KYBERNETIKA — VOLUME 30 (1994), NUMBER 5, PAGES 551 – 562

ON L-ESTIMATORS VIEWED AS M-ESTIMATORS

JAN ŠINDELÁŘ

Arithmetical mean and median usually serve as basic examples of M-estimators ([5]). Both of them are L-estimators. Thus there is a natural question whether there are some other L-estimators which are M-estimators as well. We shall show that, with rare exceptions, this is not the case. More precisely, we shall show that the arithmetical mean and empirical quantiles are the only L-estimators with nonnegative coefficients having a nontrivial ψ-function.

INTRODUCTION

The presented paper has the following source of motivation.

There are many nonstatistical approaches to uncertainty, some of them resulting in their own estimators (like gnostical theory of uncertain data, cf. [4]). Moreover, many people develop their own "problem oriented" estimators. Statistical properties of new-developed estimators are of interest from viewpoint of statistics as well as for practical purposes. E. g. it should be favorable to circumscribe (qualify) the field of successfull applicability of the estimator. Statistics could be a largely developed and well examined source of the desirable information.

But how to find statistical properties of some estimator derived independently of statistics?

A possible way is to verify whether the estimator is an M-estimator e. g. finding some of its ψ-functions. It is a well known fact that the notion of ψ-function playes a central role in theory of M-estimators (cf. [3, 2, 7]). Most of M-estimators are defined on the basis of corresponding ψ-functions. Statistical properties of M-estimators could be derived from their ψ-functions (ibid). Hence if some ψ-function of an M-estimator is found, then the above stated question can be answered using standard statistical methods (see [2]; see also [6] for examples).

Two theorems on problem of determining ψ-functions of given estimator are stated in Section 1. General ideas are then illustrated on a specific class of estimators, namely on the class of L-estimators in Section 2. It is shown that the arithmetical mean and empirical quantiles are the only L-estimators with nonnegative coefficients having nontrivial ψ-functions.

1. M-ESTIMATORS, L-ESTIMATORS

The concept of estimator plays a central role in statistics. Various approaches to this notion can be found in the literature. For instance, an estimator could be a mapping from a sample space into a parametric space (see [5]), or a mapping from a set of probability distribution functions (containing empirical distribution functions) into a set of probability distribution functions (see [7]). For purposes of the presented text we shall view estimators as mappings ascribing reals to sequences of real-valued observations. Hence, with $n \in N = \{1, 2, 3, \ldots\}$ fixed, an estimator T is a mapping from R^n into R, i.e.

$$T : R^n \mapsto R. \tag{1}$$

Consider a measurable space $\langle \Omega, \mathcal{A} \rangle$ equiped with a probability measure P. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed random variables.

An *M-estimator* is obtained by minimizing $\sum_{i=1}^{n} \rho(X_i, \theta)$ where ρ is a given real-valued function (cf. [7]). If ρ has a partial derivative $\psi = \frac{\partial \rho}{\partial \theta}$, then the M-estimator may be defined as a solution of the equation

$$\sum_{i=1}^{n} \psi(X_i, \theta) = 0. \tag{2}$$

The M-estimators getting out of (2) will be considered below. Hence if T is an M-estimator and $n \in N$ is fixed, then

$$\sum_{i=1}^{n} \psi(X_i, T(X_1, \ldots, X_n)) = 0 \quad \text{a.e.} \tag{3}$$

should hold.

Consider $n \in N$ and an estimator T given by (1). Assume that we want to know whether the estimator T is an M-estimator. For this purpose we should find functions ψ satisfying (3), i.e. solve the functioal equation (3) in ψ.

Finding all solutions of (3) could be quite difficult. On the other hand we are usually interested in solutions of (3) satisfying some additional regularity conditions like measurability, continuity, differentiability etc., i.e. solutions of (3) are searched for in some class of functions $\psi : R^2 \mapsto R$. Such a class of functions will be denoted by $\mathcal{F}$.

Finally, we can formulate our task of verifying whether a given estimator is an M-estimator in the following manner. *Given*

- *$n \in N$,*
- *an estimator $T : R^n \mapsto R$,*
- *a class $\mathcal{F}$ of functions ψ, $\psi : R^2 \mapsto R$,*

find all solutions ψ of the functional equation (3) *lying in the class $\mathcal{F}$.*

The set

$$\left\{ \omega \in \Omega \middle| \sum_{i=1}^{n} \psi(X_i, T(X_1, \ldots, X_n)) = 0 \right\} \tag{4}$$

may not be measurable. But this set is measurable under relatively general conditions laid on ψ and T. For instance, if both the function ψ and the estimator T are measurable functions, or, more generally, if

$$\left\{ \langle x_1, \ldots, x_n \rangle \in R^n \,\middle|\, \sum_{i=1}^{n} \psi(x_i, T(x_1, \ldots, x_n)) = 0 \right\} \tag{5}$$

is a Borel subset of R^n, then the set (4) is measurable. For a fixed estimator T, the symbol $\mathcal{F}_T$ denotes the set of all mappings $\psi : R^2 \mapsto R$ the set (4) is measurable which for.

