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BILINEAR SYSTEMS AND CHAOS

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A single-input homogeneous bilinear control system in R^3 is investigated to determine a class of bilinear control systems for which the feedback can lead to the chaotic behavior of the corresponding closed loop system. A simple and clearly formulated conjecture on the positive solution of the above problem is stated and investigated, both by analytical methods and by means of the computer simulation. It is shown that a rich class of bilinear systems with the chaotic behavior of the closed loop system exists, i. e. very simple by their form nonlinear systems can have very complicated behavior.

1. INTRODUCTION

The increasing interest in the investigation of the bilinear systems during the end of the 60ties and during the 70ties was mainly motivated by their relative simplicity and similarity to the linear systems, that provided a good opportunity for the transition from the linear control theory to the nonlinear one (see e. g. [15]). However, as the result of the same simplicity, we can observe during the last ten years substantial decrease of the number of publications on this topic. In other words, bilinear systems are presently considered at the theoretical level as too restrictive and exhausted model.

We aim, however, to present here a class of bilinear systems in R^3 with the extremely complex structure, that gives opportunity for the synthesis of the chaotic behavior. In other words, slightly nonlinear by its form system can give rise to such a complicated, strongly nonlinear behavior.

Besides the purely theoretical interest there are several reasons, why to try synthesize the chaotic behavior. The chaotic behavior is practically long term unpredictable due to the so-called sensitive dependence on initial data, nevertheless, it is *bounded.* The first motivation to study synthesis of the chaotic behavior is therefore the pragmatic one: bounded behavior may be considered as a reasonable substitute for the stabilization to a single equilibrium point.

Nevertheless, deeper ideas supporting the chaos synthesis are presently available as the result of the change of the general paradigm on the role of the chaos in the

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natural processes: the chaos is now evaluated as the positive phenomenon rather then the negative one (cf. [4, 8, 16, 18]), e. g. as the attribute of the healthy behavior of the biological systems or good mixing.

Before going into the detailed exposition we give some notions and notations that will be frequently used.

$M_n(R)$ stands for the vector space of real $n \times n$ matrices and $so(n) \subset M_n(R)$ denotes its subspace of skew-symmetric matrices. As usual, $A \in M_n(R)$ is called stable (unstable) if all eigenvalues of A have negative (positive) real parts. The matrix is called semistable if it is neither stable, nor unstable and all their eigenvalues has nonzero real parts.

For a given smooth vector field $f(x) \in V(R^n)$ we define dynamical system as the following differential equation

$$\dot{x} = f(x), \qquad x \in R^n. \tag{1}$$

Trajectory of the dynamical system starting from $x_0 \in R^n$ in the zero time will be denoted as $x(t, x_0)$, or $x(t)$ when the initial state is obvious. Some basic facts from the dynamical systems theory are collected in the Appendix.

Throughout the paper we consider the single-input homogeneous bilinear control system (BLS) of the form

$$\dot{x} = Ax + Bxu, \quad x \in R^3, \quad A, B \in M_3(R), \quad u \in R. \tag{2}$$

For a given smooth nonconstant feedback

$$u = \alpha(x)$$

we obtain nonlinear dynamical system referred as the closed loop BLS (2) corresponding to the above feedback.

The chaotic behavior of dynamical systems has been extensively studied in the past twenty years and many different approaches have been followed based on mathematical analysis as well as on experimental observations and on computer simulations (cf. e. g. [12, 10, 17] for basic information and further references). At the same time, the number of papers that deal with chaos in control systems is substantially lower (cf. [1, 14, 11, 18, 5, 6, 7]) and only the papers [2, 19, 20] are related to the problem of the chaos synthesis in the continuous time nonlinear control systems.

We state here a conjecture on the chaos synthesis for a subclass of BLS (2) and use some results and ideas from global bifurcation theory (cf. [9, 21]) as well as computer simulation to support this conjecture.

We conclude this section by giving more precise description of the notion 'chaotic behavior'. The field of chaos theory is very extensive and numerous views and definitions are possible. Intuitively, 'chaotic' is understood as deterministic, but practically unpredictable long term behavior. In this paper we will deal with the deterministic continuous dynamical systems described by the system of the ordinary differential equations. In this framework the term 'chaotic behavior' or 'chaotic motion' could be understood as nonvanishing but bounded motion with the sensitive dependence on initial conditions. The last property particularly means that the motion is neither periodic nor quasiperiodic. The sensitive dependence on initial conditions means that we are not able to guarantee that arbitrarily small changes in the initial conditions would not cause principally different long term behaviors of the appropriate bounded trajectories (the short term behaviors are of course closed each to other due to the smooth dependence on initial conditions). Another property of the chaotic motion is that it fills densely a certain (generalized) volume in the state space. Practically, during the long term behavior we are only able to predict that the state should be somewhere in this volume.

We will not give the complete and precise definition in this paper, since numerous definitions are possible and all of them require relatively extensive exposition to the dynamical systems and global bifurcations theory (see [9, 21]). The paper is therefore organized in such a way that except Section 5, where some analytical results based on global bifurcation theory are given, the remaining parts of the paper operates only with the just intuitively described notion of the chaos. Another reason for doing so is the fact that except Section 5 the justification is performed mainly by computer simulations. Interested reader may find more detailed characterization of chaos in the Appendix.

2. ROTATED SEMISTABLE BILINEAR SYSTEMS. MAIN CHAOS CONJECTURE

Let us consider BLS (2) with A semistable and $B \in so(3)$, such a BLS will be further referred to as the rotated semistable bilinear system (RSBLS). The class of RSBLS is fairly general, it covers all BLS (2) with matrix B having only eigenvalues with the zero real parts and matrix A semistable.

It is intuitively clear that if the axis of infinitesimal rotation defined by $B \in so(3)$ is suitably placed between expanding and contracting directions of the matrix A, than, using appropriate feedback, one can force trajectories of the corresponding closed loop system to leave the expanding directions region and enter the contracting directions region of the matrix A. As a result, one may expect at least bounded behavior of the closed loop system and in some cases even to stabilizability of the RSBLS (see [3]) .

In this paper we aim to focus on the former possibility rather than on the latter one, since using the feedback, that made the closed loop system behave in a bounded, but locally unstable way, it is possible to introduce very complicated motions.

Now we are prepared to state a conjecture on chaos in RSBLS. First, let us introduce the following condition for eigenvalues of the matrix A :

$$\left.\begin{array}{lll} & -\lambda_2 > \lambda_1 > -\lambda_3 > 0 & \text{for} \quad \lambda_1, \lambda_2, \lambda_3 \in R, \\ \text{or} & & \\ & \lambda_1 > -\rho > 0 & \text{for} \quad \lambda_1 \in R, \quad \lambda_{2,3} = \rho \pm i\omega \in C. \end{array}\right\} \tag{3}$$

MAIN CONJECTURE ON CHAOS:

RSBLS with eigenvalues of the matrix A satisfying conditions (3) has an open region of directions of the axis of infinitesimal rotation defined by the matrix B, such that there is a feedback which leads to the chaotic behavior of the corresponding closed loop system.

