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EXACT MODELLING OF SCALAR 2D ARRAYS[1]

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In this paper we analyse the problem of modelling a noise free finite 2D scalar array. We introduce the concept of unfalsified model and of undominated unfalsified model. We give some methods for verifying if a model is unfalsified and we study the properties of undominated unfalsified models. We define moreover a special class of models that are called coroborated and we give a condition ensuring the uniqueness of the corroborated undominated unfalsified model. Finally we show how to obtain such model, when it exists.

1. INTRODUCTION

The problem of modelling finite time series by linear dynamical systems is very important from the application point of view. In general, in the process of modelling one starts from a set of data and tries to find the dynamical relations among them. These relations must be represented by a mathematical description, that is called model. Consequently the first step in modelling is to fix the model set, namely what are the relations on the data we want to find. The second step consists in finding a subset of models that can be considered compatible with the data which are called unfalsified models. Finally we have to choose one of the unfalsified models.

Many different modelling procedures have been developed. One first distinction can be done between the ideal case when data are supposed to be exact and noisy free and the more realistic case when we have to cope with noisy data. Though the first case does not provide algorithms which can be directly used in the applications, it is mathematically and historically important, since it gives some light also to the solution of the noisy case. In the classical paper [1] by Kalman many connections are shown between this topic and many other famous mathematical problems.

The problem of modelling can be formalized as follows: given a finite array of data, find a linear system with minimum McMillan degree whose Markov coefficients match the given array where the array is defined. This is equivalent to another well known problem which is the Padé approximation. Another formulation of modelling has been proposed by J.C. Willems in [2]. In his approach the model has no apriori input/output structure and so in the linear discrete case its description is given by a

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set of difference equations rather than by a transfer function. A more detailed analysis of the exact modelling of finite time series is given in [3], where another question is investigated: when do the data present evidence of obeying a law? For instance consider the finite array $(1,1,1,1,1,1,1)$. It is clear that in this case there is a clear evidence that this array obeys some law, while if we have the array $(0,0,0,0,0,0,1)$, then such evidence does not exist. This question is solved introducing the concept of corroboration which is based on the idea of considering only the generic behaviour of the models. By this new idea it is possible to construct an algorithm with many good properties. It can be seen that the concept of corroboration can be interpreted as the data compression of the model. More precisely a model is considered acceptable if it provides an efficient description of the data.

In this paper we will deal with the problem of modelling finite 2D arrays. This case is remarkably more complicate as we will show in the sequel. This is the reason why we will consider only the scalar case. There exist few papers on 2D modelling in literature even in the classical input/ output framework. All the algorithms proposed present disadvantages and the problem can be considered open. An interesting algorithm has been proposed by Sakata in [4, 5]. Even if this algorithm presents some limitations, it is very efficient and so very suitable for the real applications. This algorithm is the direct extension of the Berlekamp Massey algorithm to the 2D case. In this paper we will formulate the 2D modelling problem and analyze it from various points of view. We will try to extend the concept of corroborated model to the 2D case and we will give a condition which guarantees the uniqueness of the this model. Finally we will propose a procedure providing the corroborated model of a 2D array, when it exists. This algorithm is an extension of the Sakata algorithm.

2. BEHAVIOURAL THEORY OF DYNAMICS

In this paper we will follow the behavioural approach to dynamical systems for describing models. According to this approach, introduced by Willems in [2], there is no a priori distinction between inputs and outputs and the external data are characterized only by a family of laws describing what signals are allowed.

A dynamical system is described by a triple $\Sigma = (T, W, \mathcal{B})$, where T is the time set, W is the signal set and $\mathcal{B}$ is a subset of W^T, called behaviour, which describes what trajectories of W^T can occur. In this paper we will be concerned with dynamical systems with $T = \mathbb{N}^2$, $W = \mathbb{R}^q$ and $\mathcal{B} = \ker R(\sigma_1, \sigma_2)$, where

$$R(\sigma_1, \sigma_2) = \sum_{ij} R_{ij} \sigma_1^i \sigma_2^j, \tag{1}$$

$R_{ij} \in \mathbb{R}^{n \times q}$, is a polynomial operator that acts on a signal $w \in (\mathbb{R}^q)^{\mathbb{N}^2}$ as follows

$$R(\sigma_1, \sigma_2) w = \sum_{ij} R_{ij} \sigma_1^i \sigma_2^j w \tag{2}$$

and σ_1, σ_2 are the usual shift operators acting on the two directions of $\mathbb{N}^2$, i.e. acting on w as follows $\sigma_1 w(t_1, t_2) = w(t_1 + 1, t_2)$ and $\sigma_2 w(t_1, t_2) = w(t_1, t_2 + 1)$ for all $(t_1, t_2) \in \mathbb{N}^2$. A detailed description of the properties of these systems can be found in [6, 7]. It can be shown that the set of all behaviours that admit the previous representation, coincides with the set of all the linear, shift invariant and closed (w.r. to the pointwise convergence) subspaces of $(\mathbb{R}^q)^{\mathbb{N}^2}$, the space of all the signals. One important property of the previous representation is the following: if we have two behaviours $\mathcal{B}_1 = \ker R_1$ and $\mathcal{B}_2 = \ker R_2$, then we have that $\mathcal{B}_1 = \mathcal{B}_2$ if and only if the polynomial row module generated by the rows of R_1 coincides with the polynomial row module generated by the rows of R_2.

