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INTERPRETATIONS OF THE GAP TOPOLOGY: A SURVEY[1]

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We explore the interconnections between various ways of introducing the gap topology for linear time-invariant input/output systems. Specifically, we consider:

1. the topology defined by the gaps between the graphs of transfer functions
2. Vidyasagar's graph topology
3. the weakest topology in which the closed loop behavior of the standard feedback interconnection is continuous
4. the topology of uniform convergence of the associated Martin–Hermann mappings from $\mathbf{C}^+$ to the Grassmannian manifold Grass$(m, m+p)$ ('pointwise gap')
5. the gap topology defined by the gaps between the associated $L^2(-\infty, 0)$-behaviors.

We also compare some *different* gap topologies.

1. INTRODUCTION AND PRELIMINARIES

The gap topology for linear systems, originally introduced in [18], has recently been characterized in a variety of ways. Our purpose in this paper is to collect these characterizations and to show in an efficient way how they are connected. The framework we shall use is that of L_2-behaviors (cf. [17]).

Let us introduce some notation and terminology first. Given a finite-dimensional linear system in standard input/state/output form with parameters $\Sigma=(X, U, Y; A, B, C, D)$, the associated L_2-behavior ([17, § XI.3], [15, Ch. 3]) is the set $\mathcal{B}(\Sigma) \subset L_2(\mathbf{R}; Y \times U)$ defined by

$$\mathcal{B}(\Sigma) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in L_2(\mathbf{R}; Y \times U) \,|\, \exists\, x \in L_2(\mathbf{R}; X) : \dot{x} = Ax + Bu, \quad y = Cx + Du \right\}.$$

For our purposes it will often be convenient to use 'driving-variable' representations [17, p. 275] rather than input/state/output representations. In a DV representation, the external variables w need not be split into inputs u and outputs y, and so we use one external variable space W instead of a product $Y \times U$. An auxiliary input v is introduced, and for $\Sigma = (A, B, C, D)$ we write

$$\operatorname{im} \Sigma = \{ w \in L_2(\mathbf{R}; W) \mid \exists\, x \in L_2(\mathbf{R}; X), v \in L_2(\mathbf{R}; V) : \dot{x} = Ax + Bv, w = Cx + Dv \}.$$

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For a set of state space parameters Σ with state space X, we write $\deg\Sigma = \dim X$. Time axes that we shall use are $T = \mathbf{R}$, $T = (0,\infty)$, and $T = (-\infty, 0)$; for brevity, we shall often write $L_2(T)$ instead of $L_2(T; W)$. The *concatenation* of an element f_1 of $L_2^+ = L_2(0,\infty)$ and an element f_2 of $L_2^- = L_2(-\infty, 0)$ is defined by

$$\begin{aligned}(f_1 \wedge f_2)(t) &= f_1(t) \quad (t < 0)\\ &= f_2(t) \quad (t \geq 0).\end{aligned}$$

The *standard embedding* of L_2^+ into $L_2(-\infty,\infty)$ is defined by the mapping $i_+ : f \mapsto 0 \wedge f$. The *standard projection* of $L_2(-\infty,\infty)$ onto L_2^+ is denoted by $\mathcal{P}_+$ and defined by $(\mathcal{P}_+ f)(t) = f(t)$ $(t \geq 0)$. For any real number d, let σ_d and τ_d denote the forward and the backward shift by d. By abuse of notation, we use the same symbols to denote the forward and backward shifts on L_2^- and L_2^+; so for instance $(\sigma_d f)(t) = f(t-d)$ for $f \in L_2$, whereas for $f \in L_2^+$ we have $(\sigma_d f)(t) = f(t-d)$ if $t \geq d$, and $(\sigma_d f)(t) = 0$ otherwise. By $\mathcal{L}$ we shall indicate the standard Fourier-Laplace transform isomorphism from $L_2(-\infty,\infty)$ to $L_2(i\mathbf{R}) = H_2^- \oplus H_2$.

Besides state space representations, other representations such as transfer functions and Martin–Hermann mappings may be used for L_2-behaviors, as will be discussed below. These correspond to various 'approaches' that are in the present context all mathematically equivalent (note that L_2-behaviors are automatically controllable in the sense of J.C. Willems, see [17, p. 280] or [15, Thm. 3.8]). In this light, the choice of a starting point could be viewed as arbitrary. However, we shall follow J.C. Willems in taking the notion of 'behavior' as fundamental. In the L_2-context, this means that a 'system' with external variable space W is a closed shift-invariant subspace of $L_2(\mathbf{R}, W)$. The set of all such subspaces will be denoted by $\mathcal{S}(W)$. The space of external variables W will always be taken to be a Euclidean space, so that $L_2(\mathbf{R}; W)$ is a real Hilbert space; in connection with Fourier transforms we will need the complexification of W, which will be used without specific mention.

We start by giving a criterion for a system in $\mathcal{S}(W)$ to allow a finite-dimensional state space. For this we first need some further notation, and a lemma. Let a closed shift-invariant subspace V of $L_2(\mathbf{R})$ be given. We can associate two subspaces of L_2^+ with V, namely the projection of V onto L_2^+,

$$V_+ := \mathcal{P}_+ V = \left\{ w \in L_2^+ \mid \exists\, w' \in L_2^- \text{ s.t. } w' \wedge w \in V \right\},$$

and the inverse image of V under the embedding of L_2^+ into L_2,

$$V_+^0 := i_+^{-1} V = \{ w \in L_2^+ \mid 0 \wedge w \in V \}.$$

(If we would identify L_2^+ with its embedding into L_2, we could simply write $V_+^0 = V \cap L_2^+$.) Obviously V_+^0 is a subspace of V_+. The space V_+ is invariant under the backward shifts because $\tau_d \mathcal{P}_+ = \mathcal{P}_+ \tau_d$, and the space V_+^0 is invariant under the forward shifts because $i_+ \sigma_d = \sigma_d i_+$. The dimension of the orthogonal complement $V_+ \ominus V_+^0$ of V_+^0 in V_+ will be denoted by $[V_+ : V_+^0]$. We shall write

$$\mathcal{S}_{\mathrm{fd}}(W) = \left\{ V \in \mathcal{S}(W) \mid [V_+ : V_+^0] < \infty \right\}.$$

It will be seen in Prop. 1.2 below that this set indeed singles out the finite-dimensional systems in $\mathcal{S}(W)$, as suggested by the notation; but first we present a lemma that will be needed in the proof of the proposition.

1.1. Lemma. If V and $\tilde{V}$ are behaviors in $\mathcal{S}_{\mathrm{fd}}(W)$ such that $V_+^0 = \tilde{V}_+^0$, then $V = \tilde{V}$.