If some additional regularity conditions are laid on an estimator T, on desirable solutions of (3) and on an underlying statistical model, then solution of the functional equation (3) can be reduced to solution of a more simple functional equation. Let us discuss this topic in detail.

Consider $n \in N$ fixed. The random vector $\langle X_1, \ldots, X_n \rangle$ induces a Borel measure on the σ-field $\mathcal{B}_n$ of Borel subsets of R^n denoted by

$$P_{X_1,\ldots,X_n}.$$

Its support will be denoted by

$$Sp\, P_{X_1,\ldots,X_n}.$$

Solution of (3) can be reduced to solution of a more simple functional equation if, for instance,

- T is a continuous mapping,
- continuous solutions of (3) are searched for.

The following two theorems are devoted to the topic.

Theorem 1.1. Suppose that $\psi \in \mathcal{F}_T$. Then ψ is a solution of (3) if

$$\forall \langle x_1, \ldots, x_n \rangle \in Sp\, P_{X_1,\ldots,X_n} : \sum_{i=1}^{n} \psi(x_i, T(x_1, \ldots, x_n)) = 0. \tag{6}$$

Proof. Consider $\psi \in \mathcal{F}_T$. Then the set (3) is measurable. Moreover $\{\omega \in \Omega \mid \langle X_1, \ldots, X_n \rangle \in Sp\, P_{X_1,\ldots,X_n}\}$ is measurable and has the probability one, so that (3) follows from (6). □

We shall call solutions of (6) as *T-solutions.*

Theorem 1.2. Consider $\langle x_1, \ldots, x_n \rangle \in Sp\, P_{X_1,\ldots,X_n}$. Suppose that T is continuous at $\langle x_1, \ldots, x_n \rangle$, ψ is continuous at points $\langle x_i, T(x_1, \ldots, x_n) \rangle$ for $i = 1, \ldots, n$. If ψ is a solution of (3), then

$$\sum_{i=1}^{n} \psi(x_i, T(x_1, \ldots, x_n)) = 0.$$

Proof. Consider a function $f : R^n \mapsto R$ defined by

$$\forall\langle y_1, \ldots, y_n\rangle \in R^n : f(y_1, \ldots, y_n) = \sum_{i=1}^{n} \psi\left(y_i, T(y_1, \ldots, y_n)\right).$$

Hence f is continouus at $\langle x_1, \ldots, x_n\rangle$. We want to show that $f(x_1, \ldots, x_n) = 0$. Consider $0 < \varepsilon$. The interval $I = (f(x_1, \ldots, x_n) - \varepsilon, f(x_1, \ldots, x_n) + \varepsilon)$ is a neighbourhood of $f(x_1, \ldots, x_n)$ and f is continuous at $\langle x_1, \ldots, x_n\rangle$, hence there is an open neighbourhood U of $\langle x_1, \ldots, x_n\rangle$ such that $f(U) \subseteq I$. Now $0 < P_{X_1,\ldots,X_n}(U)$, because $\langle x_1, \ldots, x_n\rangle \in Sp\, P_{X_1,\ldots,X_n}$. So that there is some $\langle y_1, \ldots, y_n\rangle \in U$ such that $f(y_1, \ldots, y_n) = 0$, as follows from (3). Hence $0 \in I$, because $f(U) \subseteq I$. Now $0 < \varepsilon$ is arbitrary and $0 \in (f(x_1, \ldots, x_n) - \varepsilon, f(x_1, \ldots, x_n) + \varepsilon)$, so that $f(x_1, \ldots, x_n) = 0$. □

Corollary 1.1. Suppose that T is a continuous estimator, $\psi : R^2 \mapsto R$ is a continuous function. Then the conditions (3) and (6) are equivalent.

Assume moreover that P_{X_1} is equivalent to the Lebesgue measure on $\mathcal{B}_1$. Then (3) takes place iff

$$\forall\langle x_1, \ldots, x_n\rangle \in R^n : \sum_{i=1}^{n} \psi\left(x_i, T(x_1, \ldots, x_n)\right) = 0.$$

Let us turn to L-estimators.

