S$ with at least three elements we have

(*) X is a non-conflicting set with respect to R if and only if $X \subseteq V$ and X is independent in G.

One implication is clear; any independent set $X \subseteq V$ in G with $|X| \geq 3$ is non-conflicting with respect to R since nothing can be derived from X.

Let us prove the other implication. Let $X \subseteq S$ be a non-conflicting set with respect to R and let $|X| \geq 3$.

a) First, we shall show that $X \subseteq V$. Since X is non-conflicting and $\{p, q, r\} \in R$, we get that $\{p, q, r\} \not\subseteq X$ (see Corollary 1). Since $|X| \geq 3$ we have that X contains at least one element v of V. Note that v is adjacent to at least one other vertex $w \in V$. Now, assume for contradiction, that $X \cap \{p, q, r\}$ is non-empty. Without loss of generality we assume that $p \in X$. If $w \in X$ then $\{v, w, p\} \subseteq X$, a contradiction (see Corollary 1). If $w \notin X$ consider the following two derivations from X:

(1) $(v, p, w), (v, w, q), (p, q, r)$

(2) $(v, p, w), (v, w, r)$

They are clearly minimal and non-equivalent. Thus X is conflicting, a contradiction. Therefore $X \cap \{p, q, r\}$ is empty and $X \subseteq V$.

b) It remains to prove that X is an independent set of vertices in G. Assume that there exist $v, w \in X$ with $\{v, w\} \in E$. Then the following two minimal derivations of r from X are non-equivalent:

(3) $(v, w, p), (v, w, q), (p, q, r)$

(4) (v, w, r)

Thus again X is conflicting, a contradiction.

Hence, we have proved $(*)$ which concludes the proof of the Theorem. □

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RNDr. Jiří Demel, CSc., katedra inženýrské informatiky, Stavební fakulta ČVUT (Department of Informatics, Faculty of Civil Engineering, Czech Technical University), Thákurova 7, 166 29 Praha 6. Czech Republic.

Doc. RNDr. Marie Demlová, CSc., katedra matematiky, Elektrotechnická fakulta ČVUT (Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University), Technická 2, 166 27 Praha 6. Czech Republic.