Proof. We first show that V_+ can be reconstructed from V_+^0 as the smallest backwards invariant subspace containing V_+^0. It has already been noted above that V_+ is backwards invariant and contains V_+^0, so it only remains to show that any backwards invariant subspace containing V_+^0 must also contain V_+. Take any w_+ in V_+. By the Willems controllability of the behavior V [17, p. 280], there exists a $w_- \in L_2^-$ such that $w := w_- \wedge w_+ \in V$ and $w_-(t) = 0$ for all t smaller than some $T > 0$. Then $w_+ = \tau_{-T}\mathcal{P}_+\sigma_{-T}w$ where $\mathcal{P}_+\sigma_{-T}w$ is in V_+^0, so that any backwards invariant subspace in L_2^+ containing V_+^0 must also contain w_+.
Now, let V and $\tilde{V}$ be as in the statement of the lemma. By what has just been said, we must have $V_+ = \tilde{V}_+$ and therefore also $V_+ \ominus V_+^0 = \tilde{V}_+ \ominus \tilde{V}_+^0 =: X$. Define the following two seminorms on $V_+ = \tilde{V}_+$:

$$\|w_+\|_- := \min\{\|w_-\|_{L_2(-\infty,0)} \mid w_- \wedge w_+ \in V\}$$

and

$$\|w_+\|_-^{\sim} := \min\{\|w_-\|_{L_2(-\infty,0)} \mid w_- \wedge w_+ \in \tilde{V}\}$$

When restricted to X, these seminorms are actually norms. Since X is finite-dimensional, both norms must be equivalent and so there exists a constant c such that $\|w_+\|^{\sim} \leq c\|w_+\|$ for all $w_+ \in X$, from which the same inequality also follows for w_+ in $V_+ = \tilde{V}_+$.
Take $w \in V$, and let $\varepsilon > 0$. We can find $d > 0$ such that $\|\mathcal{P}_-\sigma_d w\| < \varepsilon$. Then clearly $\|\mathcal{P}_+\sigma_d w_-\| < \varepsilon$ so that $\|\mathcal{P}_+\sigma_d w_-\|^{\sim} < c\varepsilon$. It follows that there exists a $\tilde{w} \in \tilde{V}$ such that $\|w - \tilde{w}\| < (c+1)\varepsilon$. Because $\tilde{V}$ is closed, this shows that $V \subset \tilde{V}$, and the reverse inclusion follows by symmetry. □

1.2. Proposition. If $V \in \mathcal{S}(W)$ has a finite-dimensional DV representation Σ, then $\deg\Sigma \geq [V_+ : V_+^0]$ and so in particular $[V_+ : V_+^0]$ is finite. Conversely, any $V \in \mathcal{S}_{\mathrm{fd}}(W)$ has a finite-dimensional DV representation $\Sigma = \Sigma(A, B, C, D)$ with $\deg\Sigma = [V_+ : V_+^0]$.

Proof. Concerning the first claim, observe that V_+^0 is the largest closed forward shift-invariant subspace of V_+. Let $L \subset V_+$ be the space of outputs of Σ that can arise with an initial state $x(0) = 0$. Since we are imposing $\deg\Sigma$ linear constraints, we must have $[V_+ : L] \leq \deg\Sigma$. It is clear that L is invariant for the forward shifts. Therefore L must be a subspace of V_+^0 and it follows that $\deg\Sigma \geq [V_+ : L] \geq [V_+ : V_+^0] = n$.
To prove the second statement, let $X_0 = \mathcal{L}V_+^0$. Because X_0 is a closed shift-invariant subspace of H_2, there exists by the Beurling–Lax theorem [8] an inner function $\Theta(s)$ such that $X_0 = \Theta H_\infty$. We shall construct a finite-dimensional driving-variable representation of V by a state-space realization of the Beurling symbol Θ. To this end we must show that $\Theta(s)$ is rational. Let K be the orthogonal complement of $X_1 := \mathcal{L}V_+$. Since V_+ is invariant for the backward shift, K is a shift-invariant subspace of H_2. Let Θ_K be a Beurling symbol for K. Now the inner matrix

$$\Omega = [\Theta \quad \Theta_K]$$

is such that $\operatorname{im}\Omega = X_0 \oplus K$. So $H_2 \ominus \Omega H_2 = X_1 \ominus X_0 = \mathcal{L}(V_+ \ominus V_+^0)$. If $[V_+ : V_+^0]$ is finite, it follows that the matrix Ω must be square and rational, and has McMillan degree equal to $[V_+ : V_+^0]$ [9, p. 33]. This immediately implies that $\Theta(s)$ is rational with McMillan degree at most equal to $[V_+ : V_+^0]$. Now choose a minimal state space realization Σ of Θ, and write $\tilde{V} := \operatorname{im}\Sigma$. We have $\tilde{V}_+^0 = V_+^0$ by construction, and so it follows from the preceding lemma that $V = \tilde{V}$. □

In view of the proposition, a natural choice of a state space for an L_2-behavior V would be $V_+ \ominus V_+^0$ (the orthogonal complement of V_+^0 in V_+). This will be used in Proposition 7.3 below. The whole construction can of course also be carried out with respect to the left halfline, leading to a state space $V_- \ominus V_-^0$.

Let us now discuss how the graph of a transfer function can be viewed as a representation of an L_2-behavior. If the space W is written as $Y \times U$, in such a way that U is an input space for the system V [17, §VIII], then there exists a rational transfer function G from U to Y such that the Beurling symbol $\Theta \in H_\infty$ of V_+^0 is of the form $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$, with M invertible and $G = NM^{-1}$. The $L_2(i\mathbf{R})$-graph of a transfer function is of course defined as

$$\mathcal{G}_{L_2}(G) = \{(y,u) \in L_2(i\mathbf{R}) \mid y(s) = G(s)\,u(s)\}\,.$$

This means that we have

$$\mathcal{L}V = \mathcal{G}_{L_2}(G).$$

Since this relation determines V uniquely in terms of G, we can use it to define V and V_+ as the 'behaviors of G' in $L_2(\mathbf{R})$ and L_2^+ respectively. We write

$$\mathcal{B}(G) = V = \mathcal{L}^{-1}(\mathcal{G}_{L_2}(G))$$

and

$$\mathcal{B}_+(G) = V_+.$$

We also introduce the H_2-graph of a transfer function, where H_2 is the Hardy space of W-valued functions on the right half plane. This graph will be denoted simply by $\mathcal{G}(G)$ and is defined by

$$\mathcal{G}(G) = \{(y,u) \in H_2 \mid y(s) = G(s)u(s)\}.$$

We have

$$\mathcal{G}(G) = \mathcal{L}(\mathcal{B}_+^0(G)).$$

Of course we can also consider $\mathcal{B}_-(G) = V_-$ and $\mathcal{G}_-(G) = \mathcal{L}(\mathcal{B}_-^0(G))$. The reader should bear in mind that it is possible for any V in the class we consider to find a division of the external variables into inputs and outputs such that $V = \mathcal{B}(G)$ for some rational G. So considering graphs of transfer functions entails no loss of generality.

