KYBERNETIKA — VOLUME 30 (1994), NUMBER 6, PAGES 645-658

THE PARTIAL NON INTERACTING PROBLEM: STRUCTURAL AND GEOMETRIC SOLUTIONS

JUAN CARLOS MARTÍNEZ GARCÍA, MICHEL MALABRE AND RABAH RABAH

The aim of this paper is to propose a new contribution in the domain of the famous Decoupling Problem for Linear Time-Invariant systems: we introduce here and solve the so-called kth-order Partial Non Interacting Problem (PNIP(k)), which amounts to diagonalizing the first k Markov parameters of the compensated plant. This contribution is based on classical results on exact decoupling and the partial treatment is inspired from a similar control problem, namely the partial model matching problem.

1. INTRODUCTION

The Decoupling Problem (also sometimes called Non Interacting Problem) is certainly one of the most famous problems in Control Theory which amounts to reducing the control of a "complex" multivariable process to that of several single-input, single-output ones. Some intensive treatment of this problem can be found in [17]. However, the structural requirements for that problem to be solvable may be quite demanding (see for instance [4], [3], and recently [13]). This is the reason why a *partial* version of this problem is introduced here, which amounts to obtaining non interaction only through the first k Markov parameters of the compensated plant. We present here the geometic and structural solutions for this problem.

2. NOTATION AND BASIC CONCEPTS

Throughout the paper we shall essentially follow the notational conventions of [17]. Script capital $(\mathcal{X}, \mathcal{Y}, \ldots)$ denote finite-dimensional vector spaces over the field of real numbers $\mathbb{R}$, and $\dim(\mathcal{X})$, $\dim(\mathcal{Y})$, $\ldots$, denote their dimensions. The notation $\mathcal{X} \simeq \mathcal{Y}$ means $\dim(\mathcal{X}) = \dim(\mathcal{Y})$. If $\mathcal{V} \subset \mathcal{X}$, then $\mathcal{X}/\mathcal{V}$ denotes the quotient space $\mathcal{X}$ modulo $\mathcal{V}$.

Italic capitals $(A, B, \ldots)$ denote interchangeably linear maps and their matrix representations in particular bases. The ith row of a matrix C is denoted by c_i. We shall use $\overline{C}_i$ to denote the matrix C without the ith row c_i. The image of a map B is written as $\operatorname{Im} B$ and its kernel as $\operatorname{Ker} B$. The identity map on a n-dimensional space is denoted by I_n.

The set of positive integer numbers is denoted by $\mathbb{N}$.

Given the maps $A:\mathcal{X} \to \mathcal{X}$, $B:\mathcal{U} \to \mathcal{X}$, $C:\mathcal{X} \to \mathcal{Y}$ ($\dim(\mathcal{X}) = n$, $\dim(\mathcal{U}) = m$, $\dim(\mathcal{Y}) = p$), associated with the linear time-invariant system:

$$\left\{ \begin{array}{rcll} \dot{x}(t) & = & Ax(t) + Bu(t) & t \geq 0, \\ y(t) & = & Cx(t) & t \geq 0, \end{array} \right. \tag{1}$$

that we shall denote by (A, B, C), it will be assumed here that the reader is familiarized with the concepts of $(A, \mathcal{B})$-invariant, $(\mathcal{C}, A)$-invariant and controllability subspaces [17] (see also [1]). We shall mainly use the following.

Let $\mathcal{B} = \operatorname{Im} B$ and $\mathcal{C} = \operatorname{Ker} C$. A subspace $\mathcal{W} \subset \mathcal{X}$ is said to be $(A, \mathcal{B})$-invariant if there exists a map $F:\mathcal{X} \to \mathcal{U}$ satisfying $(A+BF)\mathcal{W} \subset \mathcal{W}$.

A subspace $\mathcal{W} \subset \mathcal{X}$ is said to be $(\mathcal{C}, A)$-invariant if there exists a map $K:\mathcal{Y} \to \mathcal{X}$ satisfying $(A+KC)\mathcal{W} \subset \mathcal{W}$.

Given any subspace $\mathcal{K} \subset \mathcal{X}$, the supremal $(A, \mathcal{B})$-invariant subspace included in $\mathcal{K}$ is given as the limit, say $\mathcal{V}^*(\mathcal{K})$, of the *Invariant Subspace Algorithm* (ISA):

$$\left\{ \begin{array}{lcl} \mathcal{V}^0 & := & \mathcal{X} \\ \mathcal{V}^\mu & := & \mathcal{K} \cap A^{-1}\left(\mathcal{V}^{\mu-1} + \mathcal{B}\right), \quad \mu \geq 1. \end{array} \right. \tag{2}$$

When $\mathcal{K} = \ker C$ the limit of ISA is noted as $\mathcal{V}^*$.
When $\mathcal{K} = \ker c_i$ the μth step of ISA is noted as $\mathcal{V}_i^\mu$ and its limit is noted as $\mathcal{V}_i^*$.
When $\mathcal{K} = \ker \overline{C}_i$ the μth step of ISA is noted as $\mathcal{W}_i^\mu$ and its limit is noted as $\mathcal{W}_i^*$.