The conditions (3) are necessary in order to have possibility to apply appropriatte results from global bifurcation theory (cf. [21]). These results in a significant manner use symmetries, so the mentioned feedback will be usually chosen to introduce a symmetry in the corresponding closed loop system. Generally, this feedback will be quadratic, resulting in a symmetry $x \to -x$ or in some cases even *linear* feedback will be sufficient, introducing more special types of symmetry.

We finish this section by stating without the proof the following simple property of the RSBLS.

Lemma 1. Consider RSBLS and apply to it two linear feedbacks $l_2(x), l_1(x)$, resulting in two closed loop systems, such that $l_2(x) = vl_1(x), v \in R, v \neq 0$. Then the trajectories of the corresponding closed loop systems are related as $x^1(t) = vx^2(t)$. The analogous statement holds also for the case of quadratic feedback, where $q_2(x) = v^2 q_1(x)$ leads to $x^1(t) = vx^2(t)$ for the corresponding trajectories.

Particularly, Lemma 1 means that being able to impose to the RSBLS bounded behavior we are also able to make this behavior arbitrarily close to the origin (of course, the price for doing this will be great amplification in the feedback loop.)

3. LORENZ EQUATIONS AND RSBLS

In [13] Lorenz, a former student of Birkhoff and then a meteorologist, presented an analysis of coupled set of three quadratic ordinary differential equations, representing the model for fluid convection in a two dimensional layer heated from below. Since that time a great effort was given to study extremely complicated behavior of these equations (cf. e.g. [9, 10, 17] for details and further references), both by analytical methods and by computer simulations. The Lorenz equations have the following form

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{aligned} \tag{4}$$

where $(x, y, z) \in R^3$ and $\sigma, r, b > 0$ are physical parameters. It was shown that there exist values of parameters σ, r, b for which the equations (2) present chaotic behavior, i.e. bounded, nonvanishing motions with strong dependence on initial data.

Lorenz equations (further LE) can be interpreted using the class of RSBLS, namely, consider BLS (2) with

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5}$$

and apply the linear feedback of the form

$$u = c^{\mathrm{T}} x, \quad c \in R^n, \tag{6}$$

with $n = 3$, $c^{\mathrm{T}} = (1, 0, 0)$. Then the LE is immediately obtained. It can be easily seen that for $r > 1$ matrix A has real eigenvalues of both positive and negative signs, so the LE can be viewed as the result of application of a *linear* feedback to RSBLS. This fact constitutes partial motivation for the main conjecture formulated in the previous section. Let us note that for $r \leq 1$ is LE (4) globally asymptotically stable to the origin. It is known that increasing r the motions of LE (4) became more and more complicated and finally result into a chaotic behavior.

On the other hand, RSBLS (2), (5) exhibits the following simple, but little bit surprising property.

Assertion 1. RSBLS (2), (5) is globally asymptotically stabilizable by the *constant* feedback for any positive values of parameters $\sigma, r, b \in R$.

Proof. For sufficiently large $u_s \in R$ the function $V(x) = (1/2)(x_1^2 + \sigma x_2^2 + \sigma x_3^2) - (\sigma(1 + r)/u_s)x_1 x_3$ is obviously the Lyapunov one for the equation $\dot{x} = (A + u_s B)x$. □

In other words, for BLS (2), (5) we can have both stable and chaotic closed loop system. Generally it is not true, we will show that there are nonstabilizable RSBLS for which the feedback leading to chaos in the corresponding closed loop system is possible.

We finish this section by introducing a subclass of the class of RSBLS that is generalization of BLS (2), (5). Namely, consider BLS (2) with

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{7}$$

where submatrix $\tilde{A} = (a_{ij}) \in M_2(R)$ has eigenvalues $\lambda_1, \lambda_2 \in R$, $-\lambda_2 > \lambda_1 > -\lambda_3 > 0$. Such a system will be further referred to as the bilinear system of the Lorenz type (BLSLT). The closed loop system resulting from the BLS (2), (7) after application of the linear feedback (6) will be called as the generalized Lorenz equation (GLE).

4. BILINEAR SYSTEM OF THE LORENZ TYPE

We describe here some basic properties of the BLSLT and GLE. We introduce also various simplifying transformations of these models.

Assertion 2. The BLSLT (2), (7) is constant feedback asymptotically stabilizable iff

$$\lambda_1 + \lambda_2 + \lambda_3 < a_{11} < 0. \tag{8}$$

Proof. One can easily check that characteristic polynomial of the matrix $(A + uB)$, where A, B are as in (7), $u \in R$, has the following form

$$P_{(A+uB)}(x) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) + (a_{11} - \lambda)u^2.$$

Routh–Hurwitz criterion gives after minor calculations the following three inequalities as the necessary and sufficient condition for negativeness of the real parts of roots of the equation $P_{(A+uB)}(\lambda) = 0$

$$\begin{gathered}\lambda_1\lambda_2\lambda_3 + a_{11}u^2 < 0,\\ \lambda_1 + \lambda_2 + \lambda_3 < 0,\\ (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3) < (a_{11} - (\lambda_1 + \lambda_2 + \lambda_3))u^2.\end{gathered}$$

The assertion follows immediately from these inequalities since $(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_1+\lambda_3) > 0$ and $\lambda_1\lambda_2\lambda_3 > 0$. □

It is not difficult to see that (8) is equivalent to

$$a_{11} < 0 \quad \text{and} \quad a_{22} < -\lambda_3 \tag{9}$$

since $a_{11} + a_{22} = \lambda_1 + \lambda_2$. It can be shown that the violation of the first of these inequalities leads to the unbounded behavior of the corresponding closed loop systems. On the other hand, the second of these inequalities is necessary only for constant feedback stabilization.

Assertion 3. Suppose BLSLT (2), (7) is given and let $a_{11} < 0, \lambda_1 > a_{22} \geq 0, a_{12} \neq 0$. Then, applying the feedback $u = x_1$, we obtain bounded behavior of the corresponding closed loop system. More exactly there exists bounded set $\mathcal{E} \subset R^n$ such that for any $x_0 \in R^n$ the trajectory of the closed loop system with this initial state enter $\mathcal{E}$ and remain within it.

The following simple lemma stated without proof will be useful for the verification of Assertion 3.

