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STATISTICAL ANALYSIS AND APPLICATIONS OF LOG-OPTIMAL INVESTMENTS[1]

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Known facts about the existence and uniqueness of the log-optimal investment portfolio are presented in a simpler and more complete form than in previous publications. Five examples illustrate the problems around its existence and uniqueness and provide an intuitive insight into mathematical properties of the log-optimal portfolio and the associated optimal doubling rate.

1. INTRODUCTION

Let $\mathbf{X} = (X_1, \ldots, X_m)$ be the random *return vector* for one stock market day, i. e. X_j is the ratio of the closing to opening price for stock $j \in \{1, \ldots, m\}$. Hence the support of the distribution F of $\mathbf{X}$ is a subset of $[0, \infty)^m$. Furthermore, let $\mathbf{b} = (b_1, \ldots, b_m)$ be an *investment portfolio*, i.e., each b_j is the fraction of one's initial wealth invested in stock j. Then

$$\mathbf{bX} = \sum_{j=1}^{m} b_j X_j \tag{1}$$

is one's terminal wealth at the end of the day. Being involved in a continuing process of reinvestment it seems reasonable to maximize the expected continuous compound rate of growth

$$\phi(\mathbf{b}) = \mathsf{E} \log \mathbf{b}\,\mathbf{X}$$

or, for short, the so-called *doubling rate* (since we use here and in the sequel $\log = \log_2$) by selecting an appropriate element of the set $\mathbb{B} = \{\mathbf{b} = (b_1, \ldots, b_m) : b_j \geq 0,\ \sum b_j = 1\}$ of all portfolios. Such an element $\mathbf{b}^* \in \mathbb{B}$ is called a *log-optimal portfolio* and $\phi(\mathbf{b}^*)$ is called the *optimal doubling rate*.

For the statistical model of optimal investments, and in particular for the motivation of log-optimal portfolios, we refer to Algoet and Cover [1]. These authors consider a general statistical model of consecutive realizations of the return vectors

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$\mathbf{X}_1, \mathbf{X}_2, \ldots$. Here we restrict ourselves to i. i. d. realizations in which case the conditional distributions of return vectors considered by the quoted authors simply reduce to the unconditional distribution F of $\mathbf{X}$ on $[0,\infty)^m$.

As mentioned on page 877 of the cited paper, "a log-optimum portfolio always exists and is unique if the distribution F has full support not confined to a hyperplane in $\mathbb{R}^m$". The hyperplane condition is neither sufficient nor necessary, as the following two simple examples indicate.

Example 1: Bank savings. One can either keep money in the pocket ("investment" $j = 1$) or save it in the bank (investment $j = 2$). The investment portfolio $\mathbf{b} = (b_1, b_2)$ determines how one distributes his available wealth between these two possibilities. If the bank's interest rate is $100 \cdot \delta$ % per "day" (per month, per year), with $\delta > 0$, then the return vector $\mathbf{X} = (X_1, X_2)$ is deterministic, i. e. $P(X_1 = 1,\, X_2 = 1 + \delta) = 1$ and hence

$$\phi(\mathbf{b}) = \log(b_1 \cdot 1 + b_2(1+\delta)) \leq \log(1+\delta)$$

with equality iff $\mathbf{b} = \mathbf{b}^* = (0,1)$. (Although $\mathbf{X}$ is obviously confined to the straight line $x_2 = (1+\delta)x_1$ in $\mathbb{R}^2$, the log-optimal portfolio $\mathbf{b}^*$ is unique.)