Given any subspace $\mathcal{K} \subset \mathcal{X}$, the infimal $(\mathcal{K}, A)$-invariant subspace containing $\mathcal{B}$ is given as the limit of the *Conditioned Invariant Subspace Algorithm* (CISA):

$$\left\{ \begin{array}{lcl} \mathcal{S}^0 & := & 0 \\ \mathcal{S}^\mu & := & \mathcal{B} + A\left(\mathcal{K} \cap \mathcal{S}^{\mu-1}\right), \quad \mu \geq 1. \end{array} \right. \tag{3}$$

When $\mathcal{K} = \ker C$ the limit of CISA is noted as $\mathcal{S}^*$.
When $\mathcal{K} = \ker c_i$ the μth step of CISA is noted as $\mathcal{S}_i^\mu$ and its limit is noted as $\mathcal{S}_i^*$.
When $\mathcal{K} = \ker \overline{C}_i$ the μth step of CISA is noted as $\overline{\mathcal{S}}_i^\mu$ and its limit is noted as $\overline{\mathcal{S}}_i^*$.

Given any subspace $\mathcal{K} \subset \mathcal{X}$, the maximal $(A, \mathcal{B})$-controllability subspace contained in $\mathcal{K}$ is given as the limit of the *Controllability Subspace Algorithm* (CSA):

$$\left\{ \begin{array}{lcl} \mathcal{R}^0 & := & 0 \\ \mathcal{R}^\mu & := & \mathcal{V}^*(\mathcal{K}) \cap \left(A\mathcal{R}^{\mu-1} + \mathcal{B}\right), \quad \mu \geq 1. \end{array} \right. \tag{4}$$

When $\mathcal{K} = \ker C$ the limit of CSA is noted as $\mathcal{R}^*$, which is equal to $\mathcal{V}^* \cap \mathcal{S}^*$.
When $\mathcal{K} = \ker c_i$ the μth step of CSA is noted as $\mathcal{R}_i^\mu$ and its limit is noted as $\mathcal{R}_i^*$, which is equal to $\mathcal{V}_i^* \cap \mathcal{S}_i^*$
When $\mathcal{K} = \ker \overline{C}_i$ the μth step of CSA is noted as $\mathcal{T}_i^\mu$ and its limit is noted as $\mathcal{T}_i^*$, which is equal to $\mathcal{W}_i^* \cap \overline{\mathcal{S}}_i^*$.

Infinite Zero Structure

Given any system (A, B, C) described by (1) or equivalently by its strictly proper $p \times m$ transfer function matrix $T(s) := C(sI_n - A)^{-1}B$, its *structure at infinity* is described by the *multiplicity orders* of its *zeros at infinity*. From an algebraic point of view, this structure can be derived from the so-called *Smith–McMillan Form at infinity* of $T(s)$, say $\wedge_\infty$, which is a canonical form under right and left *biproper* transformations (see for instance [15]). Indeed, there exist biproper matrices, $B_1(s)$ and $B_2(s)$, such that:

$$B_1(s)\,T(s)\,B_2(s) = \wedge_\infty = \left[\begin{array}{c|c} \Delta_\infty & 0 \\ \hline 0 & 0 \end{array}\right],$$

where $\Delta_\infty = diag\{s^{-n_1},\, s^{-n_2},\, \ldots,\, s^{-n_r}\}$, $r := \text{rank}\,(T(s))$.

The non increasing list of integers $\{n_1,\ n_2, \ldots, n_r\}$ is the list of the orders of the zeros at infinity of the system. This list is frequently called *global structure at infinity* of (A, B, C). From a geometric point of view, various equivalent definitions have been given for this structure. The original one, due to [2], is:

$$n_i \;=\; \text{card}\{p_\mu \geq i\}, \quad \forall\, i \in \{1,\, 2,\, \ldots,\, r\}, \tag{5}$$

where *card* stands for cardinal (number of elements in the set) and with:

$$p_\mu \;:=\; \dim\left(\frac{\mathcal{V}^* + \mathcal{S}^\mu}{\mathcal{V}^* + \mathcal{S}^{\mu-1}}\right), \quad \forall\, \mu \geq 1. \tag{6}$$

Other geometric characterizations have been given in [10]. A particularly interesting one is given by:

$$p_\mu \;:=\; \dim\left(\frac{\mathcal{B} \cap \mathcal{V}^{\mu-1}}{\mathcal{B} \cap \mathcal{V}^*}\right), \quad \forall\, \mu \geq 1. \tag{7}$$

For the system (A, B, c_i), which denotes the ith row of the transfer function matrix $T(s)$, the order of its zero at infinity is noted as n'_i and is given by:

$$n'_i \;=\; \dim\left(\frac{\mathcal{S}_i^* + \mathcal{V}_i^*}{\mathcal{V}_i^*}\right). \tag{8}$$

This list $\{n'_1,\ n'_2,\ \ldots,\ n'_p\}$ is called the *row structure at infinity* of (A, B, C). The elements of this list are also given by:

$$n'_i = \min\{j : c_i A^{j-1} B \neq 0,\ \ j = 1, 2, \ldots\}, \quad \forall\, i \in \{1,\, 2,\, \ldots,\, p\}. \tag{9}$$

3. PROBLEM STATEMENT

The kth-order Partial Non Interacting Problem (PNIP(k))

Definition 1. (PNIP(k)) Given a system (A, B, C) and a positive integer k, the kth-order Partial Non Interacting Problem has a solution if and only if there exists a static state feedback control law:

$$u(t) \;=\; Fx(t) + Gv(t) \tag{10}$$

with G non singular, such that the first k Markov parameters of the closed-loop system have all their non-diagonal elements equal to zero, i. e.:

$$C\,(A+BF)^j\,BG \;=\; \{\textit{diagonal matrix}\}\,,\ \ \forall\, j\in\{0,\,1,\,\ldots,\,k-1\}. \tag{11}$$

Note that some diagonal elements of the corresponding matrices may be zero. Moreover, we do not require any full rank property for the considered system.

