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OPTIMAL RECONSTRUCTION OF STATE VECTOR IN 2-D SYSTEMS

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In the paper a 2-D Kalman filter algorithm is derived as a result of effective transport of 1-D optimal state estimation theory to the discrete two-dimensional setting and an algorithm which could be directly applied for image processing purposes is obtained.

1. INTRODUCTION

Searching for new effective methods in image processing, the attention has been directed to recursive estimators, leading naturally to the subject of 2-D Kalman filter. Kalman filter is a very well known and useful algorithm in one-dimensional digital signal processing, so there arises an interesting and important goal of generalization of this algorithm into two-dimensions taking advantage of the recently introduced 2-D systems models. The main questions posed while extending the standard 1-D recursive filtering techniques to the 2-D case are:

- how to establish a suitable 2-D recursive model by defining a proper state vector?
- how to reduce the dimensionality of the resulting state vectors by reasonable approximation?
- how to speed up the obtained Kalman filtering procedure by processing signals in parallel?

The early papers devoted to image modelling by state-space techniques were given by Nasi and Assefi [13], Habibi [7], and by Aboutalib, Murphy and Silverman [1]. In the following years there were several attempts to generalize a Kalman filter algorithm to two dimensions. The first one, so called Reduced Update Kalman Filter (RUKF) was due to Woods and Radewan [18] and was continued in several subsequent papers [17], [9], [5]. This approach has been based on autoregressive (AR) two-dimensional models with general or nonsymmetric half-plane (NSHP) coefficient support. Other approach to Kalman filtering in two dimensions with applications to image processing has been proposed by Chen [6] and this was rather straightforward application of 1-D Kalman filter to a line-by-line scanned image. Next important contribution to the subject has been by Porter and Aravenna [14]. They considered a Roesser

Model converted to wave advance process model. The solution has been restricted to a local state realization. This idea has been utilized by Marszałek [12] to present a state estimation algorithm for 2-D systems based on well-known Bayes theorem. Recently Angwin and Kauffman [2] proposed the Reduced Order Model Kalman Filter (ROMKF) which is based on a lower order state space model of an image and is similar to the approach of Woods and Radewan. The low dimension of this system results in decreased computation time. Several other authors have proposed different 2-D Kalman filtering schema for restoration of images degraded by both blur and noise. Suresh and Shenoi [16] proposed the Kalman strip filtering with modelling the blur by 2-D state-space structure. Wu [19] employed three-dimensional state space models to develop another strip filtering model for the degraded image with NSHP support. Later on, Azimi-Sadjadi and Wong [3], [4] presented the two-dimensional block Kalman filtering scheme. The 2-D block state space model considered in this approach takes into account the correlations of the image data in successive neighbouring blocks and reduces the edge effects. However, the optimal Kalman filters for strip observations as well as for the block observations are characterized by complexities and large computational requirements. Most recent results in the field are by Zhang and Steenart [20]. They presented a simple 2-D Kalman filter for two 2-D state space structures cascaded to form a composite state space dynamic model. At last, quite different approach to the problem has been proposed by Šebek [15]. He formulated and solved the Kalman filtering problem via 2-D polynomial methods. In this paper we introduce a new Kalman filter algorithm for a general model of 2-D systems given by Kurek [11]. The reduction technique due to Klamka [10] and Kaczorek [8] is essential in this approach and resulted in an interesting recursive estimator's algorithm.

2. PROBLEM STATEMENT

Let us consider the linear, discrete 2-D system general state space model in the form [11]

$$\begin{aligned} x(i+1,j+1) &= A_0x(i,j) + A_1x(i+1,j) + A_2x(i,j+1) + B_0u(i,j) \\ &\quad + B_1u(i+1,j) + B_2u(i,j+1) + w(i,j) + w(i+1,j) + w(i,j+1) \\ y(i,j) &= Cx(i,j) + v(i,j), \end{aligned} \tag{1}$$

where:

i, j	integer valued horizontal and vertical coordinates respectively and $0 \leq i \leq N$, $0 \leq j \leq M$,
$x(i,j) \in \mathcal{R}^n$	local state vector at the point (i,j),
$u(i,j) \in \mathcal{R}^m$	input vector,
$y(i,j) \in \mathcal{R}^p$	output vector,
$w(i,j) \in \mathcal{R}^n$	"white" Gaussian noise with covariance matrix W (state disturbance),
$v(i,j) \in \mathcal{R}^p$	"white" Gaussian noise with covariance matrix V (output or observation disturbance),

and A_k, B_k for $k = 0, 1, 2$ and C are real valued matrices of appropriate dimensions.