Example 2: All but one portfolio log-optimal. Consider $m = 2$ and a random return vector $\mathbf{X} = (X_1, X_2)$ such that

$$P(X_1 = 0,\, X_2 = 1) = p \quad \text{and} \quad P(X_1 = 1, X_2 \geq 0) = 1 - p, \quad 0 < p < 1,$$

with a conditional distribution $P(X_2 < y \,|\, X_1 = 1)$ satisfying $\mathsf{E}\left(\log(1 + X_2) \mid X_1 = 1\right) = \infty$. Then

$$\begin{aligned}\phi(\mathbf{b}) &= p\log(1-b_1) + (1-p)\mathsf{E}\log\left(b_1 + (1-b_1)X_2 \,|\, X_1 = 1\right) = \\ &= \begin{cases} \infty & \text{for } \mathbf{b} = (b_1, 1-b_1), b_1 \in [0,1) \\ -\infty & \text{for } \mathbf{b} = (1,0). \end{cases}\end{aligned}$$

Hence all portfolios $\mathbf{b}^* = (b, 1-b)$, $b \in [0,1)$ are log-optimal. (Although $\mathbf{X}$ is obviously not confined to a straight line in $\mathbb{R}^2$, there are infinitely many log-optimal portfolios.)

The main emphasis of the present paper is to treat the problem of existence and uniqueness of the log-optimal portfolio rigorously. In particular, it demonstrates that if and only if the stock market return vector $\mathbf{X}$ satisfies the condition

$$\mathsf{E}\left|\log \sum_{j=1}^{m} X_j\right| < \infty \tag{2}$$

then there exists a log-optimal portfolio $\mathbf{b}^*$ with finite optimal doubling rate $\phi(\mathbf{b}^*)$. Furthermore, it demonstrates that, provided (2) is satisfied, the log-optimal portfolio $\mathbf{b}^*$ is unique if the following condition holds.

The distribution F is not confined to a hyperplane in $\mathbb{R}^m$ containing the diagonal $D = \{(d, \ldots, d) \in \mathbb{R}^m : d \in \mathbb{R}\}$. (3)

Note that the finiteness of the optimal doubling rate is very desirable. Indeed, consider i. i. d. realizations $\mathbf{X}_i$ of $\mathbf{X}$ on the days $i = 1, 2, \ldots$. Then it follows from the main result of [1] that the wealth

$$W_n = \prod_{i=1}^{n} \mathbf{b}_i(\mathbf{X}_1, \ldots, \mathbf{X}_{i-1})\, \mathbf{X}_i$$

resulting after n days from an arbitrary non anticipating measurable investment strategy $\mathbf{b}_1 \in \mathbb{B}$ and $\mathbf{b}_i(\mathbf{X}_1, \ldots, \mathbf{X}_{i-1}) : [0, \infty)^{m(i-1)} \mapsto \mathbb{B}$, $i = 2, 3, \ldots$, satisfies the relation

$$W_n \le 2^{n(\phi(\mathbf{b}^*)+\varepsilon)} \quad \text{eventually with probability 1 for any } \varepsilon > 0. \tag{4}$$

On the other hand, the wealth

$$W_n^* = \prod_{i=1}^{n} \mathbf{b}^* \mathbf{X}_i$$

resulting from the constant log-optimal strategy $\mathbf{b}_i^*(\mathbf{X}_1, \ldots, \mathbf{X}_{i-1}) = \mathbf{b}^*$ satisfies the relation

$$W_n^* \ge 2^{n(\phi(\mathbf{b}^*)-\varepsilon)} \quad \text{eventually with probability 1 for any } \varepsilon > 0. \tag{5}$$

(Note that the relation $2^{n(\phi(\mathbf{b})-\varepsilon)} \le W_n \le 2^{n(\phi(\mathbf{b})+\varepsilon)}$ eventually with probability 1, valid for every constant strategy $\mathbf{b}_i(\mathbf{X}_1, \ldots, \mathbf{X}_{i-1}) = \mathbf{b} \in \mathbb{B}$ with finite $\phi(\mathbf{b})$, motivates the use of the term "doubling rate": if $\phi(\mathbf{b}) = 1$ then W_n is approximately doubled each day.)

Relations (4) and (5) are equivalent to the fact that the wealth W_n^* corresponding to the log-optimal investments is asymptotically at least as large as the wealth W_n corresponding to an arbitrary technically acceptable investment strategy. More precisely, the maximal achievable growth rate of the wealth W_n,

$$\limsup_{n\to\infty} \frac{1}{n} \log W_n,$$

is a. s. not greater than the growth rate of W_n^*,

$$\lim_{n\to\infty} \frac{1}{n} \log W_n^* = \phi(\mathbf{b}^*).$$

For an infinite optimal doubling rate $\phi(\mathbf{b}^*)$, there is no analogy of (4) and (5) and the interpretation of the log-optimal portfolio $\mathbf{b}^*$ becomes rather problematic.