It is often useful to have a description of the orthogonal complement of a graph or behavior. In $L_2(\mathbf{R})$, the orthogonal complements of shift-invariant spaces are shift-invariant themselves, and the adjoint of a multiplication operator is also a multiplication operator, so the orthogonal complement of a behavior in $L_2(\mathbf{R})$ is easy to describe. Let $\tilde{G}(s) = G^T(-s)$ as usual.

1.3. Lemma. $\mathcal{B}(G)^{\perp} = \mathcal{B}(-\tilde{G})$.

Proof. Write

$$\mathcal{L}(\mathcal{B}(G)) = \mathbf{M}_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]} L_2(i\mathbf{R}),$$

where $G = NM^{-1}$ is a coprime factorization over L_∞, and $\mathbf{M}_\Theta$ is the multiplication operator by Θ. The adjoint of the multiplication operator by $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is of course the operator

$$\mathbf{M}_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]}^{*} = \mathbf{M}_{[\tilde{N} \;\; \tilde{M}]},$$

so $\mathcal{L}(\mathcal{B}(G))^{\perp} = \ker \mathbf{M}_{[\tilde{N} \;\; \tilde{M}]}$, and the latter space is equal to $\mathcal{L}(\mathcal{B}(-\tilde{G}))$. □

The orthoplement of an H_2-graph is somewhat more complicated to describe. We relate it here to the L_2^+-behavior of the 'dual' system.

1.4. Proposition. $\mathcal{G}(G)^{\perp} = \mathcal{L}(\mathcal{B}_+(-\tilde{G}))$.

Proof. For closed subspaces V, W of a Hilbert space, one has in general $V \ominus (V \cap W) = V \cap (V \cap W)^{\perp} = V \cap (V^{\perp} + W^{\perp}) = \mathcal{P}_V(W^{\perp})$. Applying this to $V = H_2$, $W = \mathcal{L}(\mathcal{B}(G))$, we get the result by noting that $W^{\perp} = \mathcal{L}(\mathcal{B}(-\tilde{G}))$, $V \cap W = \mathcal{G}(G)$, and $\mathcal{P}_V(W^{\perp}) = \mathcal{L}(\mathcal{B}_+(-\tilde{G}))$. □

We next come to another way of representing L_2-behaviors. Instead of describing a system in the frequency domain by its transfer function, one can also identify it with the associated *Martin–Hermann mapping* to a Grassmannian manifold. This mapping is defined as follows. Let Grass(m, q) denote the Grassmannian manifold of m-dimensional subspaces of a q-dimensional linear space. For any $q \times m$ rational matrix $F(s)$, the associated mapping

$$f : s \mapsto \operatorname{im} F(s) \in \text{Grass}(m, q)$$

is defined initially only for those s that are not poles or zeros of $F(s)$, but it can be extended in a unique way to a regular mapping from the extended complex plane $\mathbf{C}_\infty$ to the Grassmannian [6]. This mapping is continuous with respect to the spherical metric of $\mathbf{C}_\infty$ and the gap metric on Grass(m, q). The map can be restricted to a given subset Ω of $\mathbf{C}_\infty$; the set of all mappings that are obtained this way will be denoted by $\mathcal{R}(\Omega, m, q)$. In particular it is of interest in the theory of robust stability to take $\Omega = \mathbf{C}^+$ as the domain of definition, where $\mathbf{C}^+$ denotes the closed right half plane (including the point at infinity). We shall define the *Martin–Hermann mapping* of the system V as the mapping associated in this way to the Beurling symbol Θ of V_+^0.

Note that, in the context of input/output systems, the Beurling symbol corresponds to a normalized right coprime factorization of the transfer matrix. Since the factors in any coprime factorization are related to each other by right multiplication by an RH_∞-unimodular matrix, the Martin-Hermann mapping may also be defined in terms of an arbitrary RH_∞-coprime factorization $G(s) = N(s)M^{-1}(s)$ by

$$s \mapsto \operatorname{im} \begin{bmatrix} N(s) \\ M(s) \end{bmatrix}.$$

Another way to obtain the Martin–Hermann mapping associated with V is [11]

$$s \mapsto \{g(s) \mid g \in \mathcal{L}V_+^0\}.$$

An even more direct method is to write

$$s \mapsto \{w \mid \text{the mapping } t \mapsto e^{st}w \text{ belongs to (the complexification of) } V\}$$

although this defines the Martin–Hermann mapping only on the open right half plane; extension to the closed right half plane can then be made by continuity.
A slightly different viewpoint, which is in some senses closer to classical control theory, is obtained by looking at subspace-valued functions defined only on the extended imaginary axis. Let $f(i\omega)$ be such a function, and suppose that there exists an RH_∞-matrix $F(s)$ having full column rank throughout the right half plane such that $f(i\omega) = \operatorname{im} F(i\omega)$ for all $\omega \in \mathbf{R} \cup \{\infty\}$. To the curve $f(i\omega)$, we then associate the Martin–Hermann mapping $s \mapsto \operatorname{im} F(s)$. It has to be verified that this is indeed a valid definition. Note in the first place that the values of $F(s)$ in the open right half plane are determined by the values of $F(s)$ on the imaginary axis through the Poisson integral formula. Furthermore, suppose that $F_1(s)$ and $F_2(s)$ are both representations of $f(i\omega)$ of the indicated type. Because $\operatorname{im} F_1(i\omega) = \operatorname{im} F_2(i\omega)$ for all $\omega \in \mathbf{R}$, there exists a uniquely determined matrix function $M(i\omega)$ such that $F_1(i\omega) = F_2(i\omega)M(i\omega)$ for all $\omega \in \mathbf{R}$. By the full rank assumption, $F_2(s)$ has a left inverse $F_2^+(s)$ that is analytic on the right half plane. It follows that $M(s) = F_2^+(s)\,F_1(s)$ provides an analytic extension of $M(i\omega)$ into the right half plane. By the uniqueness of extensions, we must have $F_1(s) = F_2(s)\,M(s)$ for all s in the right half plane and in particular it follows that $\operatorname{im} F_1(s) \subset \operatorname{im} F_2(s)$ for all s in $\mathbf{C}^+$; the reverse inclusion follows by symmetry. So we see that indeed a unique Martin–Hermann mapping on the right half plane can be associated with a curve $f(i\omega)$ as above. A continuous curve $f(i\omega)$ on the Grassmannian such that $f(i\omega) = \operatorname{im} F(i\omega)$ for some H_∞-matrix $F(s)$ of constant rank on the right half plane might reasonably be called a *Nyquist curve*; indeed, the usual Nyquist curve for single-input-single-output systems is obtained via the standard identification of the Grassmannian manifold Grass$(1,2)$ with the extended complex plane by the mapping $\operatorname{im}\left[\begin{smallmatrix} s \\ 1 \end{smallmatrix}\right] \mapsto s$, $\operatorname{im}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \mapsto \infty$.
It should be noted that the Martin–Hermann mapping (or the Nyquist curve) associated with V specifies V uniquely. Indeed, if