Lemma 2. Transformation $y = Sx$, $S = \operatorname{diag}\{1, v, v\}$, $v \in R \setminus \{0\}$ takes BLSLT (2), (7) into the following form

$$\tilde{A} = SAS^{-1} = \begin{pmatrix} a_{11} & a_{12}/v & 0\\ va_{21} & a_{22} & 0\\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \tilde{B} = SBS^{-1} = B. \tag{10}$$

Proof of Assertion 3. Due to Lemma 2 it is sufficient to prove the statement only for the case when $a_{12} = a_{22} + \beta$, $\beta > 0$. Consider quadratic function $V(x) = (1/2)[\epsilon x_1^2 + (x_1 - x_2)^2 + (x_3 - k)^2]$ where $\epsilon > 0$ and $k \in R$ will be further specified. We have for the trajectories of the investigated closed loop systems $x(t)$:

$$\begin{gathered}\tfrac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = V_x\dot{x}(t) = ((1+\epsilon)a_{11} - a_{21} + x_3)x_1^2 - \beta x_2^2 +\\ ((1+\epsilon)(a_{22}+\beta) + a_{21} - a_{11} - a_{22} - k)x_1x_2 + \lambda_3 x_3(x_3 - k).\end{gathered}$$

Choosing $k = (1+\epsilon)\beta + \epsilon a_{22} - a_{11} + a_{21}$ one obtain $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = ((1+\epsilon)a_{11} - a_{21} + x_3)x_1^2 - \beta x_2^2 + \lambda_3 x_3(x_3 - k)$, or $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = ((1+\epsilon)a_{11} - a_{21} + x_3)x_1^2 - \beta x_2^2 + \lambda_3(x_3 -$

$\frac{k}{2})^2 - \lambda_3 \frac{k^2}{4}$. Suppose $x_3 \leq a_{21} - (1+\epsilon)a_{11} - \mu$ for some $\mu > 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq -\mu x_1^2 - \beta x_2^2 + \lambda_3 (x_3 - \frac{k}{2})^2 - \lambda_3 \frac{k^2}{4}.$$

Let us consider ellipsoid E given by $\mu x_1^2 + \beta x_2^2 - \lambda_3(x_3 - \frac{k}{2})^2 = -\lambda_3(k^2/4) + \nu$, $\nu > 0$. Now, obviously, for all solid ellipsoids E_c given by $V(x) \leq C$, $C \geq \tilde{C} > 0$, such that $E_{\tilde{C}}$ contains E, we have that trajectories can leave these ellipsoids only if they cross the plane P given by $x_3 = a_{21} - (1+\epsilon)a_{11} - \mu$, $\mu > 0$.

Further, consider bounded sets $\tilde{P_C} \subset P$ with the boundary $E_C \cap P$ (it may happen that $E_{\tilde{C}} \cap P_{\tilde{C}} \neq \emptyset$, in this case $E_{\tilde{C}}$ is the desired bounded set $\mathcal{E}$), obviously $\tilde{P}_C = \tilde{P}^C_{OUT} \cup \tilde{P}^C_{TO}$, where trajectories intersecting $\tilde{P}^C_{OUT}$ enter halfspace $x_3 > a_{21} - (1+\epsilon)a_{11} - \mu$ while trajectories intersecting $\tilde{P}^C_{TO}$ enter the opposite one.

Now, let us note, that:

(i) for $x_3 \geq a_{21} - a_{11}a_{22}/(a_{22} + \beta)$ the eigenvalues of the matrix

$$\begin{pmatrix} a_{11} & a_{22} + \beta \\ a_{21} - x_3 & a_{22} \end{pmatrix} \tag{11}$$

has only the negative real parts.

(ii) for $x_1 x_2 < -\lambda_3 x_3$ we have that $\dot{x}_3(t) < 0$.

It follows from (i), (ii) that any trajectory that leave a solid ellipsoid E_C, $C \geq \tilde{C}$, through its crossing the plane P should after some time return to this ellipsoid.

Finally, let us consider bounded set B defined as $B = \left\{ y \in R^3 : \exists\, t \geq 0, x_0 \in \tilde{P}^{\tilde{C}}_{OUT}\right.$, $\left. y = x(t, x_0) \right\}$. Now, it is clear that we can take $\mathcal{E} = B \cup E_{\tilde{C}}$ to obtain the desired assertion. □

Remark 1. Assertion 3 deals only with case when $a_{22} \geq 0$, $a_{11} < 0$, $a_{22} + a_{11} = \lambda_1 + \lambda_2 < 0$. For the case $a_{22} < 0, a_{11} < 0$ we can prove the same result in a much more trivial way, that is fully analogous to the proof of the boundedness of LE (cf. [17]). Moreover, by Assertion 2, we have for this case even constant feedback asymptotical stabilization.

Remark 2. For $a_{11} \geq 0$ and/or $a_{12} = 0$ there obviously always exists unbounded trajectory of the corresponding GLE.

We will further concentrate on the BLSLT with $a_{12}a_{21} > 0$. The reason for doing so is that in this case we can use a simple linear transformation to obtain more suitable parameterization of the BLSLT.

Lemma 3. Consider BLSLT (2), (7) with $a_{12}a_{21} > 0$. There exists linear transformation of the state space transformation $z = Sx$ that takes this system into the form

$$\dot{z} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} z + u \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix} z, \tag{12}$$

$$c = (c_1, c_2, c_3)^{\mathrm{T}} = (\cos\alpha, \sin\alpha, 0)^{\mathrm{T}}$$

Proof. First, apply the linear transformation $y = S_1 x, S_1 = \operatorname{diag}\left\{1, \sqrt{a_{12}/a_{21}}, \sqrt{a_{12}/a_{21}}\right\}$ and obtain due to Lemma 2 that $S_1 B S_1^{-1} = B$, and that the matrix $S_1 A S_1^{-1}$ is symmetric. We have therefore the eigenvectors v_1, v_2, v_3 of the matrix $S_1 A S_1^{-1}$ pairwise orthogonal. Let $\alpha \in [0, \pi/2]$ be the angle between positive direction of the y_1-axis and eigenvector v_1 corresponding to the eigenvalue $\lambda_1 > 0$. Then obviously the transformation $z = S_2 S_1 x$, where

$$S_2^{-1} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

takes the system (2), (7) into the desired form (12). □

Remark 3. It can be shown that

$$\begin{aligned} a_{11} &= \lambda_1 \cos^2\alpha + \lambda_2 \sin^2\alpha, \\ a_{22} &= \lambda_1 \sin^2\alpha + \lambda_2 \cos^2\alpha \end{aligned} \tag{13}$$

The angle $\alpha \in [0, \pi/2]$ is the angle between the z_1-axis (i.e. unstable direction of $S^{-1}AS$) and the axis of the infinitesimal rotation defined by matrix $S^{-1}BS \in so(3)$. It can be also easily seen that introducing $\alpha \in [\pi/2, \pi]$ gives behaviors symmetric to those of $\alpha \in [0, \pi/2]$.

Corollary 1. The GLE with $a_{12}a_{21} > 0$ is taken by the transformation from Lemma 3 into the form:

$$\dot{z} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} z + cz \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}, \tag{14}$$

$$c = (c_1, c_2, c_3)^{\mathrm{T}} = (\cos\alpha, \sin\alpha, 0)^{\mathrm{T}}$$

Proof. Obviously $x = S^{-1}z$, $S = S_2 S_1$, i.e. $x_1 = cz$. □

5. GENERALIZED LORENZ EQUATION — QUANTITATIVE ANALYSIS

This section will be devoted to the qualitative analysis of the GLE which can be viewed as the result of the application of the linear feedback (6) to the BLSLT (2), (7). We will exploit its form (14). The following lemma follows directly from Assertion 3, Remark 1 and Remark 3.