An order statistics corresponding to $X_1, \ldots, X_n$ will be denoted by $X_{(1)}, X_{(2)}, \ldots$ $\ldots, X_{(n)}$ (see [5], p. 40).

An *L-estimator* has the form

$$\sum_{i=1}^{n} w_i X_{(i)}, \tag{7}$$

where w_i are constants satisfying

$$\sum_{i=1}^{n} w_i = 1 \tag{8}$$

(cf. [5] , pp. 368–369). It is convenient to define $w_1, \ldots, w_n$ by means of a probability distribution on $\langle 0, 1\rangle$ (ibid). In such a case an L-estimator equals (7) with nonnegative $w_1, \ldots, w_n$. Further on, $w_1, \ldots, w_n$ may depend on the value of $X_1, \ldots, X_n$; they are constant (fixed) if observations $X_1, \ldots, X_n$ are different. Then

$$X_{(1)} < X_{(2)} < \cdots < X_{(n)}. \tag{9}$$

In the following we assume that the distribution function of X_1 is equivalent to the Lebesgue measure. Hence (9) is true almost shurely.

Let us use an L-estimator (7) in (3). We obtain

$$\sum_{i=1}^{n} \psi\left(X_{(i)}, \sum_{i=1}^{n} w_i X_{(i)}\right) = 0 \quad \text{a.e.} \tag{10}$$

From our viewpoint an unknown parameter in the equation (10) is the function ψ. Thus the functional equation (10) with "known" w_i and X_i should be solved. The task of the presented text could be thus formulated as follows.

What are the L-estimators for which (10) is solvable (in ψ); how do the solutions of (10) look like?

We shal use the symbols x and y for n-tuples of observed values, i. e. $x = \langle x_1, \dots, x_n \rangle$, where $x_1, \dots, x_n \in R$. We introduce an auxiliary set

$$\mathcal{S} = \langle x \in R^n \,|\, x_1 < x_2 < \cdots < x_n \rangle \tag{11}$$

of ordered and different observations. This is motivated by the fact that $w_1, \dots, w_n$ are fixed for different observations only.

Consider an L-estimator T. Hence

$$\forall x \in \mathcal{S} : T(x) = \sum_{i=1}^{n} w_i \, x_i \tag{12}$$

is true, where $w_1, \dots, w_n$ are some nonnegative constants satisfying (8). Assume moreover that $\psi : R^2 \mapsto R$ is a T-solution. Then

$$\forall \; x \in \mathcal{S} : \; \sum_{i=1}^{n} \psi(x_i, \, T(x)) = 0 \tag{13}$$

is true. Hence any solution of (13) is a candidate for a T-solution.

The estimator T is continuous on $\mathcal{S}$. Therefore any continuous solution of (3) has to satisfy (13), as follows from Theorem 1.2.

We shall found all solutions of (13) *below. For the sake of simplicity we limit ourselves for the case when at least three observations are given, i. e. when $n \geq 3$.*

We shall show that the L-estimators

$$\textit{arithmetical mean} \quad \left(w_1 = w_2 = \cdots = w_n = \frac{1}{n} \right)$$

and

$$\textit{empirical quantile} \quad (w_k + w_{k+1} = 1 \quad \textit{for some } k)$$

are the only L-estimators leading to nontrivial solution of (13). *Moreover we shall found all solutions of* (13).

2. ON T-SOLUTIONS OF L-ESTIMATORS

The functional equation (13) will be solved in ψ having the domain R^2. The class of all such solutions of (13) will be denoted by

$$\Psi_{T,R^2}.$$

Clearly, (13) is true if and only if

$$\forall \; x \in T^{-1}(t) \cap \mathcal{S} : \qquad \sum_{i=1}^{n} \psi(x_i, t) = 0 \tag{14}$$

holds for all $t \in R$.

The following convention will be used repeatedly. If t is not speficied, then $t \in R$ is arbitrary but fixed.

Using this convention we find that each $\psi \in \Psi_{T,R^2}$ satisfies (14).

Two main cases will be considered concerning coefficients $w_1, \dots, w_n$.

(C0) *There are at most two positive consecutive elements among* $w_1, \dots, w_n$ (i.e. at least one of w_{k-1}, w_k, w_{k+1} equals zero for all $k = 2, \dots, n-1$).