$$\operatorname{im} \begin{bmatrix} N_1(s) \\ M_1(s) \end{bmatrix} = \operatorname{im} \begin{bmatrix} N_2(s) \\ M_2(s) \end{bmatrix}$$

for all $s \in \mathbf{C}^+$, and both matrices have full column rank everywhere on $\mathbf{C}^+$, then there exists an RH_∞-unimodular matrix $T(s)$ such that $N_1(s) = N_2(s)T(s)$ and $M_1(s) = M_2(s)T(s)$, so that both systems have the same transfer function. It is also easy to see that every element of $\mathcal{R}(\mathbf{C}^+, m, q)$ can be obtained as the Martin–Hermann mapping associated to some system V in $\mathcal{S}_{\mathrm{fd}}(W)$. We may therefore identify $\mathcal{R}(\mathbf{C}^+, m, q)$ with $\mathcal{S}_{\mathrm{fd}}(W)$.
In order to define topologies, we shall frequently use the *gap* function. The gap between two closed subspaces X, Y of a Hilbert space is given by

$$\delta(X,Y) := \|\mathcal{P}_X - \mathcal{P}_Y\|,$$

where $\mathcal{P}_X$ is the orthogonal projection on X, or equivalently by

$$\delta(X,Y) = \max(\vec{\delta}(X,Y), \vec{\delta}(Y,X)),$$

where

$$\vec{\delta}(X,Y) = \sup_{x\in X, \|x\|=1} d(x,Y).$$

The gap is a metric and hence can be used to define a topology on the set of closed subspaces of H. We shall use the gap both in finite- and in infinite-dimensional contexts.

We are ready to introduce most of the definitions of topologies we shall consider. For purposes of comparison and to avoid technicalities, the topologies that will be discussed below will all be defined on the set $\mathcal{S}_{\mathrm{fd}}(W)$, even though some of the definitions apply as well to $\mathcal{S}(W)$ as a whole. The definition below is due to [18].

1.5. Definition. The *gap topology* on $\mathcal{S}_{\mathrm{fd}}(W)$ is the topology induced by the distance function

$$\delta_{H_2}(U,V) = \delta(\mathcal{L}U_+^0, \mathcal{L}V_+^0).$$

It is natural also to consider the gaps between solution sets. Several options are possible, which will be discussed in section 7. To get equivalence with the other topologies defined in this section, we need to consider the gap between the behaviors on the *left* half-line. This characterization does not seem to have appeared in the literature before.

1.6. Definition. The topology $\mathcal{O}_{L_2^-}$ on $\mathcal{S}_{\mathrm{fd}}(W)$ is the topology induced by the gap

$$\delta_-(U,V) = \delta(U_-, V_-).$$

The third definition that we shall consider is due to [11]. For the purposes of this definition, we replace $\mathcal{S}_{\mathrm{fd}}(W)$ by $\mathcal{R}(\mathbf{C}^+, m, q)$ which is allowable by the remarks above.

1.7. Definition. Put

$$\delta_{\mathrm{sup}}(f,g) := \sup\,\{\delta(f(s), g(s)) \mid s \in \mathbf{C}^+\}.$$

The *pointwise gap topology* is the topology on linear systems induced by δ_{sup} on $\mathcal{R}(\mathbf{C}^+, m, q)$.

The final definition that we shall consider in this section is that of the 'graph topology'. We rephrase Vidyasagar's original definition of this topology in the setting of Martin–Hermann maps:

1.8. Definition. The *graph topology* on $\mathcal{R}(\mathbf{C}^+, m, q)$ is defined by the open neighborhoods $U_{\epsilon,F}(f)$ for f in $\mathcal{R}(\mathbf{C}^+, m, q)$, defined as follows. F is a rational matrix in $H_\infty^{n\times q}(\mathbf{C}^+)$ such that $f(s) = \operatorname{im} F(s)$, and $U_{\epsilon,F}(f) = \{g \in \mathcal{R}(\mathbf{C}^+, m, q) : \exists G \in H_\infty^{m\times q}(\mathbf{C}^+) : g(s) = \operatorname{im} G(s) \text{ and } \|F - G\|_\infty < \epsilon\}$.

A fifth definition requires a little more preparation and will be given later, in section 4. The equivalence between the gap topology (Def. 1.5) and the graph topology (Def. 1.8) was shown in [19], whereas the equivalence between the 'pointwise gap' (Def. 1.7) and the gap of Def. 1.5 was shown in [11]. In this paper we explore the connections between the various definitions and provide new proofs. We build in part on the work in [3].

2. SOME HILBERT SPACE GEOMETRY

In this section we discuss some properties associated with angles between subspaces of a Hilbert space (cf. for instance [4]). The *maximal angle* $\theta(X,Y) \in [0, \frac{1}{2}\pi]$ between two closed subspaces V and W of some Hilbert space X is defined by

$$\theta(X,Y) = \arcsin \delta(X,Y).$$

The *minimal angle* $\phi(V,W) \in [0, \frac{1}{2}\pi]$ is defined by

$$\sin \phi(V,W) = \inf\{d(x,W) : x \in V, \|x\| = 1\}.$$

It is easy to verify that

$$\sin \phi(V,W) = \inf_{v,w} \sin \phi(v,w)$$

where v and w denote lines through the origin in V and W respectively. Two further useful facts are the following. Assume $V + W$ is closed and $V \cap W = \{0\}$. Let $\mathcal{P}_W^V$ denote the skew projection of $V + W$ along V on W. Then (cf. for instance [4, p. 339]):

2.1. Lemma. $\sin \phi(V,W) = \frac{1}{\|\mathcal{P}_W^V\|}$.

Furthermore, we have (cf. [1]): if $V \cap W = \{0\}$ then $\forall V' : \delta(V,V') < \sin \phi(V,W) \Rightarrow V' \cap W = \{0\}$. This last fact can also be obtained as a consequence of the following important lemma from [12]. We present an alternative proof.