4. MATRIX AND STRUCTURAL SOLUTION OF PNIP(k)

First of all, let us present here an easy-to-verify property of a system (A, B, C). This property will be used to establish the necessary and sufficient solvability condition of PNIP(k) in matrix terms.

Property 1. [6] Given a system (A, B, C) and from the definition (9) of n'_i we have that:

$$c_i\,(A+BF)^j \;=\; c_iA^j,\ \ \forall\, j\in\{0,\,1,\,\ldots\,n'_i-1\}$$

and:

$$c_i\,(A+BF)^j \;=\; c_iA^{n'_i-1}\,(A+BF)^{j-n'_i+1}\,,\ \ \forall\, j\in\{n'_i,\,n'_i+1,\ldots\}.$$

In what follows we shall consider that the outputs of (A, B, C) have been reordered in such a way that:

$$n'_1\le n'_2\le\,\ldots\,\le n'_p.$$

For the problem of interest (non interaction) this is obviously an unrestrictive assumption.

Let us now present our:

Theorem 1. Given a system (A, B, C) and a positive integer k, the following statements are equivalent:

i) PNIP(k) is solvable.

ii) The matrix D_k is epic, where:

$$D_k \;:=\; \begin{bmatrix} c_1A^{n'_1-1}B \\ c_2A^{n'_2-1}B \\ \vdots \\ c_lA^{n'_l-1}B \end{bmatrix} \tag{12}$$

with $n'_i\le k$, for all $i\in\{1,\,2,\,\ldots,\,l\}$.

iii) The set of all the elements of the global structure at infinity of (A, B, C) which are less than or equal to k is equal to the set of all the elements of the row structure at infinity of (A, B, C) which are less than or equal to k.

Proof. ii) $\Longrightarrow$ i): Assuming that D_k is epic, it is claimed that the static state feedback control law:

$$u(t) = F^* x(t) + G^* v(t)$$

with:

$$F^* := -G^* A^*, \tag{13}$$

and:

$$G^* := \left[\; D_k^+ \mid K_k \; \right], \tag{14}$$

where:

$$A^* := \begin{bmatrix} c_1 A^{n'_1 - 1} \\ c_2 A^{n'_2 - 1} \\ \vdots \\ c_l A^{n'_l - 1} \\ \hline 0^{(m-l) x m} \end{bmatrix}, \tag{15}$$

$$D_k D_k^+ = I_l, \tag{16}$$

$$K_k \text{ is a basis of } \operatorname{Ker} D_k, \tag{17}$$

solves PNIP(k).

From the very definition of n'_i's we have that:

$$c_i \, (A + BF^*)^j \, BG = 0, \;\; \forall j \leq n'_i - 1. \tag{18}$$

By Property 1:

$$c_i \, (A + BF^*)^{n'_i - 1} = c_i A^{n'_i - 1}, \;\; \forall i \in \{1, 2, \ldots, l\}.$$

Thus:

$$c_i \, (A + BF^*)^{n'_i - 1} \, BG^* = c_i A^{n'_i - 1} BG^* = c_i A^{n'_i - 1} B \left[\; D_k^+ \mid K_k \; \right].$$

But $c_i A^{n'_i - 1} B$, for all i in $\{1, 2, \ldots, l\}$, is the ith row of D_k and so it follows that:

$$c_i \, (A + BF^*)^{n'_i - 1} \, BG^* = [\gamma_{i1} \;\; \gamma_{i2} \;\; \ldots \;\; \gamma_{im}] \tag{19}$$

with:

$$\gamma_{ij} := \begin{cases} 1, & \text{for } j = i \\ 0, & \text{for } j \neq i \end{cases}, \;\; \forall j \in \{1, 2, \ldots, m\}.$$

Both equations (18) and (19) let us conclude that $u(t) = F^* x(t) + G^* v(t)$ solves PNIP(k), as was claimed.

i) $\Longrightarrow$ iii): Suppose that PNIP(k) is solvable, i.e. there exists a static state feedback control law $u(t) = Fx(t) + Gv(t)$, with G non singular, such that the first k Markov parameters of the closed-loop system, i.e. $(A + BF, \; BG, \; C)$, have all their non-diagonal elements equal to zero.

Let us now write the transfer function matrix of $(A + BF,\ BG,\ C)$ as follows:

$$\begin{aligned} T_{FG}(s) &:= C\left(sI_n - (A+BF)\right)^{-1} BG \\ &= T(s)\, C(s) \end{aligned}$$

with:

$$T(s) := C\left(sI_n - A\right)^{-1} B$$

and:

$$C(s) := \left(I_m - F\left(sI_n - A\right)^{-1} B\right)^{-1} G.$$

Since G is a non singular matrix, $C^{-1}(s) = G^{-1}\left(I_m - F\left(sI_n - A\right)^{-1} B\right)$ exists and is *biproper*. Hence, the global structure at infinity is the same for $T(s)$ and $T_{FG}(s)$. This is also the same for the row structure at infinity of $T(s)$ and $T_{FG}(s)$. Indeed, the order of the zero at infinity of the ith-row of $T_{FG}(s)$, i.e. $T_{FG_i}(s) := c_i\left(sI_n - (A+BF)\right)^{-1} BG$, is equal to n'_i, the order of the zero at infinity of the row-system (A, B, c_i). Let us now define:

$$\Delta_{n'}(s) := \operatorname{diag}\{s^{-n'_1},\ s^{-n'_2},\ \ldots,\ s^{-n'_l},\ s^{-n'_{l+1}},\ \ldots,\ s^{-n'_p}\}. \tag{20}$$