We shall recall now some properties of systems in the scalar case $q = 1$. For the representation of this systems we need column matrices only. In the sequel we will use the notation $\ker(p_1, \ldots, p_n)$, where $p_1, \ldots, p_n$ are polynomials in z_1, z_2, to mean

$$\ker \begin{bmatrix} p_1(\sigma_1, \sigma_2) \\ \vdots \\ p_n(\sigma_1, \sigma_2) \end{bmatrix}.$$

We need to introduce now some concepts of Gröbner basis theory [8]. Let $<_T$ be an admissible total ordering in $\mathbb{N}^2$, i.e. a total ordering compatible with the monoid structure in $\mathbb{N}^2$ and such that $(0,0)$ is the smallest element in $\mathbb{N}^2$. Given a polynomial p in $\mathbb{R}[z_1, z_2]$ we define its degree $\deg p$ as the maximum of the support of p w.r. to the total admissible ordering $<_T$. If $\mathcal{I}$ is an ideal in $\mathbb{R}[z_1, z_2]$, we define deg $\mathcal{I}$ as the set of the degrees of all the polynomials in $\mathcal{I}$ and $M(\mathcal{I})$ as the set of all the minimal elements of deg $\mathcal{I}$ w.r. to the usual partial ordering in $\mathbb{N}^2$. We say that a set $\mathcal{G} = \{g_1, \ldots, g_m\}$ of generators of $\mathcal{I}$ is a Gröbner basis of $\mathcal{I}$, if

$$M(\mathcal{I}) \subseteq \deg\ \mathcal{G}, \tag{3}$$

where $\deg \mathcal{G} = \{\deg g_1, \ldots, \deg g_m\}$. Consequently we have that

$$\deg\ \mathcal{I} = \deg \mathcal{G} + \mathbb{N}^2 := \{\deg g + t \ : \ g \in \mathcal{G}, t \in \mathbb{N}^2\}.$$

If in a Gröbner basis $\mathcal{G}$ each polynomial is in a reduced normal form w.r. to the ideal $\mathcal{I}$ or, equivalently, if the support of each polynomial g in $\mathcal{G}$ is included in $\Delta(\mathcal{I}) \cup \{\deg g\}$, where we define

$$\Delta(\mathcal{I}) := \mathbb{N}^2 \setminus \deg\ \mathcal{I},$$

then $\mathcal{G}$ is said reduced Gröbner basis of $\mathcal{I}$ [8]. Note that, if $\mathcal{G}$ is a reduced Gröbner basis, then we have that

$$M(\mathcal{I}) = \deg\ \mathcal{G}.$$

It can be shown that Gröobner bases and reduced Gröobner bases always exist and that the reduced Gröbner basis of an ideal is also unique. Moreover there exist algorithms providing Gröbner bases or the reduced Gröbner basis of an ideal $\mathcal{I}$ from any set of of generators of $\mathcal{I}$.

The behaviour $\mathcal{B} = \ker(p_1, \ldots, p_n)$ is fixed by the ideal $\mathcal{I}$ generated by $p_1, \ldots, p_n$. Consequently if $g_1, \ldots, g_m$ is a Gröbner basis of $\mathcal{I}$, then $\ker(g_1, \ldots, g_m)$ is an equivalent representation of $\mathcal{B}$. This representation has also the following operative advantage. In fact, it can be seen that $\mathcal{B}$ and $\mathbb{R}^{\Delta(\mathcal{I})}$ are isomorphic vector spaces in the sense that for every $w_\Delta \in \mathbb{R}^{\Delta(\mathcal{I})}$, there always exists a unique $w \in \mathcal{B}$ such that $w_{|\Delta(\mathcal{I})} = w_\Delta$. This means that $\Delta(\mathcal{I})$ is a good subset of $\mathbb{N}^2$ where the initial conditions can be fixed. Consequently $\mathcal{B}$ is finite dimensional if and only if $\Delta(\mathcal{I})$ is a finite subset of $\mathbb{N}^2$ and its dimension coincides with the number of points in $\Delta(\mathcal{I})$. It can be shown that, starting from a Gröbner basis, it is possible to find $\Delta(\mathcal{I})$ and whole trajectory $w \in \mathcal{B}$ starting from the initial conditions $w_{|\Delta(\mathcal{I})}$. If $\mathcal{B}$ has finite dimension d, then it admits the following state representation

$$\begin{cases} \sigma_1 x = M_1 x \\ \sigma_2 x = M_2 x \\ \quad w = Cx \end{cases}, \tag{4}$$

where M_1, M_2 are commutative matrices in $\mathbb{R}^{d\times d}$ and C is a row matrix in $\mathbb{R}^{1\times d}$. The state x is a signal in $(\mathbb{R}^d)^{\mathbb{N}^2}$ that is completely fixed by its value in the time instant $(0,0)$. Moreover $x(0,0)$ is directly computable from $w_{|\Delta(\mathcal{I})}$. Consequently for all $w \in \mathcal{B}$ here exists a unique $x(0,0) \in \mathbb{R}^d$ such that

$$w(i,j) = C\, M_1^i\, M_2^j\, x(0,0).$$

Viceversa, if $M_1, M_2 \in \mathbb{R}^{d\times d}$ are two commuting matrices and $C \in \mathbb{R}^{1\times d}$ and if we define the subspace $\mathcal{B}$ of $\mathbb{R}^{\mathbb{N}^2}$ as follows

$$\mathcal{B} := \{w \in \mathbb{R}^{\mathbb{N}^2} : \exists x \in \mathbb{R}^d,\ w(i,j) = C\, M_1^i\, M_2^j\, x, \forall\, (i,j) \in \mathbb{N}^2\},$$

then it is easy to see that $\mathcal{B} = \ker(p_1, \ldots, p_n)$ for some suitable polynomials $p_1, \ldots, p_n$. In this case we can say only that $\mathcal{B}$ is a finite dimensional vector space of dimension less than or equal to d.