2.2. Lemma. Let X,Y,Z be closed subspaces in a (real or complex) Hilbert space H. Then one has

$$\phi(Y,Z) \geq \phi(X,Z) - \theta(X,Y).$$

Proof. First of all, note that we may assume that H is a real vector space, as we may replace a complex space H by the real Hilbert space structure H' on the same set that arises by restricting the scalar multiplication to $\mathbf{R}$ and replacing the complex-valued inner product by the real-valued $\langle x,y\rangle' := \mathrm{Re}\,\langle x,y\rangle$. This transformation from H to H' preserves distances and hence also angles.
We first prove the inequality for one-dimensional (x,y,z), for which there is just one angle $\phi = \theta$. Suppose (x,y,z) violate the inequality. Let V be the span of x and z over $\mathbf{R}$. If we then let y' denote the orthogonal projection on V of y, (x,y',z) would be a set of three lines in the plane such that $\theta(x,y') + \theta(y',z) < \theta(x,z)$, which is obviously impossible.
Now in the general case, let $\varepsilon > 0$ be arbitrary, let the lines $y \subset Y$ and $z \subset Z$ be such that $\phi(y,z) \leq \phi(Y,Z) + \varepsilon$, and let $x = \mathcal{P}_X(y)$. We then have $\theta(X,Y) \geq \theta(x,y)$, so $\phi(Y,Z) + \theta(X,Y) + \varepsilon \geq \phi(y,z) + \theta(x,y) \geq \phi(x,z) \geq \phi(X,Z)$. □

We establish the fact that skew projections are 'continuous in their kernels'.

2.3. Lemma. Let (U_1, W) and (U_2, W) be pairs of closed complementary subspaces of a Hilbert space X. Suppose $\theta(U_1, U_2) < \phi(U_1, W)$. Then

$$\left\|\mathcal{P}_W^{U_1} - \mathcal{P}_W^{U_2}\right\| \leq \frac{1}{\sin\phi(U_1, W)\sin(\phi(U_1, W) - \theta(U_1, U_2))}\delta(U_1, U_2).$$

Proof. Let $\Psi_1 = \mathcal{P}_W^{U_1}$, $\Psi_2 = \mathcal{P}_W^{U_2}$, $\delta = \delta(U_1, U_2)$. Let $u_1 \in U_1$. Choose $u_2 \in U_2$ such that $\|u_1 - u_2\| \leq \delta\|u_1\|$. Then $\|(\Psi_1 - \Psi_2)u_1\| = \|\Psi_2 u_1\| = \|\Psi_2(u_1 - u_2)\| \leq \delta\|\Psi_2\|\,\|u_1\|$. For arbitrary x we have $x = u_1 + w$, $u_1 \in U_1$, $\|u_1\| \leq \|\Psi_1\|\|x\|$, and $(\Psi_1 - \Psi_2)x = (\Psi_1 - \Psi_2)u_1$, so $\|(\Psi_1 - \Psi_2)x\| \leq \|\Psi_1\|\|\Psi_2\|\,\delta\,\|x\|$. The minimal angle γ between U_2 and W is larger than or equal to $\phi(U_1, W) - \theta(U_1, U_2)$, so because $\|\Psi_2\| = \frac{1}{\sin\gamma}$ we obtain the desired formula. □

2.4. Lemma. Let $\Psi_i = \mathcal{P}_{V_i}^{U_i}$, $i = 1, 2$ where U_i, V_i are pairs of complementary subspaces. Then for U_2, V_2 in a sufficiently small neighborhood of U_1 resp. V_1, one has

$$\|\Psi_1 - \Psi_2\| \leq \frac{1}{\sin\phi(U_1, V_1)\ \sin(\phi(U_1, V_1) - \theta(U_1, U_2))}\delta(U_1, U_2) +$$

$$\frac{1}{\sin(\phi(U_1, V_1) - \theta(U_1, U_2))\sin(\phi(U_1, V_1) - \theta(U_1, U_2) - \theta(V_1, V_2))}\delta(V_1, V_2).$$

Proof. Using Lemma 2.3 and the inequality $\|\mathcal{P}_{V_1}^{U_1} - \mathcal{P}_{V_2}^{U_2}\| \leq \|\mathcal{P}_{V_1}^{U_1} - \mathcal{P}_{V_1}^{U_2}\| + \|\mathcal{P}_{V_1}^{U_2} - \mathcal{P}_{V_2}^{U_2}\|$ one easily obtains the desired formula. □

The next proposition shows that convergence in the gap topology is the same as convergence of skew projections.

2.5. Proposition. Let (U, V) be complementary. We have:

$$\{U_n \to_\delta U \text{ and } V_n \to_\delta V\} \Longleftrightarrow \{\mathcal{P}_{U_n}^{V_n} \to_{\|\cdot\|} \mathcal{P}_U^V\}.$$

Proof. $\Rightarrow$: Obvious from Lemma 2.4. $\Leftarrow$: We show $\delta(U_1, U_2)$, $\delta(V_1, V_2) \leq \|\Psi_1 - \Psi_2\|$. Choose $x \in U_1, \|x\| = 1$. Now $d(x, U_2) \leq \|x - \mathcal{P}_{U_2}^{V_2}x\| = \|\mathcal{P}_{U_1}^{V_1}x - \mathcal{P}_{U_2}^{V_2}x\| \leq \|\mathcal{P}_{U_1}^{V_1} - \mathcal{P}_{U_2}^{V_2}\|$. And since $\|\mathcal{P}_{U_1}^{V_1} - \mathcal{P}_{U_2}^{V_2}\| = \|(I - \mathcal{P}_{U_1}^{V_1}) - (I - \mathcal{P}_{U_2}^{V_2})\| = \|\mathcal{P}_{V_2}^{U_2} - \mathcal{P}_{V_1}^{U_1}\|$, the same argument also gives $\delta(V_1, V_2) \leq \|\mathcal{P}_{U_1}^{V_1} - \mathcal{P}_{U_2}^{V_2}\|$. □

Continuity of feedback interconnection of linear systems was the main reason to introduce the graph topology. Interconnection is most naturally viewed simply as intersection of behaviors. So it is essential in this context to study the continuity of the lattice operations $\cap, +, \perp$ on subspaces of a Hilbert space Z with respect to the gap topology. The defining formula $\delta(X, Y) = \|\mathcal{P}_X - \mathcal{P}_Y\|$ implies that $\perp$ is actually isometric. The behavior of the *dimensions* of $X \cap Y$ and $Z \ominus (X + Y)$ under small perturbations was already studied in the well-known book [7]. It is not difficult to extend his analysis to the continuity of the operations themselves.