The results of the present paper concerning the log-optimal portfolio are mainly based on what is proved in [7]. In order to extend the intuitive meaning of the doubling rate $\mathbf{b} \mapsto \phi(\mathbf{b})$, $\mathbf{b} \in \mathbb{B}$, of the log-optimal portfolio $\mathbf{b}^*$ and of our conditions for the existence and uniqueness of $\mathbf{b}^*$, we present a series of examples of simple investment problems. Such examples, suitable also for teaching purposes, are largely missing in the literature.

In practical applications of the statistical investment theory the mean-variance approach of Markowitz [5] still prevails. Cf. [4]) e. g. Relations of the log-optimality to this classical instrument of stock market analysts have been clarified in [1]. One can hope that the recent progress in the algorithms for evaluation of log-optimal portfolios (cf. [2, 7, 6]) will lead to a wider popularity of this non-classical and essentially information-theoretic method.

2. EXISTENCE AND UNIQUENESS

Throughout this section we use the concepts and basic agreements introduced above. In addition, we consider the subset

$$\operatorname{dom}\phi \subseteq \mathbb{B}$$

the so-called *domain of* ϕ, on which the doubling rate $\phi(\mathbf{b}) = \mathsf{E}\log \mathbf{bX}$ is well-defined, the subset

$$\operatorname{effdom}\phi \subseteq \operatorname{dom}\phi$$

the so-called *effective domain of* ϕ, on which $\phi(\mathbf{b})$ is finite and finally the interior $\overset{\circ}{\mathbb{B}}$ of the set $\mathbb{B}$.

Furthermore, let S be any nonempty subset of the set $\{1,\dots,m\}$ of stocks, let $\mathbf{1}_S$ denote the indicator function of the set S and let

$$\mathbb{B}_S = \{\mathbf{b} \in \mathbb{B} : b_j = 0 \text{ iff } j \notin S\},$$

be the subset of $\mathbb{B}$ with support S. Note that $\mathbf{1}_{\{1,\dots,m\}}$ is the unity vector $\mathbf{1}$ in $\mathbb{R}^m$, $\mathbb{B}_{\{1,\dots,m\}} = \overset{\circ}{\mathbb{B}}$ and that the set $\mathbb{B}$ and its subsets $\mathbb{B}_S$ have the following geometric interpretation.

Remark 1. $\mathbb{B}$ is a regular simplex of hight one and $\mathbb{B}_S$ are vertices ($\#S = 1$), edges ($\#S = 2$) and faces of $\mathbb{B}$ of higher dimension. Furthermore, the components $b_1, b_2, \dots, b_m$ are the barycentric coordinates, i. e. the hights of the element $\mathbf{b} \in \mathbb{B}$ with respect to the perpendicular facets $\mathbb{B}_{\{2,3,\dots,m\}}, \mathbb{B}_{\{1,3,\dots,m\}}, \dots, \mathbb{B}_{\{1,2,\dots,m-1\}}$.

Finally, let $\mathbf{c} = (c_1,\dots,c_m) \in \mathbb{R}^m$ and let $\mathbf{cX}$ be defined as in (1), such that we obtain for $\mathbf{c} = \mathbf{1}$ the abbreviation $\mathsf{E}|\log \mathbf{1X}| = \mathsf{E}|\log \sum_{j=1}^m X_j|$. As shown in Lemma 1 of [7], the following characterization of the domain and the effective domain of ϕ in terms of the finiteness of the expected value $\mathsf{E}(\log \mathbf{1X})^+$ and $\mathsf{E}(\log \mathbf{1X})^-$ of the positive respectively the negative part of $\log \mathbf{1X}$ holds.

Table 1. Characterization of dom ϕ, effdom ϕ and of $\phi(\mathbf{b})$ on dom ϕ − effdom ϕ.