Lemma 4. Let us consider GLE (14) and let $\lambda_1 \cos^2 \alpha + \lambda_2 \sin^2 \alpha < 0$. Then there exists a bounded set $\mathcal{E} \subset R^n$ such that all trajectories of the GLE (14) finally enter this set and remain within it.

For $\lambda_1 \cos^2 \alpha + \lambda_2 \sin^2 \alpha \geq 0$ we can always show trajectories of the GLE (14) going to the infinity. The following lemma explains it more clearly.

Lemma 5. Let α_C be such that $\lambda_1 \cos^2 \alpha_C + \lambda_2 \sin^2 \alpha_C = 0$. GLE (14) has the nontrivial equilibrium point iff $\alpha \in (\alpha_c, \frac{\pi}{2})$. Moreover, in this case there is a symmetric pair of equilibrium points $x^{E1} = (x_1^{E1}, x_2^{E1}, x_3^{E1}), x^{E2} = (-x_1^{E1}, -x_2^{E1}, x_3^{E1})$ where

$$\begin{aligned} x_1^{E1} &= \frac{-\lambda_2}{(\lambda_1 - \lambda_2)\cos\alpha}\sqrt{\frac{\lambda_1\lambda_2\lambda_3}{-\sin\alpha(\lambda_1\cos^2\alpha + \lambda_2\sin^2\alpha)}} \\ x_2^{E1} &= \frac{\lambda_1}{(\lambda_1 - \lambda_2)\sin\alpha}\sqrt{\frac{\lambda_1\lambda_2\lambda_3}{-\sin\alpha(\lambda_1\cos^2\alpha + \lambda_2\sin^2\alpha)}} \\ x_3^{E1} &= \frac{\lambda_1\lambda_2}{\cos\alpha\sin\alpha(\lambda_1 - \lambda_2)}. \end{aligned} \tag{15}$$

Remark 4. This lemma can be easily checked by direct calculations. Moreover, it can be shown either using Routh–Hurwitz criterion or by numerical calculations that there is $\alpha_S \in (\alpha_C, \frac{\pi}{2})$ such that the equilibrium points $x^{E1}, x^{E2} \in R^3$ are stable for $\alpha \in (\alpha_C, \alpha_S)$ and unstable for $\alpha \in (\alpha_S, \frac{\pi}{2})$. For all $\alpha \in (\alpha_C, \frac{\pi}{2})$ the corresponding eigenvalues are both real and complex and for $\alpha = \alpha_S$ there is a pair of purely imaginary eigenvalues. In the other words, system undergoes ar $x^{E1,E2}$ the *Hopf bifurcation* (see e. g. [21] for details).

Remark 5. For $\alpha \in [0, \alpha_C]$ there is only one trivial unstable equilibrium point of the GLE (14) and it can be easily seen that this equation behaves in an unbounded manner. The condition $\lambda_1 \cos^2 \alpha + \lambda_2 \sin^2 \alpha < 0$, which is necessary and sufficient for bounded behavior of GLE (14), has the simple geometric meaning: let $\tilde{x} = (\cos\alpha, \sin\alpha, 0)^{\mathrm{T}}$, i. e. $\tilde{x}$ lies on the axis of the infinitesimal rotation defined by the matrix B, than $(A\tilde{x}, \tilde{x}) = \lambda_1 \cos^2 \alpha + \lambda_2 \sin^2 \alpha < 0$, in other words the flow of the vector field Ax should act on the above defined $\tilde{x}$ in a contracting manner.

The following simple property is stated without proof.

Lemma 6. The GLE (14) is symmetric with respect to the transformation $(x_1, x_2, x_3) \to (-x_1, -x_2, x_3)$ and the axis x_3 is invariant with respect to the flow of this equation.

Homoclinic trajectories play key role in the analytical investigations of the chaos (see the Appendix for details and definitions). In this respect we have the following theorem.

Theorem 1. Let us consider GLE (14) and let us suppose that $-\lambda_2 > \lambda_1 > -\lambda_3 > 0$. Then there exists $\alpha_H \in (\alpha_C, \frac{\pi}{2})$ such that the GLE (14) with α_H has the pair of two symmetric homoclinic orbits.

Proof. It can be shown that if $\lambda_1 > -\lambda_3 > 0$ then there always exists $\alpha_L \in (\alpha_C, \frac{\pi}{2})$ such that the trajectory from the point $(\epsilon, 0, 0)$, $\epsilon > 0$ will enter the region $L = \{x \in R^3 : x_1 < 0\}$. At the same time it is clear that there is $\alpha_R \in (\alpha_C, \alpha_L)$ such that trajectory starting from the point $(\epsilon, 0, 0)$ will remain in the region $\mathcal{R} = \{x \in R^3 : x_2 > 0\}$. If we choose $\epsilon > 0$ sufficiently small, these trajectories can be considered as the parts of the unstable invariant manifolds of the corresponding GLE. Moreover we can see that for $\alpha_{\tilde{R}} < \alpha_R$ the trajectory of GLE (14)with $\alpha_{\tilde{R}}$ remains in $\mathcal{R}$ and for $\alpha_{\tilde{L}} > \alpha_L$, the trajectory of GLE (14) with $\alpha_{\tilde{L}}$ will enter L. In other words, there should be $\alpha_H \in (\alpha_C, \frac{\pi}{2})$, such that for $\alpha_{\tilde{L}} \in (\alpha_H, \frac{\pi}{2})$ trajectory enter L and for $\alpha_{\tilde{R}} \in (\alpha_C, \alpha_H)$ it remain in $\mathcal{R}$. For this α_H the trajectory obviously fit exactly stable invariant manifold and therefore we obtain homoclinic trajectory. The existence of the second one follows from Lemma 6. □

Remark 6. It can be also shown that for $0 < \lambda_1 < -\lambda_3$ the trajectory starting from the point $(\epsilon, 0, 0)^{\mathrm{T}}$ remains for all $\alpha \in (0, \frac{\pi}{2})$ in $\mathcal{R}$. The proof of Theorem 1 can be better understood if one study Figures 3–4 discussed in the next section, where results of computer simulations of the GLE (14) with $\lambda_1 = 8, \lambda_2 = -16, \lambda_3 = -1$ are shown.

Now, the following theorem on the chaos existence in the GLE (14) can be stated.

Theorem 2. There exists $\alpha_0 \in (\alpha_H, \frac{\pi}{2})$ such that for all $\alpha \in (\alpha_H, \alpha_0)$ GLE (14) presents the chaotic behavior.