3. UNFALSIFIED AND UNDOMINATED UNFALSIFIED MODELS

In this section we will present a method for modelling a scalar finite 2D array by a system of multidimensional difference equations. Given a set $I \subseteq \mathbb{N}^2$ and a 2D scalar array $w_I \in \mathbb{R}^I$, we say that $\mathcal{B}$ is an unfalsified model of w_I if there exists $w \in \mathcal{B}$ such that $w_{|I} = w_I$. By this definition we say what 2D systems can be accepted as models of w_I. Between two unfalsified models $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_1 \subseteq \mathcal{B}_2$ we consider a better model $\mathcal{B}_1$, since it is more falsifiable and so more powerful in describing the system generating w_I. In some sense it has more predictive power. According this idea we say that $\mathcal{B}$ is an undominated unfalsified model of w_I if it is one of the minimal elements in the set of all the unfalsified models w.r. to the partial ordering $\subseteq$ or, in other words, if for every unfalsified model $\mathcal{B}'$ such that $\mathcal{B}' \subseteq \mathcal{B}$, we have that $\mathcal{B}' = \mathcal{B}$. Finally we will say that $\mathcal{B}$ is the most powerful unfalsified model if it is the smallest unfalsified model, i.e. if for every unfalsified model $\mathcal{B}'$ we have that $\mathcal{B} \subseteq \mathcal{B}'$. In general a modelling procedure must choose an unfalsified model starting from the data w_I. If there exists the most powerful unfalsified model, then this can be considered the best candidate to be the model of the data. It is easy to see that when $I \neq \mathbb{N}^2$, then the most powerful unfalsified model does not exist (this is true also in the 1D case). An important issue is how to choose a model in the set of all the undominated unfalsified models. Therefore we will analyze in detail the properties of these models in order to characterize their properties.

We need to introduce some notation. Let I be a subset of $\mathbb{N}^2$ and $\mathbb{R}^I$ be the set of all the scalar arrays with support in I. Moreover let $\mathbb{R}[z_1, z_2]$ be the ring of polynomials in z_1 and z_2. Consider the following form

$$\langle \cdot, \cdot \rangle_I \,:\, \mathbb{R}[z_1, z_2] \times \mathbb{R}^I \longrightarrow \mathbb{R}$$

such that for any $w_I \in \mathbb{R}^I$ we have that

$$\langle p, w_I \rangle_I := \sum_{(i,j)\in I} p(i,j) w(i,j),$$

if

$$p = \sum_{(i,j)\in I} p(i,j) z_1^i z_2^j$$

is any polynomial with support included in I, while

$$\langle p, w_I \rangle_I := 0$$

otherwise. It is easy to see that this form is linear in $\mathbb{R}^I$ but not necessarily in the other argument. In fact we have that if p, q are polynomials with support included in I, then $\langle p+q, w_I \rangle_I = \langle p, w_I \rangle_I + \langle q, w_I \rangle_I$, but in the other cases it is not necessarily true. If $I = \mathbb{N}^2$, then we will write the bilinear form $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_{\mathbb{N}^2}$. Given $w_I \in \mathbb{R}^I$, we define the set of polynomials

$$\mathcal{A}(w_I) := \{p \in \mathbb{R}[z_1, z_2] : \langle z_1^{i_1} z_2^{i_2} p, w_I \rangle_I = 0, \forall\, (i_1, i_2) \in \mathbb{N}^2\} \tag{5}$$

and we call its elements annihilators of w_I. Note that if $J \subseteq I$ and w_J is the restriction of w_I to the set J, then we have that $\langle p, w_I \rangle_I = 0$ implies $\langle p, w_J \rangle_J = 0$ and consequently

$$\mathcal{A}(w_I) \subseteq \mathcal{A}(w_J).$$

If p is an annihilator of w_I and h is any polynomial, then the polynomial hp is an annihilator for w_I. On the other side if p and q are annihilators of w_I, then it is not necessarily true that $p + q$ is an annihilator. This is true if $I = \mathbb{N}^2$ and so in this case $\mathcal{A}(w_I)$ is an ideal for all $w_I \in \mathbb{R}^I$. It is easy to see that $w \in \ker(p_1, \ldots, p_n)$ if and only if the ideal generated by $p_1, \ldots, p_n$ is included in $\mathcal{A}(w)$. The following proposition generalizes this useful dual characterization of the unfalsified models.

Proposition 1. Let $p_1, \ldots, p_n$ be polynomials and w_I be an array in $\mathbb{R}^I$. Then $\ker(p_1, ..., p_n)$ is an unfalsified model of w_I if and only if for all the polynomials in the ideal $(p_1, \ldots, p_n)$ generated by $p_1, \ldots, p_n$ we have that $\langle p, w_I \rangle_I = 0$

Proof. ($\Rightarrow$) Trivial.
($\Leftarrow$) We want to construct a $w \in \ker(p_1, \ldots, p_n)$ such that $w_{|I} = w_I$. For this purpose let $\{s_i\}_{i=1}^{\infty}$ be a sequence of points in $\mathbb{N}^2$ covering all $\mathbb{N}^2$ and $I_k := \{s_1, \ldots, s_k\}$. For all k we construct recursively $w_k \in \mathbb{R}^{I_k}$ such that $\langle p, w_k \rangle_{I_k} = 0$ for all $p \in (p_1, \ldots, p_n)$ as follows. Suppose that we have already found w_{k-1}. Let $w_{k|I_{k-1}} = w_{k-1}$. We have to fix the value of $w_k(s_k)$. If $s_k \in I$, then let $w_k(s_k) = w_I(s_k)$. If