$\mathsf{E}(\log \mathbf{1X})^+$	$\mathsf{E}(\log \mathbf{1X})^-$	dom ϕ	effdom ϕ	$\phi(\mathbf{b})$ is on dom ϕ − effdom ϕ
$=\infty$	$=\infty$	$\subseteq \mathbb{B} - \mathring{\mathbb{B}}$	$=\emptyset$	$=-\infty$
$<\infty$	$=\infty$	$=\mathbb{B}$	$=\emptyset$	$=-\infty$
$=\infty$	$<\infty$	$\supseteq \mathring{\mathbb{B}}$	$\subseteq \mathbb{B} - \mathring{\mathbb{B}}$	$=\infty$ if $\mathbf{b} \in \mathring{\mathbb{B}}$, $\in \{-\infty, \infty\}$ otherwise
$<\infty$	$<\infty$	$=\mathbb{B}$	$\supseteq \mathring{\mathbb{B}}$	$=-\infty$

Note that Example 2 is an example for the third case of this table. A very natural example of the second case will be given in Example 3.

Since the only really interesting case is the last one of Table 1, we are going to investigate it in more detail: because one might expect the typical returns for the stock market to be close to one, one could think of the following condition to establish effdom $\phi = \mathbb{B}$.

The distribution F is confined to a cube $[1-\varepsilon,\ 1+\varepsilon]^m \subseteq \mathbb{R}^m$ for some $0 < \varepsilon < 1$.

And in fact, in the Vienna Stock Market which dates back to Maria Theresa's times, this condition is satisfied for national stocks with $\varepsilon = 0.1$. As a special case of Proposition 3 below, we obtain the following neccessary and sufficient condition guaranteeing that ϕ is defined and finite on the whole set $\mathbb{B}$.

Proposition 1. effdom $\phi = \mathbb{B}$ holds if and only if

$$\sum_{j=1}^{m} \mathsf{E}\,|\log X_j| < \infty. \tag{6}$$

By the way, this condition was the first which we took into consideration. But soon we noticed that it is to strong to cover as popular investment opportunities as the horse races or casino games like Roulette, where typically the return X_j for a bet is 0 with positive probability. And this, in turn, implies $\mathsf{E}|\log X_j| = \infty$. Condition (2) turns out to be most suitable as can be seen from the following statement. The proof of the equivalence of the first two conditions as well as the implication of the first condition from the third is easily seen from Table 1. The implication of the third from the first is achieved by the continuity of the function ϕ stated in Proposition 4.

Proposition 2. The following three statements are equivalent:

(i) $\mathsf{E}|\sum_{j=1}^{m} X_j| < \infty$.

(ii) dom $\phi = \mathbb{B}$ and effdom $\phi \supseteq \mathring{\mathbb{B}}$.

(iii) A log-optimal portfolio with finite optimal doubling rate exists.

The next statement, which follows from Lemma 3 and 4 in [7] offers an explicit way to describe the subset of $\mathbb{B}$ on which ϕ is finite.

Proposition 3. Assume (2) and let

$$\mathcal{C} = \{S \subset \{1,\dots,m\} : \mathsf{E} \mid \log \mathbf{1}_S\mathbf{X} \mid < \infty\} - \{1,\dots,m\}.$$

Then $\mathcal{C}$ is hereditary in the sense that with any nonvoid proper subset $S \subset \{1,\dots,m\}$ it contains all proper subsets $S' \subset \{1,\dots,m\}$ such that $S \subset S'$. Moreover, it holds

$$\operatorname{effdom} \phi = \mathring{\mathbb{B}} \cup \bigcup_{S \in \mathcal{C}} \mathbb{B}_S.$$

The following lemma provides an insight into the nature of condition (3).

Lemma 1. Let (2) be satisfied. Then

(i) condition (3) is neccessary for the strict concavity of ϕ,

(ii) provided $\mathbf{b}^* \in \mathring{\mathbb{B}}$ is log-optimal and (3) is violated then there are infinitely many log-optimal portfolios.