This theorem is the direct consequence of Theorem 3.2.13 of [21]. The interested reader can also find here the precise definition of the chaotic behavior. It is necessary to stress that, due to analytical nature of the methods used in [21], the α_0 should be generally close to α_H, i.e. range of parameters for which the system GLE (14) presents the chaotic behavior may be rather narrow. It will be seen in the next section that this kind of chaos, that can be justified by methods of [21], is difficult to be observed numerically. The reason is that this chaos due to global bifurcations theory bifurcates or 'explodes' from a pair of the homoclinic trajectories in their small neighborhood and therefore the corresponding chaotic attractor is small. On the contrary, numerous chaotic attractors were simulated that can not be explained by the global bifurcation theory of [21]. Actually, numerical simulations shows that α_0 can be taken equal to $\pi/2$.

6. GENERALIZED LORENZ EQUATION — NUMERICAL SIMULATIONS

In this section we collect some numerical simulations of the GLE (14).

We started by choosing the eigenvalues of the matrix A as $\lambda_1 = 8, \lambda_2 = -16, \lambda_3 = -1$, and changing parameter α gradually from 0 to $\pi/2$ to observe the evolution of the behavior of the system. Note, that the above values of the eigenvalues satisfies conditions (3) and are relatively close to those of the classical Lorenz equations considered e. g. in [17]. With a few exceptions, the initial conditions in all simulations will be taken as $x_0 = (\epsilon, 0, 0)^T$ with ϵ relatively small (say $\epsilon = 1/2$) so that the corresponding trajectory will also approximate unstable manifold of the origin.

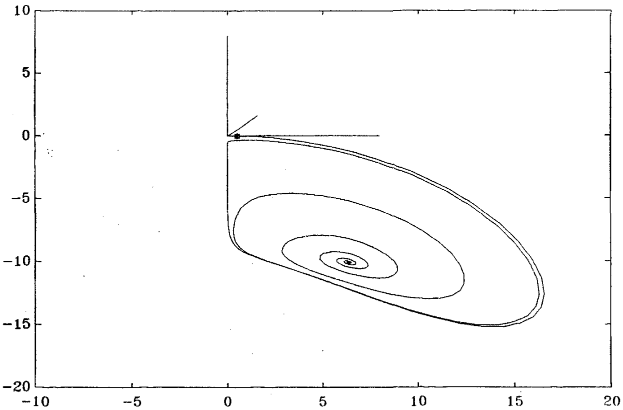

Fig. 1. Nearly homoclinic behaviour for
$\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$, $\alpha = 0.75410262387$.

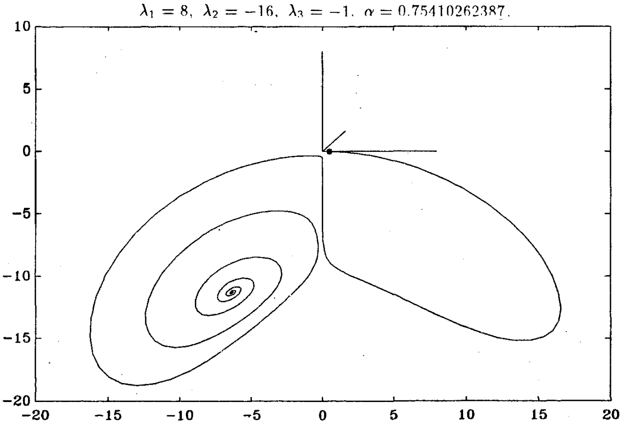

Fig. 2. Nearly homoclinic behaviour for
$\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$, $\alpha = 0.7541026238\underline{8}$.

First, numerical simulations confirms that for $\alpha_1 = 0.6 < \alpha_C = 0.6155$ (see Lemma 5 for definition of α_C) is the behavior of the corresponding GLE (14) unbounded, while for $\alpha_2 = 0.63 > \alpha_C$ it is bounded. Fig. 1 and Fig. 2 illustrate how the homoclinic trajectory appears (cf. Remark 6): we still have trivial bounded behaviors for $\alpha_3 = 0.75410262387$ and $\alpha_4 = \alpha_3 + 10^{-11}$ but in spite of the mutual closeness of the parameters α_3, α_4 their long term behaviors are principally different. Now, it is clear due to the proof of the Theorem 1 that there is a homoclinic trajectory for certain $\alpha_H \in (\alpha_3, \alpha_4)$, actually, both α_3 and α_4 can be viewed as the rather fair approximation of the homoclinic value of α. According to Theorem 2 there should be chaotic behavior for α_4 but this is practically unobservable, since the attractiveness of the stable equilibrium points x^{E1}, x^{E2} are too strong, while chaotic attractor too 'narrow', and as result the numerically simulated trajectory after some time always falls into the one of the stable equilibrium points. Actually, we can observe a typical transition to the chaos: trajectory is still converging to one of the stable equilibria but it is doing this in a more and more complicated way (see Fig. 3 where the trajectory corresponding to $\alpha_5 = 0.86$ starting from $x_0 = (-2, -2, -10)^{\mathrm{T}}$ is depicted).

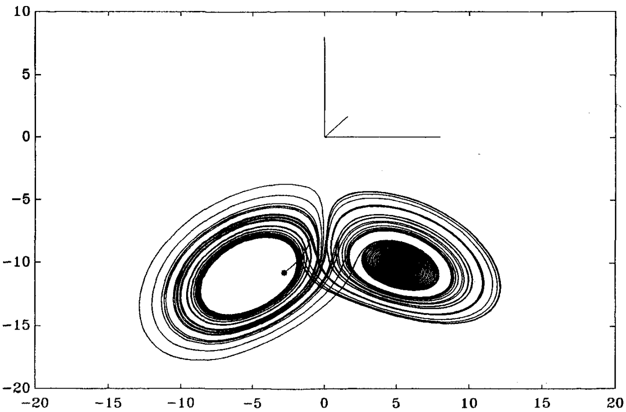

Fig. 3. Trajectory for $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$, $\alpha = 0.86$ starting at $x_0 = (-2, -2, -10)^{\mathrm{T}}$: complicated convergence to the stable equilibrium.

Remind that the equilibria x^{E1}, x^{E2} looses stability when $\alpha \geq \alpha_S$ through the so-called *subcritical Hopf bifurcation* (the appropriate unstable limit cycle can be observed also in Fig. 3) and their basins of attractivity tend to these points as $\alpha \longrightarrow \alpha_S$ (interested reader may consult e. g. [9] for bifurcation theory). Consequently, chaotic behavior becomes better numerically visible as $\alpha \longrightarrow \alpha_S$ and for $\alpha \geq \alpha_S$ is the chaotic attractor globally attracting (cf. also Lemma 4). Typical chaotic attractor is depicted in Fig. 4, this attractor is very similar to the attractor of the classical LE (cf. [17]): trajectories irregularly change a winding around x^{E1} to a winding around x^{E2} and vice versa. Such a double-scrolled structure was studied in a very detailed manner throughout an extensive amount of publications (see [9, 17, 21] for further references).