(C4) *There are at least three positive consecutive elements among* $w_1, \dots, w_n$ (i.e. w_{k-1}, w_k, w_{k+1} are positive for some $k \in \{2, \dots, n-1\}$).

The former one will be partitioned into the following three subcases

(C1) $w_k = 1,\ k \in \{1, \dots, n\}$.

(C2) $w_k + w_{k+1} = 1$ *with* $w_k,\ w_{k+1}$ *positive,* $k \in \{1, \dots, n-1\}$.

(C3) $w_j = 0$ *and there are* $k < j < \ell$ *with* $w_k,\ w_\ell$ *positive,* $j,\ k,\ \ell \in \{1, \dots, n\}$.

For $J \subseteq \{1, 2, \dots, n\}$ we denote

$$y \sim_J x$$

iff y and x differ at most in coordinates from J, i.e. iff $y_i = x_i$ holds for all $i \in \{1, \dots, n\} \setminus J$.

Lemma 2.1. Let $x,\ y \in T^{-1}(t) \cap \mathcal{S}$ and $x \sim_J y$ take place for some $J \subseteq \{1, \dots, n\}$. If $\psi \in \Psi_{T,R^2}$, then

$$\sum_{i \in J} \psi(x_i, t) = \sum_{i \in J} \psi(y_i, t). \tag{15}$$

Proof. It holds (14) and $\psi(x_i, t) = \psi(y_i, t)$ takes place for all $i \in \{1, \dots, n\} \setminus J$, thus (15) is true. □

Corollary 2.1. Let $\psi \in \Psi_{T,R^2}$.

a) If $w_1 = 0$, then $\psi(\cdot, t)$ is constant on $(-\infty, t)$.

b) If $w_n = 0$, then $\psi(\cdot, t)$ is constant on (t, ∞).

c) If $w_j = 0$ for some $1 < j < n$ and $x \in T^{-1}(t) \cap \mathcal{S}$, then $\psi(\cdot, t)$ is constant on (x_{j-1}, x_{j+1}).

Proof. We prove the part a) only. Consider x_1 and y_1 from $(-\infty, t)$. There are $x_2, \dots, x_n \in R$ such that $x \in T^{-1}(t) \cap \mathcal{S}$. Take $y_i = x_i$ for $i = 2, \dots, n$. Then $\psi(x_1, t) = \psi(y_1, t)$ is true according to Lemma 2.1, hence ψ is constant on $(-\infty, t)$. □

The following two propositions characterize T-solutions of empirical quantiles.

Proposition 2.1 (Case C1). Let $w_k = 1$ for some $k \in \{1, \ldots, n\}$. Then $\psi \in \Psi_{T,R^2}$ iff $\psi : R^2 \mapsto R$ and

$$\forall u_1, u_2, t \in R: u_1 < t < u_2 \Longrightarrow (k-1)\,\psi(u_1,t) + \psi(t,t) + (n-k)\,\psi(u_2,t) = 0. \quad (16)$$

The form of T-solution ψ for Case C1 is explained below. Consider $t \in R$ fixed.

If $k = 1$, then (16) is equivalent to

$$\forall\, u_2, t \in R: \; t < u_2 \Longrightarrow \psi(t,t) + (n-1)\,\psi(u_2,t) = 0$$

and therefore $\psi(\cdot, t)$ is constant on (t, ∞). The function $\psi(\cdot, t)$ can reach arbitrary values on the interval $(-\infty, t)$.

If $1 < k < n$, then (16) implies that $\psi(\cdot, t)$ is constant on each of the intervals $(-\infty, t)$ and (t, ∞). Hence $\psi(\cdot, t)$ can reach at most three values.

If $k = n$, then (16) implies that $\psi(\cdot, t)$ is constant on $(-\infty, t)$. Moreover the function $\psi(\cdot, t)$ can reach arbitrary values on the interval (t, ∞).

Proof. (Proposition 2.1). Assume that $w_k = 1$ for some $k \in \{1, \ldots, n\}$.

If $x \in T^{-1}(t) \cap \mathcal{S}$ is arbitrary, then both

$$x_1 < x_2 < \cdots < x_n \qquad \text{and} \qquad x_k = t$$

hold.

(only if) Let us fix $\psi \in \Psi_{T,R^2}$ and $u_1, u_2, t \in R$ satisfying $u_1 < t < u_2$. Consider $x \in T^{-1}(t) \cap \mathcal{S}$ arbitrary.

a1) Let $k = 1$. Then $w_n = 0$, thus $\psi(\cdot, t)$ is constant on (t, ∞) by Corollary 2.1 b, i.e. $\psi(x_i, t) = \psi(u_2, t)$ holds for $i = 2, \ldots, n$ and $x_1 = t$, so that

$$0 = \sum_{i=1}^{n} \psi(x_i, t) \; = \; \psi(t,t) + (n-1)\,\psi(u_2, t).$$

Therefore (16) is valid, as $k = 1$.