Proof. Let (3) be violated. Then, equivalently, there exists an element $\mathbf{c} = (c_1,\dots,c_m) \in \mathbb{R}^m$ such that $\sum_{j=1}^m c_j = 0$ and $\sum_{j=1}^m \mid c_j \mid > 0$ and $\mathbf{cX} = 0$ a. s. Now, let $\mathbf{b} \in \mathring{\mathbb{B}}$. Then there exists an $\varepsilon_0 > 0$ such that $\mathbf{b} + \varepsilon_0\mathbf{c} \in \mathbb{B}$. Hence $(\mathbf{b}+\varepsilon\mathbf{c})\mathbf{X} = \mathbf{bX}$ a. s. and therefore $\mathsf{E}\log(\mathbf{b}+\varepsilon\mathbf{c})\mathbf{X} = \mathsf{E}\log\mathbf{bX}$ for all $\varepsilon \in [0,\varepsilon_0]$. □

We conclude this section by stating the following theorem which is a consequence of the previous lemma and Lemma 5 and Theorem 1 in [7].

Proposition 4. Let (2) be satisfied. Then the function $\mathbf{b} \mapsto \phi(\mathbf{b}), \mathbf{b} \in \mathbb{B}$, is continuous and concave. Furthermore, the following two statements are equivalent.

(i) condition (3) holds, (ii) ϕ is strictly concave.

Finally, condition (3) implies that the log-optimal portfolio is unique.

3. SIMPLE APPLICATIONS OF THE RESULTS

Example 3: American Roulette. The set of possible outcomes ω of an experiment with the Roulette wheel, assumed to be a Laplace-experiment, is $\Omega = \{0,1,\dots,36\}$. In the following we consider three bets with the portfolios $\mathbf{b} = (b_1,b_2,b_3)$. Therefore the set $\mathbb{B}$ of all possible portfolios is an equilateral triangle with hight 1. Two different possibilites (a) and (b) are considered for the second and third bet.

$j = 1$: bet on "Odd",
$j = 2$: (a) bet on "1-18", (b) bet on "Even"
$j = 3$: (a) bet on "Zero", (b) bet on the "Street $0,1,2$".

Case (a). For this case the random return vector, set down in the rules of the game, and the associated probabilities are as follows:

Table 2. Return vectors for case (a) of the American Roulette.

ω	$\mathbf{X}(\omega) = (X_1(\omega), X_2(\omega), X_3(\omega))$	probability
0	$(1/2, 1/2, 36)$	1/37
$1, 3, \ldots, 17$	(2,2,0)	9/37
$2, 4, \ldots, 18$	(0,2,0)	9/37
$19, 21, \ldots, 35$	(2,0,0)	9/37
$20, 22, \ldots, 36$	(0,0,0)	9/37

It is easy to verify that $\mathsf{E}[\log(X_1 + X_2 + X_3)]^+ < \infty$ and $\mathsf{E}[\log(X_1 + X_2 + X_3)]^- = = \infty$ (so that condition (2) is violated), and $\phi(\mathbf{b}) \equiv -\infty$. Hence our betting system (a) of the American Roulette is asymptotically ruining each player at an infinite rate! In fact, we are faced with an example of the second case in Table 1.

Case (b). For this case the return vectors and the associated probabilities are given in Table 3.

Table 3. Return vectors for case (b) of the American Roulette.

ω	$\mathbf{X}(\omega) = (X_1(\omega), X_2(\omega), X_3(\omega))$	probability
0	$(1/2, 1/2, 12)$	1/37
1	(2,0,12)	1/37
2	(0,2,12)	1/37
$3, 5, \ldots, 35$	(2,0,0)	17/37
$4, 6, \ldots, 36$	(0,2,0)	17/37