$s_k \notin I$ and if there exists a $\overline{p} \in \mathcal{I}$ such that $\{s_k\} \subseteq \operatorname{supp} \overline{p} \subseteq I_k$, then let $w_k(s_k)$ be the only value such that $\langle \overline{p}, w_k \rangle_{I_k} = 0$. In all the other cases let $w_k(s_k)$ be an arbitrary value. We show now that if $p \in \mathcal{I}$, then $\langle p, w_k \rangle_{I_k} = 0$. If supp $p \not\subseteq I_k$, then $\langle p, w_k \rangle_{I_k} = 0$ by definition. If supp $p \subseteq I_{k-1}$, then $\langle p, w_k \rangle_{I_k} = \langle p, w_{k-1} \rangle_{I_{k-1}} = 0$ by induction. If supp $p \not\subseteq I_{k-1}$ and supp $p \subseteq I_k$, then $s_k \in$ supp p and so there exists $a \in \mathbb{R}$ such that $q := p + a\overline{p} \in \mathcal{I}$ is such that supp $q \subseteq I_{k-1}$. This implies that

$$\langle p, w_k \rangle_{I_k} = \langle q, w_k \rangle_{I_k} - a \langle \overline{p}, w_k \rangle_{I_k} = 0.$$

Let w be the unique element in $\mathbb{R}^{\mathbb{N}^2}$ such that $w_{|I_k} = w_k$ for all k. Then by construction we have that $w_{|I} = w_I$. Moreover if $q \in \mathcal{I}$, then $\langle z_1^{i_1} z_2^{i_2} q, w \rangle = 0$ for all $(i_1, i_2) \in \mathbb{N}^2$. In fact supp $z_1^{i_2} z_2^{i_2} q \subseteq I_k$, for some k and so $\langle z_1^{i_2} z_2^{i_2} q, w \rangle = \langle z_1^{i_1} z_2^{i_2} q, w_k \rangle_{I_k} = 0$. Therefore we have that $\mathcal{I} \subseteq \mathcal{A}(w)$ and by the previous observation this implies that $w \in \ker(p_1, \ldots, p_n)$. □

An immediate consequence of the previous proposition is the following corollary.

Corollary 1. Given $w_I \in \mathbb{R}^I$, we have that $\ker(p_1, \ldots, p_n)$ is an unfalsified model of w_I if and only if $(p_1, \ldots, p_n) \subseteq \mathcal{A}(w_I)$, where $(p_1, \ldots, p_n)$ is the ideal generated by $p_1, \ldots, p_n$.

The previous results do not allow to verify operatively if a model is unfalsified. This is possible if the set I is finite and admits the following representation

$$I = \{x \in \mathbb{N}^2 \, : \, x <_T k\} =: \Lambda(k), \tag{6}$$

for some $k \in \mathbb{N}^2$, where $<_T$ can be any admissible total ordering.

Proposition 2. Let $\{g_1, \ldots, g_m\}$ be a Gröbner basis of the ideal $\mathcal{I}$ generated by $g_1, \ldots, g_m$ w.r. to the total admissible ordering $<_T$. Let moreover $I = \Lambda(k)$ for some $k \in \mathbb{N}^2$ and $w_I \in \mathbb{R}^I$. Then $\ker(g_1, \ldots, g_m)$ is unfalsified model of w_I if and only if for all $s \in I \cap \deg \mathcal{I}$ there exists $i \in \{1, \ldots, m\}$ such that $s - \deg g_i = r = (r_1, r_2) \in \mathbb{N}^2$ and $\langle z_1^{r_1} z_2^{r_2} g_i, w_I \rangle_I = 0$.

Proof. ($\Rightarrow$) Trivial.
($\Leftarrow$) Let p be any polynomial in the ideal $\mathcal{I}$ generated by $g_1, \ldots, g_n$. If $\deg p \notin I$, then $\langle p, w_I \rangle_I = 0$. If $\deg p \in I$, then we will show that $\langle p, w_I \rangle_I = 0$ by induction on the $\deg p$, using the fact that $<_T$ must be a well ordering. Since $\{g_1, \ldots, g_m\}$ is a Gröbner basis, then there exists $i \in \{1, \ldots, m\}$ and $r = (r_1, r_2) \in \mathbb{N}^2$ such that $\deg p = \deg g_i + r$ and $\langle z_1^{r_1} z_2^{r_2} g_i, w_I \rangle_I = 0$. Since $\deg z_1^{r_1} z_2^{r_2} g_i = \deg p$, then there exists $a \in \mathbb{R}$ such that the degree of the polynomial $q := p + a z_1^{r_1} z_2^{r_2} g_i$ is less then $\deg p$. By induction we have that $\langle q, w_I \rangle_I = 0$ and so

$$\langle p, w_I \rangle_I = \langle q, w_I \rangle_I - a \langle z_1^{r_1} z_2^{r_2} g_i, w_I \rangle_I = 0.$$

□

It is immediate to prove the following corollary.

Corollary 2. In the same hypothesis of Proposition 2 we have that $\ker(g_1, \dots, g_m)$ is an unfalsified model of w_I if and only if $\{g_1, \dots, g_m\} \subseteq \mathcal{A}(w_I)$.

The previous corollary suggests an operative procedure verifying if a model $\ker(p_1, \dots, p_n)$ is unfalsified by a sequence w_I, if I admits a representation (6). Actually first we have to compute a Gröbner basis $\{g_1, \dots, g_m\}$ of the ideal $(p_1, \dots, p_n)$ and then we have to verify the conditions $\langle z_1^i z_2^j g_l, w_I \rangle_I = 0$ for every (i, j) contained in the finite sets $\Lambda(k - \deg\ g_l)$. For the other values of (i, j) the conditions are satisfied by definition of $\langle \cdot, \cdot \rangle_I$.