Then we can factorize $T_{FG}(s)$ as follows:

$$T_{FG}(s) = \Delta_{n'}(s)\overline{T}_{FG}(s),$$

where $\overline{T}_{FG}(s)$ is such that its first l rows are independent (recall that the outputs of the original system have been re-ordered in such a way that $n'_1 \leq n'_2 \leq \ \ldots\ \leq n'_p$), since $C\left(A+BF\right)^j BG = \{\textit{diagonal matrix}\}$, $\forall j \in \{0,\ 1,\ \ldots,\ k-1\}$. Thus, the algorithm which derives the global structure at infinity of a rational matrix from Laurent expansions (see [16] and [9]), let us affirm that $\{n'_1,\ n'_2,\ \ldots, n'_l\}$ is a subset of the global structure at infinity of $T_{FG}(s)$, and since the global structure at infinity of $T_{FG}(s)$ and $T(s)$ is the same, this establishes iii).

iii) $\Longrightarrow$ ii): Suppose that the set of all the elements of the global structure at infinity of (A, B, C) which are less than or equal to k is equal to the set of all the elements of the row structure at infinity of (A, B, C) which are less than or equal to k, and let us factorize the transfer function matrix of system $(A, B,\ C)$ as follows:

$$T(s) = \Delta_{n'}(s)\left(\begin{bmatrix} c_1 A^{n'_1 - 1} B \\ c_2 A^{n'_2 - 1} B \\ \vdots \\ c_l A^{n'_l - 1} B \\ c_{l+1} A^{n'_{l+1} - 1} B \\ \vdots \\ c_p A^{n'_p - 1} B \end{bmatrix} + [\,\cdot\,]s^{-1} + [\,\cdot\,]s^{-2} + \ldots\right), \tag{21}$$

where $[\,\cdot\,]$ stands for a constant matrix and with $\Delta_{n'}(s)$ as defined in (20). As above, the algorithm which derives the global structure at infinity of a rational matrix from

Laurent expansions let us conclude about the independence of at least the first l rows of the leading coefficient matrix of the right hand side of (21). This implies that D_k is epic, which concludes the proof. □

Let us remark that the results established in Theorem 1 are also valid when the outputs of the system are not re-ordered in a particular way. Re-ordering has just been used to prove the theorem in an easy-to-present way. In the sequel we do not assume any special re-ordering of the outputs of the system.

The *exact* row decoupling problem amounts to solving PNIP(k) for any possible value of $k \in \mathbb{N}$. Thanks to the previous structural condition, this gives:

Corollary 1. The exact row decoupling problem is solvable if and only if $\{n_i\} = \{n'_i\}$.

Note that this requires that the system be of full row rank p, since $\{n'_i\}$ is always formed with p integers.

Let us now write a structural solvability condition of PNIP(k) which will play a key role in the obtention of the geometric solvability condition of this problem.

Due to the correspondence (5) between both list $\{n_j\}$ and list $\{p_i\}$, which characterizes geometrically the global structure at infinity of system (A, B, C), it is quite obvious that:

$$\tau_\mu \; := \; p_1 \, - \, p_\mu, \;\; \forall \mu \in \{1, 2, \ldots, n_1\} \tag{22}$$

is the number of the zeros at infinity of (A, B, C) which order is strictly less than μ, for all $i \in \{1, 2, \ldots, n_1\}$. In particular, for the row-system (A, B, c_i) let:

$$\tau'_{i_\mu} \; := \; 1 - p'_{i_\mu}, \;\; \forall \mu \geq 1 \tag{23}$$

with:

$$p'_{i_\mu} \; = \; \dim\left(\frac{\mathcal{B} \cap \mathcal{V}_i^{\mu-1}}{\mathcal{B} \cap \mathcal{V}_i^*}\right) \;\; \text{(by (7))}. \tag{24}$$

Then τ'_{i_μ} is equal to 1 for all $\mu \geq n'_i + 1$ and is equal to zero if μ is strictly less than $n'_i + 1$. Consequently, given an integer $\mu \in \{1, 2, \ldots, k+1\}$:

$$\tau'_\mu \; := \; \sum_{i=1}^{p} \tau'_{i_\mu} \; = \; \sum_{i=1}^{p} \dim\left(\frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{V}_i^{\mu-1}}\right) \tag{25}$$

is equal to the total number of all the elements of the row structure at infinity of (A, B, C) which order is strictly less than μ.

We can now present:

Corollary 2. Let the positive integer k and system (A, B, C) be given. Then PNIP(k) is solvable if and only if:

$$\tau_\mu \; = \; \tau'_\mu, \;\; \forall \mu \in \{1, 2, \ldots, k+1\}. \tag{26}$$

Proof. To prove this corollary, it suffices to establish the equivalence between (26) and iii) in Theorem 1.

It is evident that the equality between the less than or equal to k elements of both global and row structures at infinity of system (A, B, C) implies (26).