It is easily seen that both conditions (2) and (3) are satisfied. Hence the doubling rate is defined everywhere on $\mathbb{B}$ and finite at least on the interior $\overset{\circ}{\mathbb{B}}$ and the log-optimal portfolio $\mathbf{b}^*$ is unique. This portfolio turns out to be

$$\mathbf{b}^* = (b, b, 1 - 2b) \quad \text{where} \quad b = \frac{3\left(1537 - \sqrt{4217}\right)}{9361},$$

i.e. $\mathbf{b}^* \doteq (0.491, 0.491, 0.008)$. To the dissappointment of potential casino players, the optimal doubling rate still remains negative, $\phi(\mathbf{b}^*) = -0.007$. Moreover, it is easy to verify that $\mathsf{E}\log X_j = -\infty, j \in \{1, 2, 3\}$ (consequently condition (6) is violated), that $\mathsf{E}\log(X_1 + X_3) = \mathsf{E}\log(X_2 + X_3) = -\infty$ and that $\mathsf{E}\log(X_1 + X_2)$ is finite. Therefore $\mathcal{C}$ contains only the set $\{1, 2\}$ and, by Proposition 2, the doubling rate is finite both on $\overset{\circ}{\mathbb{B}}$ and on the edge $\{\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{B} : b_3 = 0\}$ and is $-\infty$

on the remaining two edges and all three vertices of the triangle $\mathbb{B}$. For a contour plot of the function ϕ see Figure 1.

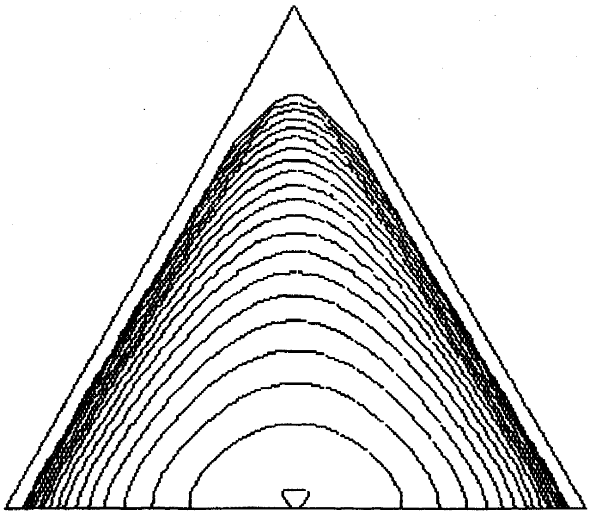

Fig. 1. Contour plot of the doubling rate $\mathbf{b} \mapsto \phi(\mathbf{b})$ for case (b) of the American Roulette (in barycentric coordinates).

The Horse Race. Cf. Cover and Thomas (1991), Chapter 6.1. Suppose m horses $j = 1, 2, \ldots, m$ race, and let $\mathbf{p} = (p_1, \ldots, p_m), p_j$ beeing the winning probability of horse j and suppose the bookmakers odds are $a_j : 1$ for the win of horse j (i. e. he pays $a_j > 0$ dollars if one bets 1 dollar on the win of this horse and nothing otherwise). Let us consider besides the bets $j = 1, \ldots, m$ on each horse the possibility of withholding money (bet 0). Then the return vectors together with the associated probabilities are as shown in Table 4.

Table 4. Return vectors of the Horse Race.

ω	$\mathbf{X}(\omega) = (X_0(\omega), X_1(\omega), \ldots, X_m(\omega))$	probability
1	$(1, a_1, 0, 0, \ldots, 0)$	p_1
2	$(1, 0, a_2, 0, \ldots, 0)$	p_2
$\vdots$	$\vdots$	$\vdots$
m	$(1, 0, 0, 0, \ldots, a_m)$	p_m

In this case $\mathbf{b} = (b_0, b_1, \ldots, b_m)$ denotes a portfolio. Before treating the general case we will recall the simple special case of