The condition imposing the set I to have the form (6) is rather restrictive and can be weakened in two different ways. First, it can be seen that the condition $I = \Lambda(k)$ in Proposition 2 and Corollary 2 can be substituted by the following two:

$$\begin{array}{rcl} I \cap \Delta(\mathcal{I}) & \supseteq & \Lambda(k) \cap \Delta(\mathcal{I}) \\ I \cap \deg\ \mathcal{I} & \subseteq & \Lambda(k) \cap \deg\ \mathcal{I}, \end{array} \tag{7}$$

where $\mathcal{I}$ is the ideal associated to the model $\mathcal{B}$.

On the other side, if the model $\mathcal{B}$ is finite dimensional, then it can be represented by a state model (4). Therefore $\mathcal{B}$ is an unfalsified model of w_I if and only if there exists $x(0,0) \in \mathbf{R}^d$ such that

$$w(i,j) = CM_1^i M_2^j x(0,0)$$

for all $(i, j) \in I$. These conditions constitute a system of linear equations with $x(0,0)$ as unknown. The model $\mathcal{B}$ is unfalsified if and only if this system has solution. In this case the only condition imposed on the set I is just its finite cardinality.

Now we can see what is the reason why the problem we want to solve is much more difficult in the 2D case than in the 1D case. In fact by the previous proposition we have that the research of unfalsified models of w_I reduces to the research of ideals in the set $\mathcal{A}(w_I)$ of the annihilators of w_I. Because of the structure of $\mathcal{A}(w_I)$ we see that if $p \in \mathcal{A}(w_I)$, then the principal ideal (p) generated by p is included in $\mathcal{A}(w_I)$. Therefore the principal ideals can be trivially extracted from $\mathcal{A}(w_I)$ and consequently it is also trivial to find all the unfalsified models in the 1D case, since the ring of the polynomials in one variable is a principal ideal domain. In the 2D case the situation is different since the polynomial ring in two variables is not a principal ideal domain any more. Moreover we have that $p_1, \dots p_n \in \mathcal{A}(w_I)$ does not imply that the ideal $(p_1, \dots p_n)$ generated by $p_1, \dots p_n$ is included in $\mathcal{A}(w_I)$. This is true if $\{p_1, \dots p_n\}$ is a Gröbner basis and this is a condition which is difficult to manipulate.

We will try now to give some characterizations of the undominated unfalsified models of a 2D scalar array w_I. We will find first that the undominated unfalsified models of w_I correspond to the maximal ideals in $\mathcal{A}(w_I)$.

Proposition 3. Let $p_1, \dots, p_n$ be polynomials and w_I be an array in $\mathbb{R}^I$. Then $\ker(p_1, \dots, p_n)$ is an undominated unfalsified model of w_I if and only if the ideal generated by $p_1, \dots, p_n$ is a maximal ideal in $\mathcal{A}(w_I)$

Proof. ($\Rightarrow$) Suppose that the polynomial p is such that the ideal $(p, p_1, \dots, p_n)$ generated by the polynomials $p, p_1, \dots, p_n$ is included in $\mathcal{A}(w_I)$. By Prop. 1 this

implies that $\ker(p, p_1, \ldots, p_n)$ is an unfalsified model of w_I and so $\ker(p, p_1, \ldots, p_n) = \ker(p_1, \ldots, p_n)$ since $\ker(p_1, \ldots, p_n)$ is undominated. This implies that $(p, p_1, \ldots, p_n) = (p_1, \ldots, p_n)$ and that $(p_1, \ldots, p_n)$ is a maximal ideal in $\mathcal{A}(w_I)$.
($\Leftarrow$) Suppose that $\ker(p, p_1, \ldots, p_n)$ is an unfalsified model of w_I. By Prop. 1 this implies that $(p, p_1, \ldots, p_n) \subseteq \mathcal{A}(w_I)$ and so $(p, p_1, \ldots, p_n) = (p_1, \ldots, p_n)$, since $(p_1, \ldots, p_n)$ is a maximal ideal in $\mathcal{A}(w_I)$. Consequently we have that $\ker(p, p_1, \ldots, p_n) = \ker(p_1, \ldots, p_n)$ and that $\ker(p_1, \ldots, p_n)$ is undominated. □

As previously mentioned, there may exist infinitely many different unfalsified undominated models of an array w_I. We want now to study under what kind of conditions the undominated unfalsified model is unique. For this pourpose it is useful the following lemma.

Lemma. Suppose that $\ker(p_1, \ldots, p_n)$ and $\ker(q_1, \ldots, q_m)$ are two undominated unfalsified models of w_I. Then $\ker(p_1, \ldots, p_n) = \ker(q_1, \ldots, q_m)$ if and only if there exists $w \in \ker(p_1, \ldots, p_n) \cap \ker(q_1, \ldots, q_m)$ such that $w_{|I} = w_I$.

Proof. One way is trivial. Suppose that there exists $w \in \ker(p_1, \ldots, p_n) \cap \ker(q_1, \ldots, q_m)$ such that $w_{|I} = w_I$. Then $\ker(p_1, \ldots, p_n) \cap \ker(q_1, \ldots, q_m)$ is unfalsified and so $\ker(p_1, \ldots, p_n) = \ker(p_1, \ldots, p_n) \cap \ker(q_1, \ldots, q_m) = \ker(q_1, \ldots, q_m)$ since both the models are undominated. □