At this stage of the considerations let us assume that state and observation noises are uncorrelated "white" noises, i. e.:

$$\begin{aligned} E[w(k,l),w(m,n)] &= W\delta_{klmn} \\ E[v(k,l),v(m,n)] &= V\delta_{klmn}, \end{aligned}$$

where

$$\delta_{klmn} = \begin{cases} 0, & \text{for } k \neq m \text{ or } l \neq n; \\ 1, & \text{for } k = m \text{ and } l = n, \end{cases}$$

for any k, l, m and n.

Boundary conditions for (1) are given by

$$x(i,0) = x_{i0} \qquad \text{for} \quad i = 0,1,\dots,N$$

and

$$x(0,j) = x_{0j} \qquad \text{for} \quad j = 0,1,\dots,M, \tag{2}$$

where: x_{i0} and x_{0j} are known vectors.

The problem to be solved may be stated as follows:

Given the values of observed signal $y(i,j)$ and system input $u(i,j)$ over a certain subset of the points forming the neighbourhood (defined differently in different approaches and applications) of given point (k,l), our task is to find a linear estimate $x(i,j)$ of the system's state vector in the point (k,l) so as to minimize the expected value of the square of estimation error i. e.

$$J = E\big(x(i,j) - \hat{x}(i,j)\big)^2.$$

3. SOLUTION TO THE KALMAN FILTER PROBLEM FOR GENERAL 2-D MODEL

For the sake of simplicity and taking into account that the model (1) represents an image, we can set $B_0 = 0$, $B_1 = 0$ and $B_2 = 0$. For the considerations to follow we assume the region of the solution of equations (1) in the form $\mathcal{Z} \times \mathcal{Z} = [0,N] \times [0,N]$ and the boundary conditions for (1) to be deterministic and given on all sides by:

$$x(i,0) = 0 \quad \text{and} \quad x(i,N) = 0 \quad \text{for} \quad i = 0,1,\dots,N$$

and

$$x(0,j) = 0 \quad \text{and} \quad x(N,j) = 0 \quad \text{for} \quad j = 0,1,\dots,M.$$

If these boundary conditions were nonzero the assumptions of $B_0 = B_1 = B_2 = 0$ would hold no more and by appropriate choice of these matrices the deterministic inputs $u(i,j)$ would account for these non-zero boundary values. For the sake of later consideration we assume that the whole image model in the area $\mathcal{Z} \times \mathcal{Z}$ is composed of two submodels: one corresponding exactly to the equations (1) for $i + j \leq N$ and the second of the form as follows

$$x(N-i-1,\, N-j-1) \;=\; \overline{A}_0 x(N-i, N-j) + \overline{A}_1 x(N-i-1, N-j)$$

$$
\begin{aligned}
&+\overline{A}_2 x(N-i, N-j-1) + \overline{B}_0 u(N-i, N-j) \\
&+\overline{B}_1 u(N-i-1, N-j) + \overline{B}_2 u(N-i, N-j-1) \\
&+w(N-i, N-j) + w(N-i-1, N-j) \\
&+w(N-i, N-j-1)
\end{aligned}
$$

$$
y(N-i,\, N-j) \quad = \quad \overline{C}x(i,j) + v(i,j) \tag{3}
$$

for $i+j > N$ with the same assumptions concerning the matrices $\overline{B}_0$, $\overline{B}_1$ and $\overline{B}_2$. This way we obtain 2-D system representation in two subareas of the region $\mathcal{Z} \times \mathcal{Z}$: lower left triangle described by equation (1) for $i+j \leq N$ and upper right triangle described by the equation (3) for $i+j > N$. First submodel is causal (quarter plane causal) with respect to the variables $i > 0$ and $j > 0$. Second one can be assumed also to be causal but with respect to the variables $\tilde{i} = N - i$, $\tilde{j} = N - j$.