Example 4 (horse race not withholding cash). For this case we can ignore the coordinate $X_0(\omega)$ and restrict ourselves to portfolios $\mathbf{b} = (b_1, \ldots, b_m)$. It is easy to check that both conditions (2) and (3) are satisfied and that $\mathcal{C} = \emptyset$. Hence $\operatorname{effdom} \phi = \mathring{\mathbb{B}}$. Let $I(\mathbf{p}, \mathbf{b}) = \sum_{j=1}^{m} p_j \log(p_j/b_j)$ be the I-divergence of the probability distributions $\mathbf{p}$ and $\mathbf{b}$. Then

$$\phi(\mathbf{b}) = \sum_{j=1}^{m} p_j \log a_j b_j = \sum_{j=1}^{m} p_j \log a_j p_j - I(\mathbf{p}, \mathbf{b})$$

and since $I(\mathbf{p}, \mathbf{b}) \geq 0$ with equality iff $\mathbf{b} = \mathbf{p}$, the unique log-optimal portfolio is $\mathbf{b}^* = \mathbf{p}$ and the optimal doubling rate is $\phi(\mathbf{b}^*) = \sum_{j=1}^{m} p_j \log a_j p_j$. We illustrate this general result by the special case

$$m = 3; \quad \mathbf{p} = \left(\frac{5}{12}, \frac{4}{12}, \frac{3}{12}\right) \quad \text{and} \quad \mathbf{a} = \left(\frac{11}{5}, \frac{11}{4}, \frac{11}{3}\right),$$

$$\text{so that} \quad \mathbf{b}^* = \left(\frac{5}{12}, \frac{4}{12}, \frac{3}{12}\right) \quad \text{and} \quad \phi(\mathbf{b}^*) = \log\left(\frac{11}{12}\right) < 0\,.$$

For a contour plot of the function ϕ see Figure 2.

Example 5 (horse race withholding cash). In this case (2) holds, whereas (3) may or may not hold. Let $\mathbf{b}' = (b'_1, \ldots, b'_m)$ be defined by

$$b'_j = \left(b_j + \frac{b_0}{a_j}\right) \cdot (1 - b_0 S(\mathbf{a}))^{-1}\,, \quad j \in \{1, \ldots, m\}$$

and let

$$\rho_0 = \left(\frac{1}{\min\{p_j a_j\}} + S(\mathbf{a})\right)^{-1} \quad \text{and} \quad S(\mathbf{a}) = 1 - \sum_{i=1}^{m} \frac{1}{a_i}\,.$$

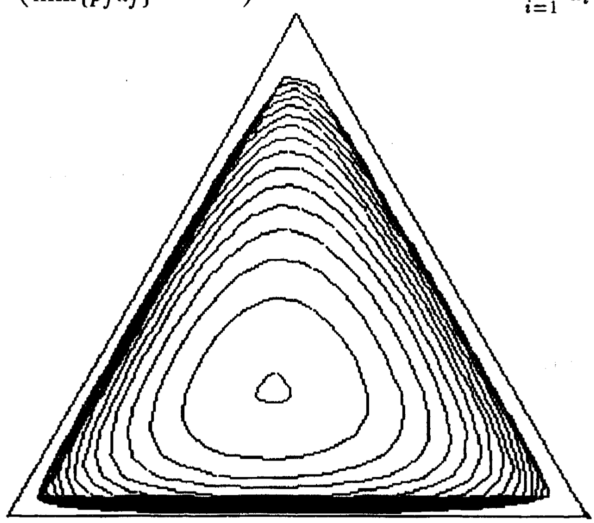

Fig. 2. Contour plot of the doubling rate $\mathbf{b} \mapsto \phi(\mathbf{b})$ for the described special case of the horse race (not withholding cash).

Then $\mathbf{b}' = \mathbf{b}'(b_o)$ is a probability distributi on on $\{1, \ldots, m\}$ and it holds $\rho_0 \in (0, 1]$. Furthermore,

$$\begin{aligned} \phi(\mathbf{b}) &= \sum_{j=1}^{m} p_j \log(b_0 + b_j a_j) = \\ &= \sum_{j=1}^{m} p_j \log\left[a_j p_j \left(1 - b_0 S(\mathbf{a})\right)\right] - I(\mathbf{p}, \mathbf{b}'). \end{aligned} \tag{7}$$

Since $I(\mathbf{p}, \mathbf{b}') \geq 0$ with equality iff $\mathbf{b}' = \mathbf{p}$, we see that the optimal portfolio $\mathbf{b}^*$ is of the form

$$\mathbf{b}^* = \left(b_0, b_1^*(b_0), \ldots, b_m^*(b_0)\right), \tag{8}$$

where

$$b_j^*(b_0) = p_j \left[1 - b_0 \left(\frac{1}{p_j a_j} + S(\mathbf{a})\right)\right], \quad j \in \{1, \ldots, m\}.$$

Here $b_0 \in [0, \rho_0]$ is still at our disposal. Now we have to distinguish the three cases "subfair odds", "fair odds", and "superfair odds" which are characterized by $S(\mathbf{a}) < 0, = 0$, and > 0 respectively.