Conversely, using combinatorial arguments, we can easily prove that the list of non-decreasing positive integers $\{\tau_\mu\}$, $\forall\,\mu \in \{1, 2, \ldots, k+1\}$ ($\{\tau'_\mu\}$, $\forall \mu \in \{1, 2, \ldots$ $\ldots, k+1\}$), characterizes an unique list of also non-decreasing τ_{k+1} (τ'_{k+1}) positive integers: the subset of the less than or equal to k elements of the global structure at infinity (row structure at infinity). Thus, (26) implies iii) in Theorem 1. □

5. GEOMETRIC SOLUTION OF PNIP(k)

In this section we shall present an alternative solvability condition of PNIP(k), established in geometric terms. To do it, we shall need two preliminary lemmas that we present here without proof in order to avoid unnecessary extension of this paper. In fact, these lemmas are some quite generalization of results given in [4] concerning the *Block Decoupling Problem*.

Lemma 1. Consider a system (A, B, C) and a positive integer k be given, if for a a given positive integer $\mu < k$ the following conditions hold:

$$\mathcal{B} = \sum_{i=1}^{p} \mathcal{B} \cap \mathcal{T}_i^\mu \tag{27}$$

$$\mathcal{V}^\mu = \bigcap_{i=1}^{p} \mathcal{V}_i^\mu \tag{28}$$

$$\mathcal{T}_i^\mu = \bigcap_{j\in\{1,\,2,\,\ldots,\,p\},\,j\neq i}^{p} \mathcal{V}_j^\mu, \quad \forall\, i \in \{1, 2, \ldots, p\}, \tag{29}$$

then (27) and (28) imply:

$$\mathcal{V}^{\mu+1} = \bigcap_{i=1}^{p} \mathcal{V}_i^{\mu+1} \tag{30}$$

and (27) and (29) imply:

$$\mathcal{T}_i^{\mu+1} = \bigcap_{j\in\{1,\,2,\,\ldots,\,p\},\,j\neq i}^{p} \mathcal{V}_j^{\mu+1}, \quad \forall\, i \in \{1, 2, \ldots, p\}. \tag{31}$$

Lemma 2. Consider a system (A, B, C) and a positive integer k be given, if for a a given positive integer $\mu \le k$ the following conditions hold:

$$\mathcal{V}^\mu = \bigcap_{i=1}^{p} \mathcal{V}_i^\mu \tag{32}$$

$$\mathcal{T}_i^\mu = \bigcap_{j\in\{1,\,2,\,\ldots,\,p\},\,j\neq i}^{p} \mathcal{V}_j^\mu, \quad \forall\, i \in \{1, 2, \ldots, p\}, \tag{33}$$

then:

$$\sum_{i=1}^{p} \dim(\mathcal{B} \cap \mathcal{T}_i^{\mu}) = \dim\left(\sum_{i=1}^{p} \mathcal{B} \cap \mathcal{T}_i^{\mu}\right) + (m-1) \cdot \dim(\mathcal{B} \cap \mathcal{V}^{\mu}), \quad \mu \leq k. \tag{34}$$

We can now present:

Theorem 2. Let the positive integer k and system (A, B, C) be given. Then PNIP(k) is solvable if and only if:

$$\mathcal{B} = \sum_{i=1}^{p} \mathcal{B} \cap \mathcal{T}_i^{\mu}, \quad \forall \mu \in \{1, 2, \ldots, k\}. \tag{35}$$

Proof. To prove this theorem, we shall follow in essence the procedure used in [4] to solve the block decoupling problem via regular static state feedback.

Let us first consider that (35) holds.

From (29) the total number of the set of elements of the row structure at infinity of (A, B, C) which order is strictly less than $\mu \in \{1, 2, \ldots, k+1\}$ is given by:

$$\tau'_{\mu} = \sum_{i=1}^{p} \dim(\mathcal{B}) - \sum_{i=1}^{p} \dim\left(\mathcal{B} \cap \mathcal{V}_i^{\mu-1}\right). \tag{36}$$

If (35) holds:

$$\mathcal{B} = \mathcal{B} \cap \mathcal{T}_i^{\mu-1} + \mathcal{B} \cap \mathcal{V}_i^{\mu-1}, \quad \forall i \in \{1, 2, \ldots, p\}, \quad \mu \in \{1, 2, \ldots, k+1\} \tag{37}$$

since $\mathcal{T}_i^{\mu-1} \subset \mathcal{V}_j^{\mu-1}$, for all i, $j \in \{1, 2, \ldots, p\}$, $j \neq i$, and $\mu \in \{1, 2, \ldots, k+1\}$.

By substitution of (37) in (36), we obtain that for all $\mu \in \{1, 2, \ldots, k+1\}$:

$$\tau'_{\mu} = \sum_{i=1}^{p} \dim\left(\mathcal{B} \cap \mathcal{T}_i^{\mu-1}\right) - \sum_{i=1}^{p} \dim\left(\mathcal{B} \cap \mathcal{T}_i^{\mu-1} \cap \mathcal{V}_i^{\mu-1}\right). \tag{38}$$

Now, from Lemma 1, we have:

$$\mathcal{T}_i^{\mu-1} = \bigcap_{i \in \{1, 2, \ldots, p\},\, j \neq i} \mathcal{V}_j^{\mu-1}, \quad \forall \mu \in \{1, 2, \ldots, k+1\}. \tag{39}$$

Then the substitution of (39) in (38) results in:

$$\tau'_{\mu} = \sum_{i=1}^{p} \dim\left(\mathcal{B} \cap \mathcal{T}_i^{\mu-1}\right) - \sum_{i=1}^{p} \dim\left(\mathcal{B} \bigcap_{i=1}^{p} \mathcal{V}_i^{\mu-1}\right). \tag{40}$$

But $\bigcap_{i=1}^{p} \mathcal{V}_i^{\mu-1} = \mathcal{V}^{\mu-1}$, for all $\mu \in \{1, 2, \ldots, k+1\}$, since (28). Then (40) can now be written as follows:

$$\tau'_{\mu} = \sum_{i=1}^{p} \dim\left(\mathcal{B} \cap \mathcal{T}_i^{\mu-1}\right) - m \cdot \dim\left(\mathcal{B} \cap \mathcal{V}^{\mu-1}\right), \tag{41}$$

for all $\mu \in \{1, 2, \ldots, k+1\}$.