4. ONE ALGORITHM FOR THE EXACT MODELLING OF FINITE 2D SCALAR ARRAYS

If I is a finite subset of $\mathbb{N}^2$, then it is easy to see that there always exist finite dimensional undominated unfalsified models. However in general also minimal dimension undominated unfalsified models may be infinitely many. Therefore one way to proceed could be finding a procedure providing one of the minimal dimension undominated unfalsified models from the 2D array. This is the method used for solving the classical partial realization problem in the 1D case. In the behavioural context this method is analyzed in [3], where it is shown that this modelling procedure lacks of many desirable properties. The proposed solution is based on the concept of corroboration. According to this idea, we have to reject an unfalsified model, if the data doesn't present an evidence of obeying the laws of that model. For instance consider the finite array $(1, 1, 1, 1, 1, 1, 1)$. It is clear that $\ker(z-1)$ can be considered a good model, while if we have the array $(0, 0, 0, 0, 0, 0, 1)$, then there is no evident reason to consider $\ker(z^7)$ a model of the array, even if it is a minimal dimension undominated model. The reason why $\ker(z^7)$ is a bad model is that the generic data compatible with this model would admit unfalsified models of smaller dimensions. Therefore if the data had been really generated by that model, we would have been very lucky to have data for which this model is a minimal dimension unfalsified model. It seems more realistic to say that there is no model that is evident from the data. A more convincing interpretation of the concept of corroboration bases on the data compression that must be provided by a good model. More precisely, if we consider an array and a model of it, then it is clear that in order to generate the array we need the parameters of the model and the initial conditions. If the sum of the number of parameters of the model and of the number of initial conditions we need is less than the number of the elements of the array, the model with the initial conditions provides a convenient description of the data and so a data compression. It is easy to see that, if we are given an 1D array $w_I \in \mathbb{R}^I$, where $I = \{0, 1, \ldots, N\}$, then a model $\mathcal{B} = \ker p$, $p \in \mathbb{R}[z]$, provides a data compression if and only if

$$\deg p + \Delta(p) \subseteq I, \tag{8}$$

where $\Delta(p) := \mathbb{N} \setminus (\deg p + \mathbb{N})$, or equivalently, if and only if $2 \deg p - 1 \leq N$.

In the 2D case the problem of finding the minimal dimension undominated unfalsified models of a finite array is still unsolved and seems a very hard problem. In this section we will propose an algorithm providing the minimal dimension undominated unfalsified model only under a condition that seems the 2D version of the corroboration condition in the 1D case. This condition is expressed below.

Fix a total ordering $<_T$ in $\mathbb{N}^2$. Given an ideal $\mathcal{I}$, consider the sets $M(\mathcal{I})$ and $\Delta(\mathcal{I})$ defined above. Consider moreover the set

$$\Sigma(\mathcal{I}) := M(\mathcal{I}) + \Delta(\mathcal{I}) = \{m + \delta : m \in M(\mathcal{I}),\ \delta \in \Delta(\mathcal{I})\}. \tag{9}$$

Note that, given a set of generators for $\mathcal{I}$, the sets $M(\mathcal{I})$, $\Delta(\mathcal{I})$ and $\Sigma(\mathcal{I})$ are effectively computable by Gröbner basis algorithms.

We say that the unfalsified model $\mathcal{B} = \ker(p_1, \ldots, p_n)$ of a 2D array $w_I \in \mathbb{R}^I$ is corroborated, if

$$\Sigma(\mathcal{I}) \subseteq I, \tag{10}$$

where $\mathcal{I}$ is the ideal generated by $p_1, \ldots, p_n$. We give now a result ensuring the uniqueness of the corroborated undominated unfalsified model. Note that (10) seems to be a good extension of the condition (8) to the 2D case.

Proposition 4. Fix a total admissible ordering $<_T$ and let $I = \{x \in \mathbb{N}^2 : x <_T k\}$ for some $k \in \mathbb{N}^2$. If w_I is any 2D array in $\mathbb{R}^I$, then there exists at most one corroborated undominated unfalsified model of w_I.

P r o o f. Suppose that $\ker(p_1, \ldots, p_n)$ and $\ker(q_1, \ldots, q_m)$ are two undominated unfalsified models of w_I and that $\Sigma(\mathcal{I})$ and $\Sigma(\mathcal{J})$, where $\mathcal{I}$ and $\mathcal{J}$ are the ideals generated by $p_1, \ldots, p_n$ and by $q_1, \ldots, q_m$ respectively, are included in I. We have to show that $\ker(p_1, \ldots, p_n) = \ker(q_1, \ldots, q_m)$. It is not restrictive to assume that $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_m\}$ are Gröbner bases w.r. to $<_T$ and that the leading power coefficients of all the polynomials are unitary. We will construct $w \in \ker(p_1, \ldots, p_n) \cap \ker(q_1, \ldots, q_m)$ such that $w_{|I} = w_I$. Suppose that we have assigned w in $\Lambda(u)$ with $u_T \geq k$. We want to show that there exist $v_1 \in M(\mathcal{I})$ and $v_2 \in M(\mathcal{J})$ such that

$$u - v_1 - v_2 = t \in \mathbb{N}^2.$$

Suppose that it is not true. Then for all $v_1 \in M(\mathcal{I})$ and $v_2 \in M(\mathcal{J})$ we have that $u - v_1 \notin \deg\ \mathcal{J}$ and $u - v_2 \notin \deg\ \mathcal{I}$. This implies that $u - v_1 \in \Delta(\mathcal{J})$ and $u - v_2 \in \Delta(\mathcal{I})$ and so there exist $\delta_1 \in \Delta(\mathcal{I})$ and $\delta_2 \in \Delta(\mathcal{J})$ such that $u = v_1 + \delta_2$ and