In the typical case of **subfair odds** $(S(\mathbf{a}) < 0)$ it is easily seen from (7) that the log-optimal portfolio is obtained by setting $b_0 = \rho_0$ in (8) and that

$$\phi(\mathbf{b}^*) = \sum_{j=1}^{m} p_j \log\left[a_j p_j \left(1 - \rho_0 S(\mathbf{a})\right)\right].$$

For the cases of **superfair odds** $(S(\mathbf{a}) > 0)$ and **fair odds** $(S(\mathbf{a}) = 0)$ one log-optimal portfolio is given by $b_0 = 0$ and hence by $\mathbf{b}^* = (0, p_1, \ldots, p_m)$, yielding an optimal doubling rate

$$\phi(\mathbf{b}^*) = \sum_{j=1}^{m} p_j \log p_j a_j \,.$$

While, however, $\mathbf{b}^*$ is unique for the case of superfair odds this is not true for the case of fair odds. The latter, since $S(\mathbf{a}) = 0$ is equivalent to the violation of condition (3). In fact, all portfolios of the form

$$\mathbf{b}^* = b_0 \cdot (1, 0, 0, \ldots, 0) + (1 - b_0)(0, p_1, \ldots, p_m), \quad b_0 \in [0, \rho_0],$$

with $\rho_0 = \min\{a_j p_j\}$, are log-optimal.

We illustrate these general results by an **Example on fair odds** with the following specifications:

$$m = 2; \quad \mathbf{p} = \left(\frac{1}{3}, \frac{2}{3}\right), \quad \mathbf{a} = \left(3, \frac{3}{2}\right) \quad \text{and hence} \quad S(\mathbf{a}) = 0 \quad \text{so that all}$$

$$\mathbf{b}^* = \left(b, (1-b)\frac{1}{3}, (1-b)\frac{2}{3}\right), b \in [0, 1], \quad \text{are log-optimal with} \quad \phi(\mathbf{b}^*) \equiv 0.$$

Notice that, of course, condition (2) is satisfied and that $\mathcal{C} = \{\{0\}, \{0,1\}, \{0,2\}, \{1,2\}\}$. Which means that the doubling rate is finite everywhere on the triangle $\mathbb{B}$ except on the two vertices $(0,1,0)$ and $(0,0,1)$ where it is $-\infty$. For a contour plot of the function ϕ see Figure 3.

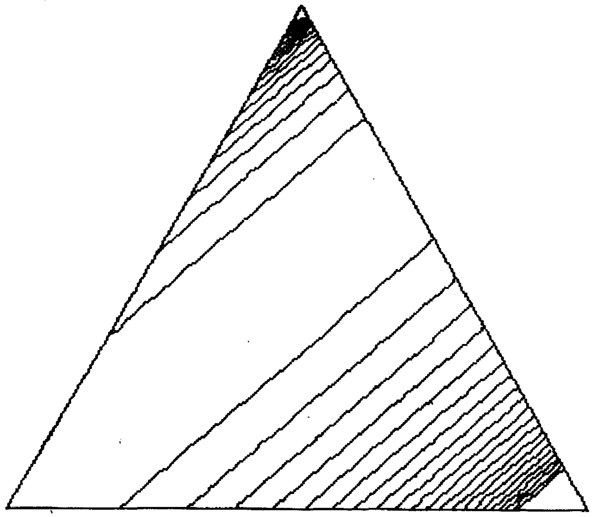

Fig. 3. Contour plot of the doubling rate $\mathbf{b} \mapsto \phi(\mathbf{b})$ for the described special case of the horse race (withholding cash).

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