Using Lemma 2 for all $\mu \in \{1, 2, \ldots, k+1\}$, and (35), (41) becomes:

$$\begin{aligned} \tau'_\mu &= \dim\left(\sum_{i=1}^{p} \mathcal{B} \cap \mathcal{T}_i^{\mu-1}\right) - \dim\left(\mathcal{B} \cap \mathcal{V}^{\mu-1}\right) \\ &= \dim(\mathcal{B}) - \dim\left(\mathcal{B} \cap \mathcal{V}^{\mu-1}\right) \\ &= \dim\left(\frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{V}^{\mu-1}}\right) =: \tau_\mu, \ \forall \mu \in \{1, 2, \ldots, k+1\}, \end{aligned}$$

and sufficiency has been proved.

For necessity we shall prove by induction that, under the assumption iii) in Theorem 1 (or equivalently: assuming that (26) holds), the following relationships hold for all $\mu \in \{1, 2, \ldots, k\}$:

$$\mathcal{V}^\mu = \bigcap_{i=1}^{p} \mathcal{V}_i^\mu, \tag{42}$$

$$\mathcal{T}_i^\mu = \bigcap_{j \in \{1, 2, \ldots, p\},\, j \neq i} \mathcal{V}_j^\mu, \tag{43}$$

$$\mathcal{B} = \mathcal{B} \cap \mathcal{T}_i^\mu + \mathcal{B} \cap \mathcal{V}_i^\mu, \ \forall i \in \{1, 2, \ldots, p\}, \tag{44}$$

$$\mathcal{B} = \sum_{i=1}^{p} \mathcal{B} \cap \mathcal{T}_i^\mu. \tag{45}$$

These relationships obviously hold for $\mu = 0$. Let us then assume that (42)-(45) hold for some $\mu \in \{1, 2, \ldots, k\}$. By Lemma 1, (42) and (43) hold for $\mu+1$. In order to establish (44) for $\mu+1$, let us write that for all $i \in \{1, 2, \ldots, p\}$:

$$\begin{aligned} \dim\left(\mathcal{B} \cap \mathcal{T}_i^{\mu+1} + \mathcal{B} \cap \mathcal{V}_i^{\mu+1}\right) &= \dim\left(\mathcal{B} \bigcap_{j \in \{1, 2, \ldots, p\},\, j \neq i} \mathcal{V}_j^{\mu+1}\right) \\ &\quad + \dim\left(\mathcal{B} \cap \mathcal{V}_i^{\mu+1}\right) \\ &\quad - \dim\left(\mathcal{B} \cap \mathcal{T}_i^{\mu+1} \cap \mathcal{V}_i^{\mu+1}\right), \end{aligned} \tag{46}$$

since (42) and (43) hold for $\mu+1$.

While developing $\dim\left(\mathcal{B} \bigcap_{j \in \{1, 2, \ldots, p\},\, j \neq i} \mathcal{V}_j^{\mu+1}\right)$ we can write that for all i which belongs to $\{1, 2, \ldots, p\}$:

$$\begin{aligned} \dim\left(\mathcal{B} \cap \mathcal{T}_i^{\mu+1} + \mathcal{B} \cap \mathcal{V}_i^{\mu+1}\right) &= \sum_{i=1}^{p} \dim\left(\mathcal{B} \cap \mathcal{V}_j^{\mu+1}\right) - \sum_{i=1}^{p-2} \dim\left(\mathcal{L}_j^{\mu+1}\right) \\ &\quad - \dim\left(\mathcal{B} \cap \mathcal{V}^{\mu+1}\right), \end{aligned} \tag{47}$$

where $\mathcal{L}_j^{\mu+1}$, for all $j \in \{1, 2, \ldots, p-2\}$, are included in $\mathcal{B}$.

Now, by Corollary 2, the solvability of PNIP(k) amounts to:

$$\tau'_\mu = \tau_\mu, \ \forall \mu \{1, 2, \ldots, k+1\},$$

which is to say:

$$\sum_{i=1}^{p} \dim \left(\frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{V}_i^{\mu+1}} \right) = \dim \left(\frac{\mathcal{B}}{\mathcal{B} \cap \mathcal{V}^{\mu+1}} \right), \ \forall \mu \in \{1, 2, \ldots, k-1\}.$$

Thus:

$$\sum_{i=1}^{p} \dim \left(\mathcal{B} \cap \mathcal{V}_i^{\mu+1} \right) = (m-1) \cdot \dim \left(\mathcal{B} \cap \mathcal{V}^{\mu+1} \right). \tag{48}$$

By substitution of (48) in (47) we obtain:

$$\begin{aligned}
\dim \left(\mathcal{B} \cap \mathcal{T}_i^{\mu+1} + \mathcal{B} \cap \mathcal{V}_i^{\mu+1} \right) &= (m-1) \cdot \dim (\mathcal{B}) - \sum_{i=1}^{p-2} \dim \left(\mathcal{L}_j^{\mu+1} \right) \\
&= \dim (\mathcal{B}) + \sum_{i=1}^{p-2} \dim (\mathcal{B}) - \sum_{i=1}^{p-2} \dim \left(\mathcal{L}_j^{\mu+1} \right) \\
&= \dim (\mathcal{B}) + \sum_{i=1}^{p-2} \dim \left(\frac{\mathcal{B}}{\mathcal{L}_j^{\mu+1}} \right)
\end{aligned}$$

and so:

$$\dim \left(\mathcal{B} \cap \mathcal{T}_i^{\mu+1} + \mathcal{B} \cap \mathcal{V}_i^{\mu+1} \right) = \dim (\mathcal{B}), \ \forall i \in \{1, 2, \ldots, p\},$$

which establishes (44) for $\mu + 1$.