Analysis of the case $k = n$ is similar.

a2) Let $1 < k < n$. Then both $w_1 = 0$ and $w_n = 0$ hold, so that $\psi(\cdot, t)$ is constant on each of the intervals $(-\infty, t)$ and (t, ∞). Therefore $\psi(x_i, t) = \psi(u_1, t)$ holds for $i = 1, \ldots, k-1$ and $\psi(x_i, t) = \psi(u_2, t)$ takes place for $i = k+1, \ldots, n$. Thus (16) is true.

(if) Assume that $\psi : R^2 \mapsto R$ satisfies (16). Let $x \in \mathcal{S}$ and $t = T(x)$. Finally, consider $u_1 < t < u_2$ arbitrary.

b1) Let $k = 1$. Then $\psi(\cdot, t)$ is constant on (t, ∞), thus

$$\sum_{i=1}^{n} \psi(x_i, t) \; = \; \psi(t,t) + (n-1)\,\psi(u_2, t). \quad (17)$$

The right-hand side of (17) equals zero, as follows from (16) and $k = 1$. Thus $\psi \in \Psi_{T,R^2}$.

Analysis of the case $k = n$ is analogical.

b2) Let $1 < k < n$. Then ψ is constant on each of the intervals $(-\infty, t)$ and (t, ∞). Thus

$$\sum_{i=1}^{n} \psi(x_i, t) = (k-1)\,\psi(u_1, t) + \psi(t, t) + (n-k)\,\psi(u_2, t),$$

i.e. $\psi \in \Psi_{T,R^2}$ by (16). □

Proposition 2.2 (Case C2). Let w_k and w_{k+1} be positive reals satisfying $w_k + w_{k+1} = 1$, $k \in \{1, \ldots, n-1\}$. Then $\psi \in \Psi_{T,R^2}$ iff $\psi : R^2 \mapsto R$ and

$$\forall\, u_1,\, u_2,\, t \in R: \; u_1 < t < u_2 \Longrightarrow k\psi(u_1, t) + (n-k)\,\psi(u_2, t) = 0. \tag{18}$$

As can be easily seen, (18) implies that $\psi(\cdot, t)$ is constant on each of the intervals $(-\infty, t)$ and (t, ∞).

Proof. (Proposition 2.2). Let w_k and w_{k+1} be positive reals satisfying $w_k + w_{k+1} = 1$, $k \in \{1, \ldots, n-1\}$.

If $x \in T^{-1}(t) \cap \mathcal{S}$ is arbitrary, then

$$t = T(x) = w_k\, x_k + w_{k+1}\, x_{k+1} \tag{19}$$

takes place. Thus $x_i < t$ holds for $i = 1, \ldots, k$ and $t < x_i$ holds for $i = k+1, \ldots, n$.

(only if) Let $\psi \in \Psi_{T,R^2}$ and u_1, u_2, $t \in R$ satisfying $u_1 < t < u_2$ be fixed.

a1) Let $k = 1$. In this case $x \in T^{-1}(t) \cap \mathcal{S}$ can be found such that $x_1 = u_1$. Moreover $w_n = 0$, thus $\psi(\cdot, t)$ is constant on (t, ∞) by Corollary 2.1 b, i.e. $\psi(x_i, t) = \psi(u_2, t)$ holds for $i = 2, \ldots, n$. Therefore $\psi \in \Psi_{T,R^2}$ implies $0 = \psi(u_1, t) + (n-1)\,\psi(u_2, t) = k\psi(u_1, t) + (n-k)\,\psi(u_2, t)$.

Analysis of the case $k + 1 = n$ is analogical.

a2) Let $1 < k$ and $k+1 < n$. In this situation $w_1 = w_n = 0$, thus $\psi(\cdot, t)$ is constant on both $(-\infty, t)$ and (t, ∞), so that $\psi \in \Psi_{T,R^2}$ implies $k\psi(u_1, t) + (n-k)\,\psi(u_2, t) = 0$.