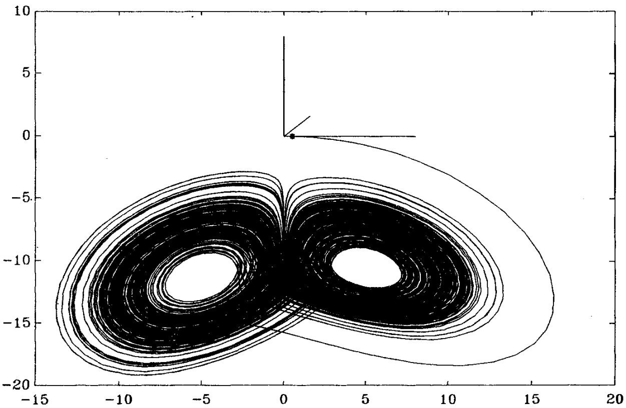

Fig. 4. Typical chaos for $\alpha = 0.9$, $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$.

An interesting phenomenon can be observed for α close to $\pi/2$ (note that for $\alpha = \pi/2$ the behavior of the GLE (14) should be unbounded). The corresponding attractor becomes larger and larger and gradually changes its topological structure: in Fig. 5 one can observe one part of this attractor for $\alpha = 1.5707$. The trajectory is now winding only on a ∞-shaped attractor, moreover, actually, there are two such attractors, second one is symmetric to that shown on Fig. 5 with respect to the symmetry $(x_1, x_2, x_3)^{\mathrm{T}} \to (-x_1, -x_2, x_3)^{\mathrm{T}}$. In other words, the novel type attractors can be simulated in the GLE (14) for α close to $\pi/2$. Note also that the scale used for Fig. 5 is 2500 times larger than in the case of Fig. 4, so the corresponding attractor is very large.

Now, we describe some simulations results for $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -7$. The reason to consider these values is that due to $|\lambda_3| \sim \lambda_1$ the numerical difficulties are not so extensive and this case is therefore more instructive; particularly a nice picture with a chaotic behavior in the neighborhood of the homoclinicity can be observed (see Fig. 7). Also homoclinic trajectory can be computed with high precision (see Fig. 6). All qualitative features of the previous case of remain without changes, except the fact that Hopf bifurcation in α_S is of the *supercritical* type (one can observe the appropriate stable limit cycle in Fig. 6).

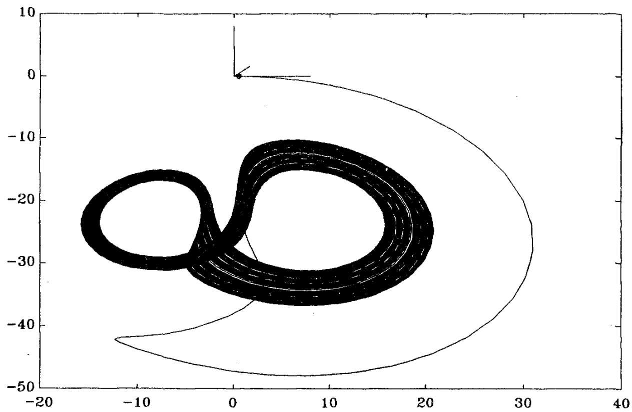

Fig. 5. Novel type attractor for $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$, $\alpha = 1.5707$, the scale is 2500 times larger.

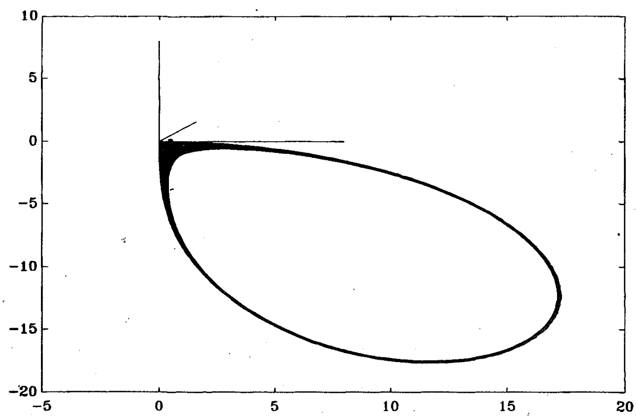

Fig. 6. Nearly homoclinic orbit and stable limit cycle for $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -7$, $\alpha = -1.440\underline{2}$, the scale is 4 times larger.

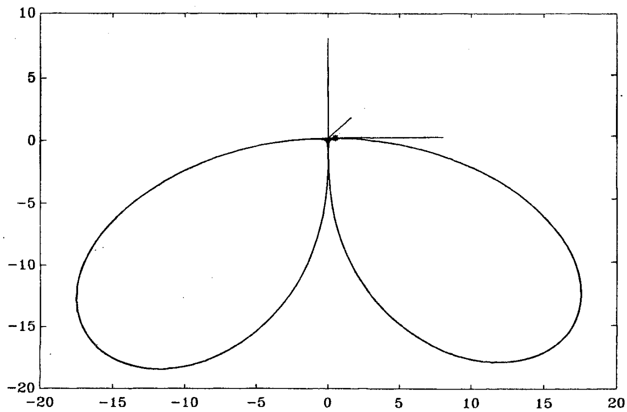

Fig. 7. Typical chaos in the neighbourhood of the homoclinicity for $\lambda_1 = 8,\ \lambda_2 = -16,\ \lambda_3 = -7,\ \alpha = -1.440\underline{3}$, the scale is 4 times larger.

The above described observations are collected in Table 1.

Table 1. Simulations for $\lambda_1 = 8, \lambda_2 = -16, \lambda_3 = -1$.

$\alpha \in [0, \alpha_C]$	No attractor, unbounded behavior.
$\alpha \in (\alpha_C, \alpha_H]$	Two stable symmetric points x^{E1}, x^{E2} as the attractors; union of their attractivity regions is the whole R^3 without stable manifold of $(0,0,0)^T$.
$\alpha \in (\alpha_H, \alpha_S]$	Attractors: two stable equilibrium points x^{E1}, x^{E2} and the chaotic attractor, their union is again globally attractive.
$\alpha \in [\alpha_S, \pi/2)$	The chaotic attractor with the global attractivity; for α close to $\pi/2$ two disjoint and mutually symmetric chaotic attractors.

7. BILINEAR SYSTEM OF THE LORENZ TYPE — CONCLUSIONS AND DISCUSSION

In this section we summarize both analytic facts proved in Sections 4, 5 and numerical observations of Section 6 to give relatively complete picture of the properties of the BLSLT (2), (7).

We have shown, that we can parameterize BLSLT (2), (7) by four parameters: $\lambda_1, \lambda_2, \lambda_3$ (eigenvalues of the matrix A) and by and α (the angle between x_1 – axis and the axis of the infinitesimal rotation of the matrix B). Due to Theorem 2 we have that for all $\lambda_1, \lambda_2, \lambda_3, \alpha$, such that

$$-\lambda_2 > \lambda_1 > -\lambda_3 > 0 \quad \text{and} \quad \alpha \in (\alpha_H, \alpha_0)$$

we should have the chaotic behavior of the closed loop system resulting from BLSLT by linear feedback. Moreover, it is fairly supported by numerical simulations, that the last inclusion can be actually replaced by $\alpha \in (\alpha_H, \pi/2)$. In other words, we have an open and extensive region in the parameter space $(\lambda_1, \lambda_2, \lambda_3, \alpha)$ for which the chaotic behavior arises in the closed loop system resulting from BLSLT (2), (7) after application of the linear feedback.