Now, starting from:

$$\mathcal{B} = \mathcal{B} \cap \mathcal{T}_i^{\mu+1} + \mathcal{B} \cap \mathcal{V}_i^{\mu+1}, \ \forall i \in \{1, 2, \ldots, p\}$$

and:

$$\mathcal{V}^{\mu+1} = \bigcap_{i=1}^{p} \mathcal{V}_i^{\mu+1} \tag{49}$$

we obviously have:

$$\begin{aligned}
\mathcal{B} &= \bigcap_{i=1}^{p} \left(\mathcal{B} \cap \mathcal{T}_i^{\mu+1} + \mathcal{B} \cap \mathcal{V}_i^{\mu+1} \right) \\
&= \sum_{i=1}^{p} \mathcal{B} \cap \mathcal{T}_i^{\mu+1} + \mathcal{B} \bigcap_{i=1}^{p} \mathcal{V}_i^{\mu+1},
\end{aligned} \tag{50}$$

since $\mathcal{B} \cap \mathcal{T}_i^{\mu+1} \subset \mathcal{B} \cap \mathcal{V}_j^{\mu+1}$, for all i, $j \in \{1, 2, \ldots, p\}$, $j \neq i$. And so, when using (49), (50) leads to:

$$\begin{aligned}
\mathcal{B} &= \sum_{i=1}^{p} \mathcal{B} \cap \mathcal{T}_i^{\mu+1} + \mathcal{B} \cap \mathcal{V}^{\mu+1} \\
&= \sum_{i=1}^{p} \mathcal{B} \cap \mathcal{T}_i^{\mu+1}
\end{aligned}$$

since $\mathcal{B} \cap \mathcal{V}^{\mu+1} \subset \mathcal{B} \cap \mathcal{T}_i^{\mu+1}$, for all $i \in \{1, 2, \ldots, p\}$.

This shows that (45) also holds for $\mu + 1$, which completes the proof. □

6. CONCLUDING REMARKS

The problem presented here is a weakened version of the famous Decoupling Problem, which corresponds to "infinite order" Partial Non Interaction: indeed, our solution brings back to the classical well-known results for that particular case (see Corollary 1). When Exact (regular) Decoupling is not solvable, our procedure gives more information on this pathology (typically we are able to know from which step the inherent couplings of the system cannot be cancelled).

A similar problem has been introduced in the early 80's [5], related to Partial Model Matching, and its geometric and structural solutions considered in [11], with also an interesting application in the field of systems with delays [12]: the present results will be a starting point for the study of the existence of non-anticipatory solutions for the partial decoupling problem of linear systems with delays ; indeed, as in [12], we can consider, for these systems, the non interacting problem with fixed (finite) horizon k.

Finally, as is done in [11] for the Partial Model Matching, the structural solvability condition of PNIP(k) can also be obtained using algebraic arguments. For that, we need to consider a more general version of the problem, related to *dynamic solutions* and denoted as DPNIP(k) (kth-order Dynamic Partial Non Interacting Problem). This corresponds to control laws of the type:

$$u(s) \;=\; F(s)\,x(s) \,+\, Gv(s),$$

with G invertible. The action of such control law on (1) is equivalent to that of the *biproper* precompensator (see [8]):

$$C(s) \;=\; \left(I_m - F(s)\,(sI_n - A)^{-1}\,B\right)^{-1} G.$$

DPNIP(k) can be formulated as follows:

Let the positive integer k and system (A, B, C) be given. Then DPNIP(k) is solvable if and only if there exists a biproper solution, say $C(s)$, to the matrix equation

$$T(s)\,C(s) \,-\, s^{-(k+1)}P(s) \;=\; T_d(s), \tag{51}$$

where $P(s)$ is any proper rational transfer function matrix and $T_d(s)$ is a proper diagonal transfer function matrix.

Equation (51) let us write:

$$\left[\; T(s) \;\middle|\; -s^{-(k+1)}I_p \;\right] \begin{bmatrix} C(s) \\ \hline P(s) \end{bmatrix} \;=\; T_d(s),$$

which translates it into a problem of Exact Decoupling, $[\ T(s)\ |\ -s^{-(k+1)}I_p\ ]$ stands for the plant and $[\ C^T(s)\ |\ P^T(s)\ ]^T$ for the precompensator which performs the exact decoupling. It is well known (see for instance [4] and [3]) that exact decoupling is solvable if and only if both the global and row structures at infinity of the plant are the same. In the present case this is true if and only if the set of all the elements of the global structure of $T(s)$ which are less than or equal to k is equal to the set of all the elements of the row structure at infinity of $T(s)$ which are less than or equal to k. Indeed, the set of all the elements of the global structure of $[\ T(s)\ |\ -s^{-(k+1)}I_p\ ]$ which are strictly greater than k coincides with the set of all the elements of the row structure of $T(s)$ which are strictly greater than k, since all these elements are equal to $k+1$, because of the presence of $-s^{-(k+1)}I_p$ in $[\ T(s)\ |\ -s^{-(k+1)}I_p\ ]$. As concerns the biproperness of $C(s)$, it can be shown that for the particular case where $k \geq \sup n'_i$ (which is the most interesting case in practice), the solution $[\ C^T(s)\ |\ P^T(s)\ ]^T$ can always be chosen such that $C(s)$ is biproper (see for instance [7]).