It has been shown by Kaczorek [8] following the idea of Klamka [10] that it is possible to rewrite the equations (1) in the form of reduced 1-D variable structure and variable dimensionality of state vector system in the following way:

$$
\begin{aligned}
\tilde{x}(k+2) \quad = \quad & F_{1,k+1}\tilde{x}(k+1) + F_{2,k}\tilde{x}(k) + G_{1,k+1}\tilde{u}(k+1) \\
& +G_{2,k}\tilde{u}(k) + H_{1,k+1}\tilde{x}(0,k+1) + H_{2,k}\tilde{x}(0,k) \\
& +T_{2,k}\tilde{w}(k) + T_{1,k+1}\tilde{w}(k+1) \\
y(k) \quad = \quad & C_k\tilde{x}(k) + \tilde{v}(k) + P_k\tilde{x}(0,k),
\end{aligned} \tag{4}
$$

where: k – nonnegative integer values

$$
\tilde{x}(k) = \begin{bmatrix} x(k-1,1) \\ x(k-2,2) \\ \vdots \\ x(2,k-2) \\ x(1,k-1) \end{bmatrix} \in \mathcal{R}^{(k-1)n}, \quad \tilde{u}(k) = \begin{bmatrix} u(k,0) \\ u(k-1,1) \\ \vdots \\ u(1,k-1) \\ u(0,k) \end{bmatrix} \in \mathcal{R}^{(k-1)m},
$$

$$
\tilde{y}(k) = \begin{bmatrix} y(k-1,1) \\ y(k-2,2) \\ \vdots \\ y(2,k-2) \\ y(1,k-1) \end{bmatrix} \in \mathcal{R}^{(k-1)p}
$$

$$
\tilde{x}(0,0) = x(0,0) \qquad \tilde{x}(0,k) = \begin{bmatrix} x(k,0) \\ x(0,k) \end{bmatrix} \in \mathcal{R}^{2n}
$$

$$
F_{1,1} = 0
$$

$$
F_{1,k+1} = \begin{bmatrix} A_2 & 0 & \dots & 0 & 0 \\ A_1 & A_2 & \dots & 0 & 0 \\ 0 & A_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_1 & A_2 \\ 0 & 0 & \dots & 0 & A_1 \end{bmatrix} \in \mathcal{R}^{kn \times (k+1)n}
$$

$$F_{2,k} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ A_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & A_0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A_0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \in \mathcal{R}^{(k-1)n \times (k+1)n}$$

$$G_{1,k+1} = \begin{bmatrix} B_1 & B_2 & 0 & \dots & 0 & 0 \\ 0 & B_1 & B_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & B_2 & 0 \\ 0 & 0 & 0 & \dots & B_1 & B_2 \end{bmatrix} \in \mathcal{R}^{(k+2)m \times (k+1)n}$$

$$G_{2,k} = \begin{bmatrix} B_0 & 0 & \dots & 0 & 0 \\ 0 & B_0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B_0 & 0 \\ 0 & 0 & \dots & 0 & B_0 \end{bmatrix} \in \mathcal{R}^{(k+1)m \times (k+1)n}$$

$$H_{1,k+1} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathcal{R}^{2n \times (k+1)n}, \quad H_{2,k} = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & A_0 \end{bmatrix} \in \mathcal{R}^{2n \times (k+1)n}$$

$$C_k = \begin{bmatrix} C & 0 & \dots & 0 \\ 0 & C & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & C \end{bmatrix} \in \mathcal{R}^{(k-1)p \times (k+1)n}$$

$$\tilde{w}(k) = \begin{bmatrix} w(k-1,1) \\ w(k-2,2) \\ \vdots \\ w(2,k-2) \\ w(1,k-1) \end{bmatrix}, \quad \tilde{v}(k) = \begin{bmatrix} v(k-1,1) \\ v(k-2,2) \\ \vdots \\ v(2,k-2) \\ v(1,k-1) \end{bmatrix},$$

$$T_{1,k+1} = \begin{bmatrix} \mathcal{I} & \mathcal{I} & 0 & \dots & 0 & 0 \\ 0 & \mathcal{I} & \mathcal{I} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \mathcal{I} & 0 \\ 0 & 0 & 0 & \dots & \mathcal{I} & \mathcal{I} \end{bmatrix} \in \mathcal{R}^{(k+2)n \times (k+1)n}, \quad T_{2,k} = \mathcal{I} \in \mathcal{R}^{(k+1)n \times (k+1)n}$$