It is rather typical that the chaotic behavior confirmed theoretically based on Theorem 2 is not numerically observable and vice versa, i.e. numerically observable chaos can not be theoretically supported by Theorem 2.

As it was mentioned in the introduction, the chaotic behavior of the corresponding closed loop system may be viewed as a substitute for the asymptotical stabilization, since it is nonvanishing but bounded behavior of the system. Moreover, the corresponding bounded set can be made arbitrarily small by the corresponding amplifying of the feedback (cf. Lemma 1). The corresponding assertions and lemmas of the Section 4 analyze both the possibility of asymptotic constant feedback stabilization of the BLSLT (2), (7) and the possibility to make the corresponding closed loop systems behave in a arbitrarily bounded manner. It is appropriate to collect this analysis in Table 2.

Table 2. Comparison of the stabilization and chaotic behavior of the BLSLT (remind that $-\lambda_2 > \lambda_1 > -\lambda_3 > 0$, $\lambda_1 \cos^2 \alpha_C + \lambda_2 \sin^2 \alpha_C = 0$, $\lambda_2 \cos^2 \alpha_N + \lambda_1 \sin^2 \alpha_N = -\lambda_3$, cf. also Remark 3 and (9)).

	Asymptotic constant feedback stabilizability	Boundedness by means of linear feedback
$\alpha \in [0, \alpha_C]$	NO	NO
$\alpha \in [\alpha_C, \alpha_N)$ (here belongs the classical LE)	YES	YES (for $\alpha > \alpha_H$ partially chaotic, cf. Table 1)
$\alpha \in [\alpha_N, \pi/2]$	NO	YES (for $\alpha > \alpha_S$ completely chaotic, cf. Table 1)

The following conclusions can be stated on the basis of the above discussion:

1. The main conjecture on the chaos (see end of the Section 2) is valid for the subclass of the BLSLT.
2. There is an open, extensive range of parameters values for which there is no asymptotical stabilizability and at the same time a chaotic behavior can be imposed to the BLSLT using linear feedback.
3. Generalized Lorenz equations, that are viewed as the corresponding closed loop dynamical system, have for an open, extensive range parameters values qualitatively different behavior than the classical Lorenz equation. Moreover, the BLSLT that corresponds to LE is constant feedback asymptotically stabilizable.

The third conclusion particularly means, that the results of this paper do not consist only in a straightforward interpretation of the known results from dynamical systems theory for the control theory but may be also of interest from the point of view of the chaos theory.

8. ROTATED SEMISTABLE BILINEAR SYSTEM — SOME COMMENTS TO THE GENERAL CASE

The main chaos conjecture that was stated at the end of the Section 2 for the case of the general RSBLS (see Section 2 for the definition) has been till now investigated only for the particular case — the bilinear system of the Lorenz type (RSBLS). In this section we aim to give some facts, computer simulations and comments on the general case.

First of all we describe the general form of the RSBLS.

Lemma 7. There exists basis in R^3 such that the RSBLS takes in this basis one of the following two forms:

$$1) \qquad A = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} \\ 0 & \lambda_2 & a_{21} \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad B \in so(3)$$

$$2) \qquad A = \begin{pmatrix} \mu & a_{12} & a_{13} \\ 0 & \rho & -\omega \\ 0 & \omega & \rho \end{pmatrix}, \quad B \in so(3)$$

This lemma follows directly from the definiton of the RSBLS. Actually, there are two levels of generalization with respect to the BLSLT. First, the axis of the infinitesimal rotation may have arbitrary direction, while for RSBLS it should lie in (x_1, x_2) plane. Secondly, eigenvectors of the matrix A need not be orthogonal in the basis where $B \in so(3)$, while for RSBLS we managed to make A orthogonal with B remaining skew-symmetric.

Moreover, to obtain a pair of symmetric homoclinic orbits that are needed for application of theory from [21], it is necessary to consider a quadratic feedback to obtain in the corresponding closed loop system the symmetry $(x_1, x_2, x_3)^{\mathrm{T}} \to (-x_1, -x_2, -x_3)^{\mathrm{T}}$. Due to this quadratic feedback a qualitatively different behavior can be obtained in comparison with the case of BLSLT.

To illustrate the situation we consider the case 2) of Lemma 7 with $a_{12} = a_{13} = 0$. Let us remind that due to condition (3) we should assume $\mu > -\rho > 0$. Inspite of the fact that the direction of the axis of the infinitesimal rotation defined by the general matrix $B \in so(3)$ is given by two angles, it is clear that due to the structure of the matrix A only one of them is necessary. In Figures 8 – 9 one can observe simulations with $\mu = 1$, $\rho = -0.8$, $\omega = 1$ and $\alpha_{11} = 1,20436$, $\alpha_{12} = 1,20437$ (here α is the angle between the positive direction of the x_1 axis and the axis of the infinitesimal rotation assuming that the latter one lies in the (x_1, x_2) plane).

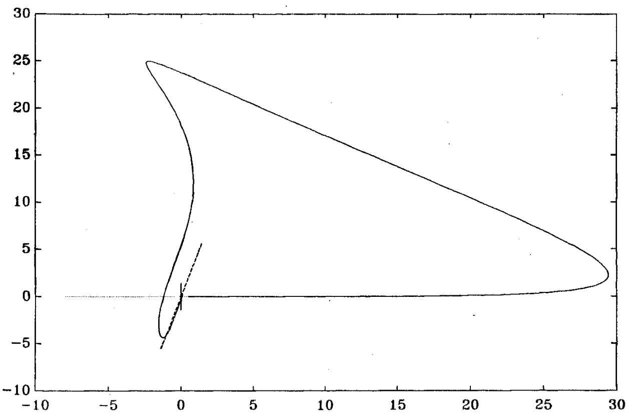

Fig. 8. Homoclinic orbit of RSBLS with quadratic feedback for $\rho = -0.8$, $\omega = 1$, $\mu = 1$, $\alpha = -1.20436$, the scale is 10 times smaller.

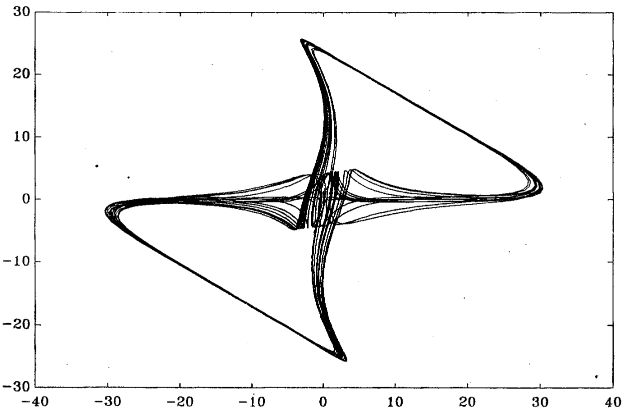

Fig. 9. Chaos for perturbed homoclinic orbit, $\rho = -0.8$, $\omega = 1$, $\mu = 1$, $\alpha = -1.2043\underline{7}$, the scale is 10 times smaller.