2.6. Definition. Let U, V be subspaces of a Hilbert space X. Then the *minimal gap* $\gamma(U, V)$ is defined as

$$\gamma(U,V) := \inf_{v \in V \setminus U} \frac{d(v,U)}{d(v,U\cap V)}.$$

2.7. Definition. Two closed subspaces U, V of a Hilbert space Z are *in general position* if $U + V$ is closed and either $U \cap V = \{0\}$ or $U + V = Z$.

2.8. Proposition. Intersection is continuous for spaces in general position. For U, V in general position one has (for U' in a sufficiently small neighborhood of U) $\delta(U \cap V, U' \cap V) \leq \frac{1}{\gamma(U,V)}\delta(U, V)$.
The proof turns out to be easier in the dual form:

2.9. Proposition. Linear summation is continuous for spaces in general position. For U, V in general position one has (for U' in a sufficiently small neighborhood of U) $\delta(U + V, U' + V) \leq \frac{1}{\gamma(U,V)}\delta(U, V)$.

Proof. We first prove separate continuity. For $x \in U + V$ with $\|x\| = 1$ we have $x = u + v, u \in U, v \in V$ with $\|v\| \leq 1/\gamma(U, V)$. It follows that in general

$$\vec{\delta}(U + V, U + V_1) \leq \delta(V, V_1)/\gamma(U, V).$$

We prove that the assumptions $\delta(V, V_1) < \gamma(U, V)$ and $(U + V = Z) \vee (U \cap V = \{0\})$ imply that this estimate also holds for the undirected gap $\delta(U + V, U + V_1)$. If $U + V = Z$, this also holds for V_1 with $\delta(V, V_1) < \gamma(U, V)$, cf. [7], and the estimate trivially holds. So we may assume that $U \cap V = \{0\}$. Suppose $\vec{\delta}(U + V_1, U + V) > a := \delta(V, V_1)/\gamma(U, V)$. So there is $x = u + v_1 \in U + V_1$ with $\|x\| = 1$ such that $d(x, U + V) > a$. Let $v = \mathcal{P}_V(v_1)$. Then the triple $U' = \mathrm{span}(u)$, $V' = \mathrm{span}(v)$, $V_1' = \mathrm{span}(v_1)$ is such that $\dim(U' + V') = \dim(U' + V_1')$ and $\vec{\delta}(U' + V', U' + V_1') \neq \vec{\delta}(U' + V_1', U' + V')$. For $\vec{\delta}(U' + V_1', U' + V') \geq d(u + v_1, U + V)$, whereas $\vec{\delta}(U' + V', U' + V_1') \leq 1/\gamma(U', V')\delta(V', V_1') \leq a$; and v_1 and u are obviously linearly independent. Of course u and v are independent because of the assumption $U \cap V = \{0\}$. So we have arrived at a contradiction.
To prove joint continuity, first notice that our assumptions imply that either $\gamma(U, V) = \sin\phi(U, V)$ or $\gamma(U, V) = \sin\phi(U^\perp, V^\perp)$. Define

$$\alpha(U, V) := \arcsin \gamma(U, V).$$

We may use Lemma 2.2 to see that

$$\alpha(U, V_1) \geq \alpha(U, V) - \theta(V, V_1).$$

Hence for any U_1 close enough to U, we finally have

$$\begin{aligned}
\delta(U_1 + V_1, U + V) &\leq \delta(U + V_1, U + V) + \delta(U + V_1, U_1 + V_1) \\
&\leq \frac{1}{\gamma(U,V)}\delta(V, V_1) + \frac{1}{\gamma(U,V_1)}\delta(U, U_1) \\
&\leq \frac{1}{\gamma(U,V)}\delta(V, V_1) + \frac{1}{\sin(\alpha(U,V) - \theta(V,V_1))}\delta(U, U_1). \quad \square
\end{aligned}$$

3. EQUIVALENCE OF POINTWISE GAP AND GRAPH TOPOLOGY

We begin by showing that subspace-valued functions that are close in the uniform topology have matrix representations that are close in the sense of H_∞.

3.1. Lemma. Let $\Omega = \mathbf{C}^+$ or $\Omega =$ some disk D, and let $f \in \mathcal{R}(\Omega, m, q)$. If $F \in H_\infty(\Omega)$ is such that $f(s) = \operatorname{im} F(s)$, then for all $\delta > 0$ we can find ε such that for all g:

$$\delta_{\text{sup}}(f,g) < \varepsilon \Rightarrow \exists G \in H_\infty(\Omega) : g(s) = \operatorname{im} G(s) \text{ and } \|F - G\|_\infty < \delta.$$

P r o o f. Let $f(s) = \operatorname{im} F(s)$. Choose $Y(s)$ solving the Bézout equation $YF = I$. This implies that $\ker Y(s)$ and $f(s)$ are complementary for all $s \in \overline{\Omega}$. Now for $g(s) = \operatorname{im} G(s)$ in a sufficiently small neighborhood of $f(s)$, ker $Y(s)$ and $g(s)$ must also be complementary for all $s \in \overline{\Omega}$, which implies that YG is unimodular in $H_\infty(\Omega)$. It follows that a representation $g(s) = \operatorname{im}\ G'(s)$ can be chosen such that $YG' = I$ (take $G' = G(YG)^{-1}$). We have

$$\|F - G'\| = \|FYF - G'YF\| \leq \|FY - G'Y\|\|F\|.$$

The function $\|F(s)\|$ is of course bounded on $\overline{\Omega}$; furthermore, because $YF = I$ and $YG' = I$, it follows that FY equals the skew projection $\mathcal{P}^{\operatorname{im} F}_{\ker Y}$, and $G'Y$ equals $\mathcal{P}^{\operatorname{im} G'}_{\ker Y}$. By Lemma 2.3, we obtain

$$\|F(s) - G'(s)\| \leq \frac{\|F(s)\|}{\sin\phi(s)\sin(\phi(s) - \alpha(s))}\,\delta_{\text{sup}}(f,g)$$

where $\alpha(s) \in [0, \frac{1}{2}\pi]$ is such that $\sin\alpha(s) = \delta(f(s), g(s))$, and $\phi(s)$ is the minimal angle between $f(s)$ and $\ker Y(s)$. A compactness argument shows that $\sin\phi$ and $\sin(\phi - \alpha)$ are bounded away from zero on $\overline{\Omega}$ when α is sufficiently small (note that it follows from Lemma 2.1 and Lemma 2.4 that $\phi(s)$ is continuous). □

Let $\lambda(f)$ be defined as $\inf\{\sigma_{\min}(F) : s \in \mathbf{C}^+\}$, where F is an inner matrix (i.e. $F(s)^*F(s) = I$ on $i\mathbf{R}$) of full column rank in $H_\infty(\mathbf{C}^+)$ such that $f(s) = \operatorname{im} F(s)$. Let $g(s) = \operatorname{im} G(s)$.