$\mathcal{I}$ — identity matrix of appropriate dimensions.

Initial conditions for the system given by (4) are $x(0) = 0$ and $x(1) = 0$. The system described by equations (4) can be rewritten in more convenient way as 1-D first order, linear, variable structure discrete system as follows

$$\overline{x}(k+1) \quad = \quad \overline{F}_k \overline{x}(k) + \overline{G}_k \overline{u}(k) + \overline{T}_k \overline{w}(k) + \overline{H}_k \overline{x}(0,k)$$

$$y(k) \quad = \quad \overline{C}_k \overline{x}(k) + \overline{v}(k) + P_k \overline{x}(0,k), \tag{5}$$

where

$$\overline{x}(k) = \begin{bmatrix} \tilde{x}(k) \\ \tilde{x}(k+1) \end{bmatrix}, \quad \overline{w}(k) = \begin{bmatrix} \tilde{w}(k) \\ \tilde{w}(k+1) \end{bmatrix}, \quad \overline{v}(k) = \begin{bmatrix} \tilde{v}(k) \\ \tilde{v}(k+1) \end{bmatrix}$$

$$\overline{x}(0,k) = \begin{bmatrix} \tilde{x}(0,k) \\ \tilde{x}(0,k+1) \end{bmatrix}, \quad \overline{u}(k) = \begin{bmatrix} \tilde{u}(k) \\ \tilde{u}(k+1) \end{bmatrix}$$

$$\overline{F}_k = \begin{bmatrix} \mathcal{O} & \mathcal{I} \\ F_{2,k} & F_{1,k+1} \end{bmatrix}, \quad \overline{H}_k = \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ H_{2,k} & H_{1,k+1} \end{bmatrix}, \quad \overline{T}_k = \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ T_{2,k} & T_{1,k+1} \end{bmatrix}$$

$$\overline{G}_k = \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ G_{2,k} & G_{1,k+1} \end{bmatrix}, \quad \overline{C}_k = \begin{bmatrix} C_k & \mathcal{O} \\ \mathcal{O} & C_k \end{bmatrix}, \quad \overline{P}_k = \begin{bmatrix} P_k & \mathcal{O} \\ \mathcal{O} & P_{k+1} \end{bmatrix}$$

$\mathcal{O}$ — matrix composed of zero elements of appropriate dimensions.

Now for the system given by equations (5) we may assume that $\overline{w}(k)$ is the state "white" noise with known covariance matrix $\overline{W}$ and $\overline{v}(k)$ is the output noise with known covariance matrix $\overline{V}$. The properness of this assumptions is easy to check. We note that

$$\begin{aligned} E[\overline{w}(k), \overline{w}(l)] &= \overline{W}\delta_{kl}, \\ E[\overline{v}(k), \overline{v}(l)] &= \overline{V}\delta_{kl}, \end{aligned}$$

where

$$\delta_{kl} = \begin{cases} 0, & k \neq l \\ 1, & k = l. \end{cases}$$

These note is a direct result of the assumptions posed on the noise of original 2-D system (1). So, for the system described by the equations (5) we can derive Kalman filter equations. As the reduction process (transforming the system (1) to (5)) demonstrates, the original system (1) has a natural half-plane causality, so the estimator obtained without other conditions would be half-plane causal. To obtain a quarter-plane causality we have to impose some additional constraints on the form of Kalman filter gain matrix.