For α_{11} we have the high precision approximation of the homoclinic orbit and for $\alpha_{12} = \alpha_{11} + 10^{-5}$ we can observe a very complicated chaotic motion. Let us note that the theory described in [21] covers also the case of a pair of complex eigenvalues of A, again using a pair of homoclinic orbits, so that the theorem analogous to Theorem 2 would be available.

9. CONCLUDING REMARKS

A contribution of this paper may be viewed as a further step towards the chaos synthesis in nonlinear systems. A simple conjecture on a the chaos synthesis by feedback for a class of the bilinear single-input systems were stated and fairly supported both analytically and numerically.

In spite of the fact that the chaos synthesis is of its own interest, it was shown, that in many cases it can be viewed as the best, what can be achieved, i. e. the corresponding system is not stabilizable by any feedback but at the same time a bounded, nonvanishing behavior can be imposed to this system using either linear or quadratic feedback.

Although we restricted ourselves only to the three dimensional case and although we considered in the detail only a subclass of the rotated semistable bilinear systems, we think that the basic idea of combining semistable uncontrolled dynamics with rotation-type controlling one may be very fruitful also in a more general cases.

10. APPENDIX: ELEMENTS OF DYNAMICAL SYSTEMS THEORY

Here we collect some necessary facts from the dynamical systems theory. For more detailed information as well for the proofs is the interested reader referred to [9], [21]. Appendix considers the dynamical system of the form (1).

Definition 1A. Consider dynamical system (1): its flow will be denoted by $\Phi_t^f(x)$, it is defined as the trajectory at time t of the differential equation (1) starting from point $x \in R^n$.

Definition 2A. The point $x_E \in R^n$, such that $f(x_E) = 0$ is called as the equilibrium point of fixed point of the dynamical system (1). This point will be called the hyperbolic one if all eigenvalues of the $f_x(x_E)$ have nonzero real parts (f_x stands for Jacobian of $f(x)$). The linear differential equation (or linear dynamical system)

$$\dot{\xi} = f_x(x_E)\xi, \;\; \xi = x - x_E, \tag{1A}$$

will be further referred to as the linear part (or the approximate linearization) at x_E of the dynamical system (1).

It is obvious, that for the hyperbolic fixed point of the dynamical systems (1) we can find linear subspaces E_S, E_U : $E_S \oplus E_U = R^n$, such that all trajectories of the (1A) starting on E_S (E_U) tends to x_E for $t \to \infty$ ($t \to -\infty$). E_S (E_U) is called as the stable (unstable) subspace of the linear dynamical system. The following theorem generalizes this fact to the case of nonlinear dynamical system.

Theorem 1A. Suppose $x_E \in R^n$ is the hyperbolic fixed point of dynamical system (1). Then there exists neighborhood U_{x_E} and local submanifolds of $R^n, N_S^l, N_U^l \subset U_{x_E}$, such that trajectories starting on N_S (N_U) tend to x_E as the $t \to \infty$ ($t \to -\infty$). Moreover, E_S, E_U coincides with the tangent spaces to N_S^l, N_U^l at x_E, correspondingly, particularly, dim N_S^l + dim $N_U^l = n$.

Definition 3A. N_S^l (N_U^l) of Theorem 1A is called local stable (unstable) manifold.

Definition 4A. Submanifold (or set) $M \subset R^n$ is called as the invariant with respect to the dynamical system (1) if for any $x \in M$ and $t \in R$ it holds $\Phi_t^f(x) \in M$. One dimensional invariant submanifolds are often referred to as the orbits of dynamical systems.

Definition 5A. Global stable (unstable) submanifold N_S (N_U) is the maximal invariant submanifold that contains N_S^l (N_U^l).

Principal role plays the notion of the homoclinic trajectory (or orbit, cycle).

Definition 6A. Trajectory $\Gamma(t)$ is called as the homoclinic to the equilibrium point $x_E \in R^n$ if $\lim_{t\to+\infty} \Gamma(t) = \lim_{t\to-\infty} \Gamma(t) = x_E$. The union $\Gamma(t) \cup x_E$ is usually referred to as the homoclinic orbit or cycle.

It can be immediately seen that the homoclinic orbit is the intersection of the global stable and unstable manifolds.

Definition 7A. Consider dynamical system (1) together with its closed invariant set $M \subset R^n$ and let M_ε, $\varepsilon > 0$ is ε-neighborhood of M: $M_\varepsilon = \{x \in R^n \mid \exists y \in M \; \|x-y\| < \varepsilon\}$. M is called *Lyapunov-stable* or shortly just *stable* if for all $\varepsilon > 0, t > 0$ there exists $\delta(\varepsilon) > 0$ such that $\Phi_t^f(M_\delta) \subset M_\varepsilon \; \forall t \geq 0$. M is called *asymptotically stable* invariant set of (1) if it is stable and if there exists open set $\widetilde{M} \supset M, \widetilde{M} \neq M$, such that $\Phi_t^f(\widetilde{M}) \to M, t \to \infty$, i.e. $\forall \varepsilon > 0 \; \exists t(\varepsilon) > 0$ such that $\Phi_{t(\varepsilon)}^f(M) \subset M_\varepsilon$. The asymptotically stable set M is also called an *attracting set* of the dynamical system and the maximal with respect to the inclusion set $\widetilde{M}$ *domain (or basin) of attraction* of the attracting set M. The invariant set M is called *globally asymptotically stable* or *globally attracting* if its domain of attraction is the whole R^n. The set M is called a *repelling (antistable) set* of system (1) if it is the attracting set for the system $\dot{x} = -f(x)$. The set, which is neither stable nor antistable, is called *semistable*. Finally, attracting (repelling) invariant set, that contains orbit dense in it, is called *attractor (repeller)* of the system (1).

Definition 8A. Consider system (1), we say that it exhibits the chaotic behavior if it holds:

(i) there exists bounded stable subset $\mathcal{M}$ of R^n with basin of the attraction $\mathcal{B} \supset \mathcal{M}$

(i.e. all trajectories starting at points belonging $\mathcal{B}$ ultimately enter $\mathcal{M}$ and remain within it thereafter);

(ii) there is no attractor in the sense of Definition 7A being a subset of $\mathcal{M}$.

Intersection of all such stable sets $\mathcal{M}$ is a chaotic (or strange) attractor while union of all corresponding $\mathcal{B}$ is the basin of the attraction of this chaotic attractor.

This definition is rather nontraditional, but characterizes chaotic motion as "nonending" transition process that is bounded but does not converge to any classical attractor.

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