(if) Assume that $\psi : R^2 \mapsto R$ satisfies (18). Let $x \in \mathcal{S}$ and $t = T(x)$. Finally, let $u_1 < t < u_2$ be arbitrary elements of R. Then $\psi(\cdot, t)$ is constant on each of the intervals $(-\infty, t)$ and (t, ∞), which gives

$$\sum_{i=1}^{n} \psi(x_i, t) = k\psi(u_1, t) + (n-k)\,\psi(u_2, t).$$

Thus (18) implies $\psi \in \Psi_{T,R^2}$. □

Trivial mapping from R^2 into R will be denoted by σ. Thus $\sigma(u, t) = 0$ for all $\langle u, t \rangle \in R^2$.

Proposition 2.1 (Case C3). In the Case C3 it holds $\Psi_{T,R^2} = \{\sigma\}$.

Proof. Consider $\psi \in \Psi_{T,R^2}$. Let $(a, b) \subseteq R$ be an open interval. Clearly, it suffices to prove that $\psi(\cdot, t)$ is constant on (a, b). We have $k < j < l$ with w_k, w_l positive and $w_j = 0$. Hence there is $x \in T^{-1}(t) \cap \mathcal{S}$ satisfying $x_{j-1} < a$ and $b < x_{j+1}$. Thus $\psi(\cdot, t)$ is constant on (a, b) by Corollary 2.1 c. □

It remains to analyze the Case C4 when at least three consecutive coefficients among $w_1, \ldots, w_n$ are positive. We shall show that for any T-solution ψ and any fixed $t \in R$ the function

$$\psi(t + ., t) - \psi(t, t)$$

is additive in this case. Using this fact we prove that if $\Psi_{T,R^2} \neq \{\sigma\}$, then T is the arithmetical mean.

It is worth mentioning that a function $f: R \mapsto R$ is called *additive iff*

$$f(u + v) = f(u) + f(v) \tag{20}$$

takes place for all $u, v \in R$.

In the following a slightly more general functional equation then that of (20) is analyzed, namely a special case of the so-called Pexider's equation is used (see [1], pp. 141–142).

Lemma 2.2. Let $g, f: R \mapsto R$ and $\alpha \in (0, \infty)$. If

$$g(u + v) = f(u) + g(v) \tag{21}$$

holds for any $u, v \in R$ satisfying the constraints

$$0 < u + v \quad \text{and} \quad -\alpha\,(u + v) < u < \alpha\,(u + v), \tag{22}$$

then f is additive.

Proof. Consider $s_1, s_2 \in R$ arbitrary. Let us take some $s \geq \left(\frac{1}{\alpha} + 1\right) \cdot (|s_1| + |s_2|)$. We use (21) and subsequently put $u = s_1 + s_2$ and $v = s$; $u = s_1$ and $v = s_2 + s$; $u = s_1$ and $v = s$. It is possible to do it because the constraints (22) are fulfilled in all these three cases. We add the last two obtained equations and substract the first one from the result. We find that $f(s_1 + s_2) = f(s_1) + f(s_2)$ is true. □

Corollary 2.2. Let $g: R \mapsto R$ and $\alpha, \beta \in (0, \infty)$. Assume that

$$g(u + v) = g(\beta\, u) + g(v) - g(0) \tag{23}$$

holds for any $u, v \in R$ satisfying (22). Then $g(.) - g(0)$ is additive.

Proof. The function $g(\beta \cdot .) - g(0)$ is additive by Lemma 2.2, thus $g(.) - g(0)$ is additive as well. □

Lemma 2.3. Let w_{k-1}, w_k and w_{k+1} be positive for some $k \in \{2, \ldots, n-1\}$. If $\psi \in \Psi_{T,R^2}$ and $t \leq w$, then

$$\psi(w + ., t) - \psi(w, t)$$

is additive.

Proof. a) We shall consider points $x, y \in T^{-1}(t) \cap \mathcal{S}$ satisfying $x \sim_{\{k,k+1\}} y$. Thus it should hold both

$$w_k\, x_k + w_{k+1}\, x_{k+1} = w_k\, y_k + w_{k+1}\, y_{k+1}$$

and

$$\psi(x_k, t) + \psi(x_{k+1}, t) = \psi(y_k, t) + \psi(y_{k+1}, t).$$

The differences $x_{k+1} - x_k$ and $x_{k+1} - y_{k+1}$ play key role in the proof. For this reason we rewrite the above stated equalities as

$$y_k = x_k + (x_{k+1} - y_{k+1}) \cdot \frac{w_{k+1}}{w_k} \tag{24}$$

and

$$\begin{aligned} &\psi(x_k, t) + \psi\left[x_k + (x_{k+1} - x_k), t\right] = \qquad (25) \\ &= \psi\left(x_k + (x_{k+1} - y_{k+1}) \cdot \frac{w_{k+1}}{w_k}, t\right) + \psi\left[x_k + (x_{k+1} - x_k) - (x_{k+1} - y_{k+1})\right]. \end{aligned}$$