3.2. Proposition. $\delta_{\text{sup}}(f,g) \leq \frac{1}{\lambda(f)}\|F - G\|_\infty$.

P r o o f. The statement follows from the fact that for constant matrices A, B we have $\delta(\operatorname{im} A, \operatorname{im} B) \leq (1/\sigma_{\min}(A))\|A - B\|$. To prove this, choose $y = Ax \in \operatorname{im} A$ with $\|y\| = 1$. Then obviously $\|x\| \leq 1/\sigma_{\min}(A)$. So $d(y, \operatorname{im} B) \leq \|Ax - Bx\| \leq (1/\sigma_{\min}(A))\|A - B\|$. □

3.3. Proposition. The graph topology is equivalent to the topology induced by δ_{sup}.

P r o o f. Immediate from Lemma 3.1 and Prop. 3.2. □

4. EQUIVALENCE OF $\mathcal{O}_{\text{robust}}$ AND δ_{H_2}: FIRST PROOF

In this section we discuss a fifth characterization of the gap topology, which is the one that shows most directly its relevance for the study of robustness in control systems. For a stable feedback configuration (G, K), put $P = \mathcal{G}(G) = \ker[-G \quad I], C = \mathcal{G}(K) = \ker[I \quad -K]$. The following are equivalent (cf. [5]):

(i) (G, K) is stable and well-posed,

(ii) The spaces P and C are complementary, so the skew projection $\mathcal{P}_P^C$ is a well defined bounded operator.

(iii) The *closed loop transfer function*

$$H(P,C) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I \\ G \end{bmatrix} (I - KG)^{-1} [I \quad K]$$

(cf. [14]) is in H_∞.

There is a simple relation between $H(P, C)$ and $\mathcal{P}_P^C$:

$$H(P,C) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \mathcal{P}_P^C \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

which is a consequence of the expression

$$\mathcal{P}_P^C = \begin{bmatrix} I \\ G \end{bmatrix} (I - KG)^{-1} [I \quad -K],$$

as is readily verified. (In these expressions we choose to confuse the multiplication operator with symbol Θ and the matrix Θ.) So it is clear that for convergence issues we may look at $\mathcal{P}_P^C$ instead of the closed loop transfer function $H(P, C)$.

4.1. Definition. The topology $\mathcal{O}_{\text{robust}}$ is defined by its subbasis elements

$$B_{P_0,C_0,\varepsilon} = \{P : \|H(P, C_0) - H(P_0, C_0)\| < \varepsilon\}$$

for (P_0, C_0) stable. Similarly $\mathcal{O}_{\mathcal{P}}$ is defined by the subbasis elements

$$B_{P_0,C_0,\varepsilon} = \{P : \|\mathcal{P}_P^C - \mathcal{P}_{P_0}^{C_0}\| < \varepsilon\}.$$

From the preceding remarks we have

4.2. Proposition. $\mathcal{O}_{\text{robust}} = \mathcal{O}_{\mathcal{P}}$.

We can apply Lemma 2.5 both to the pointwise gap topology and to the H_2-gap topology. Using Proposition 4.2 the main point of this section is now obvious:

4.3. Proposition. The topology induced by δ_{sup} or δ_{H_2} is equivalent to the topology $\mathcal{O}_{\text{robust}}$.

The definition of the the topology $\mathcal{O}_{\text{robust}}$ is due to [14], where also the equivalence with the graph topology was shown.

5. EQUIVALENCE OF $\mathcal{O}_{\text{robust}}$ AND δ_{H_2}: SECOND PROOF

The closed loop transfer function $H(P,C)$ from the previous section can be obtained in the following way. One considers signals $(e_1, e_2, u_1, u_2, y_1, y_2) \in Z := H_2(\mathbf{C}^+, E_1 \times E_2 \times U_1 \times U_2 \times Y_1 \times Y_2)$, and associates to the controller the subspace $\mathcal{C} = \{z \mid y_2 = Ke_2\}$ of Z and to the system G the subspace $\mathcal{P} = \{z \mid y_1 = Ge_1\}$. Then one considers the interconnection $\{e_1 = u_1 + y_2, e_2 = u_2 + y_1\}$ of the two systems. So let $\mathcal{I} := \{z \in Z \mid e_1 = u_1 + y_2, e_2 = u_2 + y_1\} \subset Z$ and let $\mathcal{H}(\mathcal{P},\mathcal{C}) := \mathcal{I} \cap \mathcal{P} \cap \mathcal{C} \subset Z$. Then the transfer function $H(P,C)$ corresponds to the operator from $H_2(U_1 \times U_2)$ to $H_2(E_1 \times E_2)$ the graph of which is the projection of $\mathcal{H}(\mathcal{P},\mathcal{C})$ on $H_2(E_1 \times E_2 \times U_1 \times U_2)$. The gaps between the H_2-graphs P, P' of G and G', and corresponding spaces $\mathcal{P}, \mathcal{P}' \subset Z$ are of course equal. Similarly, the topology defined by the operator norm on stable transfer functions $H(P,C)$ coincides with the topology defined by the gaps between the spaces $\mathcal{H}(\mathcal{P},\mathcal{C})$, since gap topology and norm topology are equivalent for bounded operators.