For the system described by the equations (5) we define the estimator system as follows:

$$\hat{\overline{x}}(k+1) = \overline{F}(k)\,\hat{\overline{x}}(k) + \overline{G}_k\overline{u}(k) + K(k)\,(\overline{y}(k) - \overline{C}(k)\,\hat{\overline{x}}(k)) \tag{6}$$

Then, subtracting (6) from (5) we get the error equation:

$$\begin{aligned} \overline{e}(k+1) \quad = \quad & (\overline{F}(k) - K(k)\,\overline{C}(k))\,\overline{e}(k) + \overline{T}(k)\,\overline{w}(k) \\ & + (\overline{H}(k) - K(k)\,P(k))\,\overline{x}(0,k) + K(k)\,\overline{v}(k), \end{aligned} \tag{7}$$

where $\overline{e}(k) = \overline{x}(k) - \hat{\overline{x}}(k)$.

In the sequel we shall calculate the covariance of $\overline{e}(k)$ assuming that $\overline{e}(0)$, $\overline{w}(k)$ and $\overline{v}(k)$ are all statistically independent. Denoting $\overline{M}(k) = \overline{F}(k) - K(k)\,\overline{C}(k)$ and $Q_e(k) = \mathrm{cov}[\overline{e}(k)]$, the covariance equation associated with (6) is as follows:

$$Q_e(k+1) = \overline{M}(k)\,Q_e(k)\,\overline{M}^T(k) + \overline{T}(k)\,\overline{W}(k)\,\overline{T}^T(k) + K(k)\,\overline{V}(k)\,K^T(k). \tag{8}$$

The Kalman filter gain is to be chosen by minimizing the appropriate optimality criterion. In this case this criterion has the form:

$$E(e(k)^2) = \mathrm{tr}Q_e(k). \tag{9}$$

This way minimization of $\mathrm{tr}Q_e(k)$ with respect to $K(k)$ yields the optimal filter gain. We shall now perform the minimization of performance index in the form:

$$J(K(k)) = \mathrm{tr}Q_e(k+1) \tag{10}$$

using variational techniques. Thus we obtain (only first order terms to be considered):

$$\begin{aligned}
\delta J &= J(K(k)+\delta K(k)) - J(K(k)) \\
&= \mathrm{tr}[(\overline{F}(k) - ((K(k)+\delta K(k))\,\overline{C}(k))\,Q_e(\overline{F}(k) \\
&\quad -(K(k)+\delta K(k))\,\overline{C}(k))^T + \overline{T}(k)\,\overline{W}(k)\,\overline{T}^T(k) \\
&\quad +(K(k)+\delta K(k))\,\overline{V}(k)\,(K(k)+\delta K(k))^T] \\
&\quad -\mathrm{tr}[(\overline{M}(k)\,Q_e\overline{M}^T(k) + \overline{T}(k)\,\overline{W}(k)\,\overline{T}^T(k) + K(k)\,\overline{V}(k)\,K^T(k)] \\
&= \mathrm{tr}[(\overline{F}(k) - K(k)\,\overline{C}(k) - \delta K(k)\,\overline{C}(k))\,Q_e(k)\,(\overline{F}(k) - K(k)\,\overline{C}(k) \\
&\quad -\delta K(k)\,\overline{C}(k))^T + \overline{T}(k)\,\overline{W}(k)\,\overline{T}^T + K(k)\,\overline{V}(k)\,K^T(k) + \delta K(k)\,\overline{V}(k)\,K^T(k) \\
&\quad +K(k)\,\overline{V}(k)\,\delta K^T(k) - \overline{M}(k)\,Q_e(k)\,\overline{M}^T(k) - \overline{T}(k)\,\overline{W}(k)\,\overline{T}^T(k) - K(k)\,\overline{V}(k)\,K^T(k) \\
&= \mathrm{tr}[-\delta K(k)\,\overline{C}(k)\,Q_e(k)\,\overline{M}^T(k) - \overline{M}(k)\,Q_e(k)\,(\delta K(k)\,\overline{C}(k))^T \\
&\quad +\delta K(k)\,\overline{V}(k)\,K^T(k) + K(k)\,\overline{V}(k)\,\delta K^T(k)] \\
&= -2\mathrm{tr}[\overline{M}(k)\,Q_e(k)\,\overline{C}^T(k)\,\delta K^T(k) - K(k)\,\overline{V}(k)\,\delta K^T(k)].
\end{aligned} \tag{11}$$