We specify points x, y mentioned above.

a1) Let us fix x_k, x_{k+1} satisfying

$$t \le x_k < x_{k+1} \tag{26}$$

(but otherwise arbitrary).

a2) Further on, let y_{k+1} satisfy

$$-\alpha(x_{k+1} - x_k) < x_{k+1} - y_{k+1} < \alpha(x_{k+1} - x_k), \tag{27}$$

where

$$\alpha = w_k.$$

Now y_k is computed using (24).

It holds $y_k < y_{k+1}$, as follows from (27) and (24) (namely, $y_k < x_k + w_{k+1}(x_{k+1} - x_k) < x_{k+1} - w_k(x_{k+1} - x_k) < y_{k+1}$).

a3) We take an auxiliary open interval (a, b), where

$$\begin{aligned} a &= x_k - w_{k+1}(x_{x+1} - x_k) \\ b &= x_{k+1} + w_k(x_{x+1} - x_k). \end{aligned}$$

All y_{k+1} satisfying (27) and all corresponding y_k computed by (24) lie in (a, b). Moreover, we introduce an auxiliary constant $x_{n+1} = +\infty$.

There are $x_1, \ldots, x_{k-1}$ less then a and $x_{k+2}, \ldots, x_{n+1}$ greater then b such that $x \in T^{-1}(t) \cap \mathcal{S}$ (which follows from $2 \le k$, $t \le x_k$ and $0 < w_{k-1}, w_k, w_{k+1}$). We put $y_i = x_i$ for all $i \in \{1, \ldots, n\} \setminus \{k, k+1\}$.

Thus $x, y \in T^{-1}(t) \cap \mathcal{S}$ and $x \sim_{\{k,k+1\}} y$ take place. Therefore under the constraints (26) and (27) the equality (25) holds.

b) Let us put

$$w = x_k, \quad u + v = x_{k+1} - x_k, \quad u = x_{k+1} - y_{k+1}.$$

With this substitution (25) converts into

$$\psi(w, t) + \psi(w + u + v, t) = \psi\left(w + \frac{w_{k+1}}{w_k} u, t\right) + \psi(w + v, t)$$

and constraints (26) and (27) convert into (22). If we set $g(.) = \psi(w + ., t)$ and $\beta = \frac{w_{k+1}}{w_k}$, we find that g satisfies (23). Thus $g(.) - g(0) = \psi(w + ., t) - \psi(w, t)$ is additive by Corollary 2.2. $\square$

Proposition 2.4 (Case C4). a) Let

$$w_1 = w_2 = \cdots = w_n = \frac{1}{n}. \tag{28}$$

Then $\psi \in \Psi_{T,R^2}$ iff $\psi : R^2 \mapsto R$ and $\psi(t + ., t)$ is additive for all $t \in R$.

b) Let $k \in \{2, \ldots, n-1\}$ be such that w_{k-1}, w_k, w_{k+1} are positive. If (28) does not hold, then $\Psi_{T,R^2} = \{\sigma\}$.

Proof. Let w_{k-1}, w_k, w_{k+1} be positive, $\psi \in \Psi_{T,R^2}$. Let us fix $t \in R$. Then $\psi(t + ., t) - \psi(t, t)$ is additive by Lemma 2.3. If $x \in T^{-1}(t) \cap \mathcal{S}$, then

$$0 = \sum_{i=1}^{n} \psi(x_i, t) = \sum_{i=1}^{n} \{\psi(t + [x_i - t], t) - \psi(t, t)\} + n\psi(t, t)$$

is true, thus

$$0 = \psi\left(t + \sum_{i=1}^{n}[x_i - t], t\right) + (n-1)\,\psi(t, t) \tag{29}$$

holds.

(Part a) Assume that (28) takes place, $x \in T^{-1}(t) \cap \mathcal{S}$. Then

$$\sum_{i=1}^{n}[x_i - t] = 0. \tag{30}$$

(only if) Let $\psi \in \Psi_{T,R^2}$. Then $\psi(t,t) = 0$ by (29) and (30), hence $\psi(t + ., t)$ is additive.