$u = v_2 + \delta_1$. Consequently we have that $2u = (v_1 + \delta_1) + (v_2 + \delta_2)$. This can not be true since $v_1 + \delta_1 \in \Delta(\mathcal{I}) \subseteq I$ and $v_2 + \delta_2 \in \Delta(\mathcal{J}) \subseteq I$ and so $v_1 + \delta_1 + v_2 + \delta_2 <_T 2u$. Since $v_1 \in M(\mathcal{I})$ and $v_2 \in M(\mathcal{J})$, there exist positive integers $i(u)$ and $j(u)$ such that $\deg p_{i(u)} = v_1$ and $\deg q_{j(u)} = v_2$. Construct the sequence $w \in \mathbb{R}^{\mathbb{N}^2}$ fixing

$$w(u) = \begin{cases} w_I(u) & \text{if } u \in I \\ -\sum_{\ell <_T v_1} p_{i(u)}(\ell) w(u - v_1 + \ell) & \text{otherwise,} \end{cases} \tag{11}$$

where $p_{i(u)}(\ell)$, $\ell = (\ell_1, \ell_2) \in \mathbb{N}^2$ is the coefficient of the term $z_1^{\ell_1} z_2^{\ell_2}$ in the polynomial $p_{i(u)}$. We have to show now that $w \in \ker(p_1, \ldots, p_n) \cap \ker(q_1, \ldots, q_m)$. First we prove that $\langle p, w \rangle = 0$, for every $p \in \mathcal{I}$ by transfinite induction on $u = \deg p$. It is not restrictive to suppose that the leading power coefficient of p is unitary. If $u <_T k$ then $\langle p, w \rangle = \langle p, w_I \rangle_I = 0$. Otherwise let $q = p - z_1^{s_1} z_2^{s_2} p_{i(u)}$, where $s = (s_1, s_2) = u - \deg p_{i(u)} \in \mathbb{N}^2$. Note that $\langle p_{i(u)} z_1^{s_1} z_2^{s_2}, w \rangle = 0$ by construction. Since $q \in \mathcal{I}$ and since $\deg q <_T u$, then by induction $\langle q, w \rangle = 0$ and so $\langle p, w \rangle = 0$.

We prove now that $\langle p, w \rangle = 0$, for every $p \in \mathcal{J}$ by transfinite induction on $u = \deg p$. It is not restrictive to suppose that the leading power coefficient of p is unitary. If $u <_T k$ then $\langle p, w \rangle = \langle p, w_I \rangle_I = 0$. Otherwise let $q = p - z_1^{r_1} z_2^{r_2} q_{j(u)}$, where $r = (r_1, r_2) = u - \deg q_{j(u)} \in \mathbb{N}^2$. First note that $\langle z_1^{r_1} z_2^{r_2} q_{j(u)}, w \rangle = 0$. Actually if $s = (s_1, s_2) = u - \deg q_{j(u)}$, then $s - \deg p_{i(u)} \in \mathbb{N}^2$ and so for all $\ell \in \mathbb{N}^2$ we have that

$$w(\ell + s) = -\sum_{d <_T \deg p_{i(u)}} p_{i(u)}(d) w(\ell + s - \deg p_{i(u)} + d).$$

Consequently we have that

$$\begin{aligned} & \langle z_1^{s_2} z_2^{s_2} q_{j(u)}, w \rangle = w(k) + \sum_{\ell <_T \deg q_{j(u)}} q_{j(u)}(\ell) w(\ell + s) = \\ = \;& w(k) - \sum_{\ell <_T \deg q_{j(u)}} q_{j(u)}(\ell) \sum_{d <_T \deg p_{i(u)}} p_{i(u)}(d) w(\ell + s - \deg p_{i(u)} + d) = \\ = \;& w(k) - \sum_{d <_T \deg p_{i(u)}} p_{i(u)}(d) \sum_{\ell <_T \deg q_{j(u)}} q_{j(u)}(\ell) w(\ell + s - \deg p_{i(u)} + d) = \\ = \;& w(k) + \sum_{d <_T \deg p_{i(u)}} p_{i(u)}(d) w(u - \deg p_{i(u)} + d) \;=\; 0, \end{aligned}$$

where we have exploited the fact that if $t = (t_1, t_2) <_T u - \deg q_{j(u)}$, then $\langle z_1^{t_1} z_2^{t_2} q_{j(u)}, w \rangle = 0$ by induction. Since $q \in \mathcal{I}$ and since $\deg q <_T u$, then by induction $\langle q, w \rangle = 0$ and so $\langle p, w \rangle = 0$. □

From the previous proposition we can argue that, since the corroborated undominated unfalsified model of a 2D array is unique when it exists, then it can be considered the most powerful unfalsified model in the class of all the the models satisfying condition (10).

We give now a procedure providing the corroborated undominated unfalsified model of a 2D array, when it exists. This algorithm is based on a procedure proposed by Sakata [4, 5] extending the classical Berlekanp Massey algorithm that solves the partial realization problem in 1D case. This procedure is characterized by a good efficiency and is recursive, namely if a new element of the array is provided the computation of the new solution uses in some way the work we have done to compute the old solutions. Its limitation consists on the fact that this algorithm provides unfalsified models only under particular conditions.

Let $<_T$ be the total degree ordering in $\mathbb{N}^2$, i. e. the total admissible ordering such that $(s_1, s_2) <_T (r_1, r_2)$ if and only if $s_1 + s_2 < r_1 + r_2$ or if $s_1 + s_2 = r_1 + r_2$ and $s_1 < r_1$. Suppose moreover that $I = \{x \in \mathbb{N}^2 : x <_T k\}$ for some $k \in \mathbb{N}^2$. If M is the set of all the minimal elements in the set $\deg \mathcal{A}(w_I)$ w. r. to the usual partial ordering in $\mathbb{N}^2$, then a minimal polynomial set $\mathcal{F}$ for w_I is a subset of $\mathcal{A}(w_I)$ such that $\deg \mathcal{F} = M$. It is easy to see that there always exists minimal polynomial sets $\mathcal{F}$ such that each polynomial $f \in \mathcal{F}$ is monic and in a reduced normal form w. r. to the ideal generated by $\mathcal{F}$ (or equivalently that the support of f is included in $\Delta(\mathcal{F}) \cup \{\deg f\}$, where $\Delta(\mathcal{F}) := \mathbb{N}^2 \setminus (\deg \mathcal{F} + \mathbb{N}^2)$, and so we will consider only minimal polynomial sets satisfying this requirement.