It is well known from optimization theory that such a variation is to be nonnegative for all $\delta K(k)$. The minimum value it achieves at $\delta J = 0$, so there exists a unique $K(k)$ such that:

$$\delta J = \mathrm{tr}[\overline{M}(k)\,Q_e(k)\,\overline{C}^T(k) - K(k)\,\overline{V}(k)]\,\delta K^T(k) = 0. \tag{12}$$

Since the second order terms in (11) are also nonnegative, that unique $K(k)$ will determine the optimal state estimation. Replacing the values for $\overline{M}(k)$ the expansion for δJ becomes:

$$\begin{aligned}
\delta J &= \mathrm{tr}\{[(\overline{F}(k) - K(k)\,\overline{C}(k))\,Q_e(k)\,\overline{C}^T(k) - K(k)\,\overline{V}(k)]\,\delta K^T(k)\} \\
&= \mathrm{tr}\{[\overline{F}(k)\,Q_e(k)\,\overline{C}^T(k) - K(k)\,\overline{C}(k)\,Q_e(k)\,\overline{C}^T(k) - K(k)\,\overline{V}(k)]\,\delta K^T(k)\} \\
&= -\mathrm{tr}\{[K(k)\,(\overline{V}(k) + \overline{C}(k)\,Q_e(k)\,\overline{C}^T(k)) - \overline{F}(k)\,Q_e(k)\overline{C}^T(k)]\,\delta K^T(k)\}
\end{aligned} \tag{13}$$

we note that $K(k)$ must satisfy:

$$\delta J = -\mathrm{tr}\{[K(k)\,(\overline{V}(k) + \overline{C}(k)\,Q_e(k)\,\overline{C}^T(k)) - \overline{F}(k)\,Q_e(k)\,\overline{C}^T(k)]\delta K^T(k)\} = 0. \tag{14}$$

A direct solution of (14) yields the desired optimal filter gain:

$$K(k) = \overline{F}(k)\,Q_e(k)\,\overline{C}^T(k)\,(\overline{C}(k)\,Q_e(k)\,\overline{C}^T(k) + \overline{V}(k))^{-1}. \tag{15}$$

Putting together (6), (8) and (15) we obtain a complete Kalman filter algorithm expressed in terms of system given by (5):

$$\begin{aligned}
\hat{\overline{x}}(k+1) &= \overline{F}(k)\,\hat{\overline{x}}(k) + K(k)\,(\overline{y}(k) - \overline{C}(k)\,\hat{\overline{x}}(k)) \\
K(k) &= \overline{F}(k)\,Q_e(k)\,\overline{C}^T(k)\,(\overline{C}(k)\,Q_e(k)\,\overline{C}^T(k) + \overline{V}(k))^{-1} \\
Q_e(k) &= (\overline{F}(k-1) - K(k-1)\,\overline{C}(k-1))\,Q_e(k-1)\,(\overline{F}(k-1) - K(k-1)\,\overline{C}(k-1))^T \\
&\quad + \overline{T}(k-1)\,\overline{W}(k-1)\,\overline{T}^T(k-1) + K(k-1)\,\overline{V}(k-1)\,K^T(k-1) \\
x(0) &= 0 \\
Q_e(0) &= W(0).
\end{aligned} \tag{16}$$

By means of the above-presented algorithm we may obtain the estimated value $\hat{\overline{x}}(k)$ of the vector $\overline{x}(k)$ what means that this way we may obtain the values of $\tilde{x}(k)$ and $\tilde{x}(k+1)$ as well. Having estimated the whole vectors $\tilde{x}(k)$ and $\tilde{x}(k+1)$ it is immediate to extract the desired estimated values of the vector $x(i,j)$ for every spatial point (i,j) such that $i+j=k$ or $i+j=k+1$, for both models (1) and (3). Up to now this estimator presented only the property of half-plane causality but it is not quarter-plane causal in general. Examining the equations (4) we may notice that the specific structure of matrices $F_{1,k+1}$ and $F_{2,k}$ (and consequently $\overline{F}_k$) is essential for quarter-plane causality. In addition C has also a specific structure. If the state estimator is to be quarter-plane causal (and this is our goal), the structure of matrices $\overline{M}(k)$ and $K(k)\,\overline{C}(k)$ must have the same structure as $\overline{F}(k)$. Therefore the matrix $K(k)$ must have also the same specific structure:

$$\overline{K}(k) = \begin{bmatrix} \mathcal{O} & \tilde{\mathcal{I}} \\ \overline{K}_{2,k} & \overline{K}_{1,k+1} \end{bmatrix},$$

where

$$\overline{K}_{1,k+1} = \begin{bmatrix} \overline{K}_{11} & 0 & \dots & 0 & 0 \\ \overline{K}_{21} & \overline{K}_{22} & \dots & 0 & 0 \\ 0 & \overline{K}_{32} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \overline{K}_{k,k-1} & \overline{K}_{k,k} \\ 0 & 0 & \dots & 0 & \overline{K}_{k+1,k} \end{bmatrix} \in \mathcal{R}^{kn\times(k+1)n},$$

$$\overline{K}_{2,k} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ \overline{K}'_{21} & 0 & \dots & 0 & 0 \\ 0 & \overline{K}'_{32} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \overline{K}'_{k,k-1} & \overline{K}'_{k,k} \\ 0 & 0 & \dots & 0 & \overline{K}'_{k+1,k} \end{bmatrix} \in \mathcal{R}^{(k-1)n\times(k+1)n} \qquad \text{for } k>0$$

and $\mathcal{O} \in \mathcal{R}^{(k-1)\times kn}$ – is a matrix composed of zero elements, $\tilde{\mathcal{I}} \in \mathcal{R}^{kn\times kn}$ – is an identity matrix. It could be easily checked that every matrix $K(k)$ of full column rank can be reduced to the specific form $\overline{K}(k)$ by appropriate elementary matrix operations performed both on rows and columns of the matrix $K(k)$. This is equivalent to finding the pair of nonsingular matrices:

$$S(k) \in \mathcal{R}^{(2k+1)n\times(2k+1)n} \quad \text{and} \quad Z(k) \in \mathcal{R}^{(2k-1)n\times(2k-1)n},$$

such that left multiplication by $S(k)$ and right multiplication by $Z(k)$ of $\overline{K}(k)$ results in matrix $K(k)$. Thus we can write down:

$$K(k) = S(k)\overline{K}(k)\,Z(k). \tag{17}$$

So the process of optimization has to be changed, since there exists a constraint posed on the form of the matrix $K(k)$. Under such assumptions the equation (14) has the form:

$$\begin{aligned} \delta J &= \operatorname{tr}\{[S(k)\overline{K}(k)\,Z(k)\,(\overline{V}(k) + \overline{C}(k)\,Q_e(k)\,\overline{C}^T(k)) \\ &\quad -\overline{F}(k)\,Q_e(k)\,\overline{C}^T(k)]\,\delta(Z^T(k)\,K^T(k)\,S^T(k))\} = 0. \end{aligned} \tag{18}$$

Then the desired optimal filter gain in quarter-plane causal form can be obtained as follows:

$$K(k) = S^{-1}(k)\,\overline{F}(k)\,Q_e(k)\,\overline{C}^T(k)\,(\overline{C}(k)\,Q_e(k)\,\overline{C}^T(k) + \overline{V}(k))^{-1}Z^{-1}(k). \tag{19}$$

So in the optimal estimation algorithm for quarter-plane causality case we have to substitute (19) for the second equation in (16). The rest of the algorithm remains unchanged.

4. CONCLUSIONS

The 2-D Kalman filter algorithm derived in this paper is a result of effective transport of 1-D optimal state estimation theory to the discrete two-dimensional setting. It could be observed, however, that such an important feature as quarter-plane causality will not be naturally preserved when conventional 1-D theory is applied. If quarter-plane causality is essential (what is not always the case in image processing) the way of accommodation of the algorithm through constraints on the optimization process has been shown. In the very end we obtained an algorithm which could be directly applied for image processing purposes.

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