(if) Suppose that $\psi(t + ., t)$ is additive for all $t \in R$. Then

$$\sum_{i=1}^{n} \psi(x_i, t) = \sum_{i=1}^{n} \psi(t + [x_i - t], t) = \psi\left(t + \sum_{i=1}^{n}[x_i - t], t\right) \tag{31}$$

is true, so that

$$\sum_{i=1}^{n} \psi(x_i, t) = \psi(t + 0, t)$$

is valid by (30). On the other hand $\psi(t + 0, t) = 0$ follows from additivity of $\psi(t + ., t)$. Thus $\psi \in \Psi_{T,R^2}$.

(Part b) Assume for contrary that $\psi \in \Psi_{T,R^2}$ is nontrivial, i.e that $\psi(u,t) \neq 0$ for some $\langle u, t\rangle \in R^2$. Then $\psi(t + ., t) - \psi(t,t)$ is nonconstant (otherwise $\psi(t + ., t)$ is constant; there is $x \in T^{-1}(t) \cap \mathcal{S}$; thus $0 = \sum_{i=1}^{n} \psi(x_i, t) = n\psi(u, t)$, so that $\psi(u, t) = 0$ which is a contradiction).

The relation (28) does not hold, thus there is a nonempty open interval $(a, b) \subseteq R$ such that

$$(a, b) \subseteq \left\{ \sum_{i=1}^{n}[x_i - t] \,\middle|\, x \in T^{-1}(t) \cap \mathcal{S} \right\}.$$

So that $\psi(t + ., t) - \psi(t, t)$ is constant on (a, b) by (29).

Now $\psi(t + \cdot, t) - \psi(t,t)$ is nonconstant and additive, thus it is nonconstant on any nonempty open interval (e. g. on (a,b)) which is a contradiction. □

Let T_m be the arithmetical mean, i. e. let

$$T_m(x) = \sum_{i=1}^{n} \frac{1}{n} x_i$$

hold for each $x \in R^n$.

We consider T_m-solutions, i. e. "arithmetical mean"-solutions, satisfying weak additional regularity conditions. Namely, weak type of measurability of T_m-solutions will be assumed.

We denote

$$\mathcal{F} = \left\{\psi : R^2 \mapsto R \,|\, \psi(\cdot, t) \text{ is measurable for each } t \in R\right\}.$$

Proposition 2.5. It holds $\psi \in \Psi_{T_m,R^2} \cap \mathcal{F}$ *iff* $\psi : R^2 \mapsto R$ and

$$\forall u, t \in R : \psi(u,t) = (u - t) \cdot h(t), \tag{32}$$

where $h : R \mapsto R$ is arbitrary.

Proof. (only if) Let $\psi \in \Psi_{T_m,R^2} \cap \mathcal{F}$, $t \in R$. Then $\psi(t + \cdot, t)$ is both additive and measurable, thus $\psi(t+v,t) = v \cdot \psi(t+1,t)$ holds for any $v \in R$, i. e. $\psi(u,t) = (u-t) \cdot \psi(t+1,t)$ is true for all $u \in R$. Put $h(t) = \psi(t+1,t)$ for all $t \in R$.
(if) Let $h : R \mapsto R$ be arbitrary, ψ be defined by (32). Then $\psi(t + \cdot, t)$ is additive, so that $\psi \in \Psi_{T_m,R^2}$ by Proposition 2.4 a. Clearly, $\psi \in \mathcal{F}$. □

(Received March 22, 1993.)

REFERENCES

[1] J. Aczél: Lectures on Functional Equations and their Applications. Academic Press, New York – London 1966.
[2] F. R. Hampel, E. M. Ronchetti, P. J. Rousseeuw, and W. A. Stahel: Robust Statistics. The Approach Based on Influence Functions. John Wiley & Sons, New York – Chichester – Brisbane – Toronto – Singapore 1986.
[3] P. J. Huber: Robust Statistics. John Wiley & Sons, New York – Chichester – Brisbane – Toronto 1981.
[4] P. Kovanic: Gnostical theory of small samples of real data. Problems Control Inform. Theory *13* (1984), 5, 303–319.
[5] E. L. Lehman: Theory of point estimation. J. Wiley, New York – Chichester – Brisbane – Toronto – Singapore 1983.
[6] J. Novovičová: M–estimators and gnostical estimators for identification of a regression model. Automatica *26* (1990), 3, 607–610.
[7] I. Vajda: Theory of Statistical Inference and Information. Kluwer Academic Publishers, Dordrecht – Boston 1989.

RNDr. Jan Šindelář, CSc., Ústav teorie informace a automatizace AV ČR (Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic), Pod vodárenskou věží 4, 182 08 Praha. Czech Republic.