Moreover, since the transfer function matrix $[\ T(s)\ |\ -s^{-(k+1)}I_p\ ]$ does not have any finite transmission zero, it happens that finite unstable transmission zeros of $T(s)$ do not play a role in the dynamic solution of partial non interaction when the internal stability of the closed-loop system is required. Remember that this is not the case for *exact* decoupling with stabilily, as is well known (see for instance [13]). In fact, partial non interaction is a particular case of partial model matching and it has been shown in [14] that partial model matching solvability, when internal stability of the closed-loop system is required, only depends on structure-at-infinity information (of course, under the assumption of stabilizability of the system). Finite unstable transmission zeros do not play a role here for the search of internally stable solutions to DPNIP(k), namely through dynamic compensations. The existence of *static* state feedback solutions to PNIP(k) with stability appears to be much more difficult to characterize. A simple 2 inputs-2 outputs counter-example is given in [7] for which partial dynamic solutions exist with stability for all $k \geq 1$, but no static ones.

7. ACKNOWLEDGEMENT

This work has been supported by ESPRIT Basic Research Project No. 8924 (SESDIP).

(Received November 12, 1993.)

REFERENCES

[1] G. Basile and G. Marro: Controlled Invariants and Conditioned Invariants in Linear System Theory. Prentice Hall, New Jersey 1992.

[2] C. Commault and J. M. Dion: Structure at Infinity of Linear Multivariable Systems: A Geometric Approach. Presented at the 20th IEEE Conference on Decision and Control San Diego, CA 1981.

[3] J. Descusse and J. M. Dion: On the structure at infinity of linear square decoupled systems. IEEE Trans. Automat. Control *AC-27* (1982), 3, 971–974.

[4] J. Descusse, J. F. Lafay and M. Malabre: On the structure at infinity of linear block-decouplable systems: The general case. IEEE Trans. Automat. Control *AC-28* (1983), 12, 1115–1118.

[5] E. Emre and L. M. Silverman: Partial model matching of linear systems. IEEE Trans. Automat. Control *AC-25* (1980), 4, 280–281.

[6] P. L. Falb and W. A. Wolovich: Decoupling in the Design and Synthesis of Multivariable Control Systems. IEEE Trans. Automat. Control *AC-12* (1967), 12, 651–659.

[7] A. Godot: Découplage partiel avec stabilité. Mémoire de D. E. A., Laboratoire d'Automatique de Nantes, Ecole Centrale de Nantes, septembre 1994.

[8] M. L. J. Hautus and M. Heymann: Linear feedback, an algebraic approach. SIAM J. Control Optim. *16* (1978), 1, 83–105.

[9] V. Kučera and J. Descusse: On the determination of the structure at infinity of a rational matrix. Annales 1982, pp. 37–44, Ecole Nationale Superieure de Mecanique, Nantes, France, 1982.

[10] M. Malabre: Structure à l'Infini des Triplets Invariants. Application à la Poursuite Parfaite de Modèle. Lecture Notes in Control and Inform. Sci. *44* (1982), 43–53.

[11] M. Malabre and J. C. Martinez Garcia: The partial model matching or partial disturbance rejection problem: Geometric and structural solutions. IEEE Trans. Automat. Control, to appear.

[12] M. Malabre and R. Rabah: Structure at infinity, model matching and disturbance rejection for linear systems with delays. Kybernetika *29* (1993), 5, 485–498.

[13] J. C. Martínez García and M. Malabre: The row by row decoupling problem with stability. IEEE Trans. Automat. Control, to appear.

[14] J. C. Martínez García, M. Malabre and V. Kučera: The partial model matching problem with stability. Systems Control Lett., to appear.

[15] A. I. G. Vardulakis: Linear Multivariable Control: Algebraic Analysis and Synthesis Methods. John Wiley, New York 1991.

[16] C. G: Verghese: Infinite Frequency Behaviour in Generalized Dynamical Systems. PhD Dissertation, Stanford University 1978.

[17] W. M. Wonham: Linear Multivariable Control: A Geometric Approach. Third Edition. Springer–Verlag, New York 1985.

Dr. J. C. Martínez García, CINVESTAV, CIEA-IPN, Depto. de Ing. Eléctrica, Sec. de Control Automático, A. P. 14-740, México 14, D. F., C. P. 07000 Mexico. This work has been performed while the author was on leave to Laboratoire d'Automatique de Nantes. France.

Dr. Michel Malabre, Ecole Centrale de Nantes – Université de Nantes, L.A.N., URA CNRS 823, 1 rue de la Noë, 44072 Nantes Cedex 03. France.

Dr. Rabah Rabah, Ecole Centrale de Nantes – Université de Nantes, L.A.N., URA CNRS 823, 1 rue de la Noë, 44072 Nantes Cedex 03 and Ecole des Mines de Nantes, Département d'Automatique, 4, rue Alfred Kastler, La Chantrerie, 44070 Nantes Cedex 03. France.