The condition ensuring the uniqueness of the minimal polynomial set is similar to condition (10). More precisely define for every finite family of polynomials $\mathcal{F}$ the subsets

$$\begin{aligned} \Delta(\mathcal{F}) &:= \mathbb{N}^2 \setminus (\deg \mathcal{F} + \mathbb{N}^2) \\ \Sigma(\mathcal{F}) &:= \deg \mathcal{F} + \Delta(\mathcal{F}) \end{aligned} \quad . \tag{12}$$

Note that if $\mathcal{G}$ is the reduced Gröbner basis of the ideal $\mathcal{I}$ w. r. to an admissible ordering $<_T$, then we have that $\Delta(\mathcal{G}) = \Delta(\mathcal{I})$ and $\Sigma(\mathcal{G}) = \Sigma(\mathcal{I})$.

Proposition 5. [4] Fix a total admissible ordering $<_T$ and let $I = \{x \in \mathbb{N}^2 : x <_T k\}$ for some $k \in \mathbb{N}^2$. Let w_I be any 2D array in $\mathbb{R}^I$. If there exists a minimal polynomial set $\mathcal{F}$ of a 2D array w_I, such that

$$\Sigma(\mathcal{F}) \subseteq I, \tag{13}$$

then it is unique.

Given w_I, Sakata's algorithm provides the family of all the minimal polynomial sets for w_I. However we have that a minimal polynomial set is not always a Gröbner basis, which is a necessary and sufficient condition for a minimal polynomial set to give an unfalsified model. In [5] there is an example in which we have that the minimal polynomial set is unique, but it is not a Gröbner basis. The following proposition shows that if there exists the corroborated undominated unfalsified model of a 2D array w_I, then it can be extracted from a minimal polynomial set.

Proposition 6. Fix a total admissible ordering $<_T$ and let $I = \{x \in \mathbb{N}^2 : x <_T k\}$ for some $k \in \mathbb{N}^2$. Let w_I be any 2D array in $\mathbb{R}^I$ and $\mathcal{F}$ be any minimal polynomial set of w_I. If there exists the corroborated undominated unfalsified model of w_I, then there exists a subset $\mathcal{F}' = \{f_1, \ldots, f_m\}$ of $\mathcal{F}$ such that $\ker(f_1, \ldots, f_m)$ is the corroborated undominated unfalsified model of w_I.

Proof. Suppose that $\mathcal{B} = \ker(g_1, \ldots, g_m)$ is a corroborated undominated unfalsified model of w_I and that $\mathcal{G} = \{g_1, \ldots, g_m\}$ is the reduced Gröbner basis w. r.

to the admissible ordering $<_T$. Since $\mathcal{B}$ is corroborated, then $\Sigma(\mathcal{G}) \subseteq I$. Moreover there exists $w \in \ker(g_1, \ldots, g_m)$ such that $w_{|I} = w_I$. Define the following subset of the minimal polynomial set $\mathcal{F}$

$$\mathcal{F}' := \{f \in \mathcal{F} : \deg g - \deg f \in \mathbb{N}^2,\ \exists g \in \mathcal{G}\}. \tag{14}$$

We will show now that $\mathcal{F}'$ is a Gröbner basis. If it is not the case, then there exists $s = (s_1, s_2) \in \mathbb{N}^2$ and $\bar{f} \in \mathcal{F}'$ such that $\langle z_1^{s_1} z_2^{s_2} \bar{f}, w\rangle \neq 0$. By Lemma 4 in [4], for all $g \in \mathcal{G}$ we have $s - \deg g \notin \mathbb{N}^2$ and so $s \in \Delta(\mathcal{G})$. Finally, taking $\bar{g} \in \mathcal{G}$ such that $t = \deg \bar{g} - \deg \bar{f} \in \mathbb{N}^2$, we can argue that $\deg \bar{g} + s = \deg \bar{f} + s + t$ and so, since by Corollary 2 $\deg \bar{f} + s \notin I$, then $\Sigma(\mathcal{G}) \not\subseteq I$ that is against the hypothesis. □

By the previous proposition we can extract the corroborated undominated unfalsified model of a 2D array w_I, when it exists, from a minimal polynomial set $\mathcal{F}$ in the following way:

1. Let $\mathcal{F}_1, \ldots, \mathcal{F}_l$ be the subsets of $\mathcal{F}$ such that $\Sigma(\mathcal{F}_i) \subseteq I$.
2. If there exists $\mathcal{F}_i = \{g_1, \ldots, g_m\}$ that is a Gröbner basis, then $\ker(g_1, \ldots, g_m)$ is the corroborated undominated unfalsified model of w_I. Otherwise by Proposition 6 the corroborated undominated unfalsified model of w_I does not exist.

In this paper we propose an extension of the concept of corroboration to the 2D case. Its connections with the data compression is still not completely clear to us and it will be the object of our future research.

Note moreover that if I is a finite subset of $\mathbb{N}^2$, then every corroborated undominated unfalsified model must be finite dimensional and this seems too restrictive. Presently we are trying to extend the definition of corroborated models without this restriction and to develop a modelling algorithm providing the corroborated undominated unfalsified model of a 2D array w_I in this case.

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