5.1. Proposition. The topology $\mathcal{O}_{\text{robust}}$ is equivalent to the one induced by δ_{H_2}.

Proof. We have $\mathcal{H}(\mathcal{P},\mathcal{C}) \subset Z = \mathcal{I} \cap \mathcal{P} \cap \mathcal{C}$. We prove that the subspaces we intersect are in general position. The fact that $(\mathcal{P} \cap \mathcal{I}) + \mathcal{C} = Z$ is a consequence of the interpretation of stability as complementarity. Indeed every (e_2, y_2) can be written as $(e_{21}, Ke_{21}) + (Gy_{21}, y_{21})$, so $(e_1, e_2, u_1, u_2, y_1, y_2)$ is equal to $(e_1 - y_{22}, e_{21}, u_1, u_2, y_1 - Gy_{22}, Ke_{21}) + (y_{22}, Gy_{22}, 0, 0, Gy_{22}, y_{22}) \in \mathcal{C} + (\mathcal{P} \cap \mathcal{I})$. The same reasoning also gives $(\mathcal{C} \cap \mathcal{I}) + \mathcal{P} = Z$, so certainly $\mathcal{P} + \mathcal{I} = Z$. Thus we may can conclude that the closed loop behavior $\mathcal{H}(\mathcal{P},\mathcal{C})$ is continuous in $\mathcal{P}$ and $\mathcal{C}$.
Put $V = \{z \mid u_1 = 0, u_2 = 0\}$. Then from $\mathcal{H}(\mathcal{P},\mathcal{C})$ we can reconstruct $\mathcal{P} \cap V$ continuously as $\mathcal{P} \cap V = \mathcal{H}(\mathcal{P},\mathcal{C}) \oplus A$, where $A = \{z \mid y_1 = 0, e_1 = 0, u_1 = 0, u_2 = 0\}$, since $A \cap \mathcal{H}(\mathcal{P},\mathcal{C}) = \{0\}$. This implies that two systems P and P' are close to each other in the graph topology if the closed loop transfer functions $H(P,C)$ and $H(P',C)$ are close in operator norm. □

6. EQUIVALENCE OF δ_{H_2} AND $\mathcal{O}_{L_2^-}$

We show that the analysis of stability robustness can also be done in terms of the $L_2(-\infty, 0)$ behaviors of linear time-invariant systems by establishing the link between the time-domain angles between the behaviors and the angles between graphs of transfer functions in H_2. By means of the isometric operator $Jf = \overline{f(-\overline{z})}$ we can map H_2^- into H_2 and vice versa.

6.1. Proposition. Let Σ_1, Σ_2 be linear systems with transfer functions G_1 and G_2. Then

$$\delta(\mathcal{B}_-(G_1), \mathcal{B}_-(G_2)) = \delta(\mathcal{G}(-G_1^T), \mathcal{G}(-G_2^T)).$$

Proof. It follows from Proposition 1.4 that $\mathcal{L}(\mathcal{B}_-(G)) = \mathcal{G}_-(-\tilde{G})^\perp$. Because $J\mathcal{G}_-(-\tilde{G}) = \mathcal{G}_+(-G^T)$ and in general $\delta(V_1^\perp, V_2^\perp) = \delta(V_1, V_2)$, the statement follows.

6.2. Proposition. Let Σ_G, Σ_K be linear systems with transfer functions G from U to Y and K from Y to U respectively. We denote by $\mathcal{G}(G)$ and $\mathcal{G}(K)$ the H_2-graphs of G and K, both taken on the external variable space $W = Y \times U$. If the feedback interconnection of G and K is stable, we have

$$\begin{array}{ll} \text{(i)} & \sin\phi(\mathcal{G}(-G^T), \mathcal{G}(-K^T)) = \sin\phi(\mathcal{G}(G), \mathcal{G}(K)), \\ \text{(ii)} & \sin\phi(\mathcal{B}_-(G), \mathcal{B}_-(K)) = \sin\phi(\mathcal{G}(G), \mathcal{G}(K)). \end{array}$$

P r o o f. The second statement follows from the first. Let P be the graph of G, C the graph of K, and P^T, C^T the graphs of $-G^T, -K^T$. We know (again identifying a matrix with a multiplication operator) that

$$\mathcal{P}_P^C = \begin{bmatrix} I \\ G \end{bmatrix} (I - KG)^{-1} [I \quad -K]$$

and that $\|\mathcal{P}_P^C\|^{-1} = \sin\phi(P, C)$. Now

$$\mathcal{P}_{C^T}^{P^T} = \begin{bmatrix} I \\ -K^T \end{bmatrix} (I - G^T K^T)^{-1} [I \quad G^T]$$

Furthermore, also $(\mathcal{P}_P^C)^T$ is equal to this last expression, and of course the L_∞ norms of a matrix and its transpose are equal. □

6.3. Proposition. The topology induced on the set of finite-dimensional input/state/output systems by the gap between the L_2^--behaviors is the same as the gap topology.

P r o o f. The gap topology is the weakest topology on systems such that $\mathcal{P}_P^C$ is a continuous function of P. By Proposition 6.1 and the proof of Proposition 6.2, the L_2^--gap topology is the weakest topology such that $(\mathcal{P}_P^C)^T$ is continuous in P. This is obviously the same thing. □

Note that it follows that complementarity of the L_2^--behaviors is the same thing as complementarity of the H_2^+-graphs. Thus, we can also model stability robustness in terms of the L_2^--behaviors of systems. This is what we should expect, a feedback interconnection being stable iff the autonomous L_2^--behavior is $\{0\}$.

7. OTHER GAP TOPOLOGIES

So far, we have been considering various interpretations of the same topology. To conclude, it is perhaps enlightening to compare a few *different* gap topologies. The gaps $\delta_{H^+}, \delta_{H^-}$ are between the graphs in H_2^+ resp. H_2^-, and δ_L, $\delta_{L^+}, \delta_{L^-}$ refer to the gaps between the behaviors in L_2, L_2^+ resp. L_2^-; $\delta_{\mathbf{C}_\infty}$ is the gap $\delta_{\sup}$ on $\mathcal{R}(\mathbf{C}_\infty, m, q)$. The topology on transfer functions in L_∞ induced by δ_L is equivalent to the L_∞ norm topology, (and so it is weaker than the graph topology), $\delta_{L^+} \equiv \delta_{H^-}$ is uncomparable with $\delta_{L^-} \equiv \delta_{H^+}$.

7.1. Proposition. Let V_1, V_2 be closed shift-invariant subspaces of $L_2(\mathbf{R})$ and let Θ_i be the Beurling symbols of V_{i+}^0. For s on the imaginary axis let $V_i(s) = \operatorname{span}\Theta_i(s)$. Then

$$\delta_L(V_1, V_2) = \sup\{\delta(V_1(s), V_2(s)) \mid s \in i\mathbf{R}\}.$$

Proof. Let $\Psi_2(s) = \Theta_2(s)\Theta_2(s)^*\Theta_1(s)$. The matrix $\Theta_2(s)\Theta_2(s)^*$ represents the orthogonal projection on $V_2(s)$, and the associated multiplication operator represents the projection on V_2. So we have $\vec{\delta}(V_1, V_2) = \|(I - \Theta_2\Theta_2^*)\Theta_1\|_\infty = \sup\{\|\mathcal{P}_{V_2(s)^\perp}\Theta_1(s)\| \mid s \in i\mathbf{R}\} = \sup\{\|\mathcal{P}_{V_2(s)^\perp}|_{V_1(s)}\| \mid s \in i\mathbf{R}\} = \sup\{\vec{\delta}(V_1(s), V_2(s)) \mid s \in i\mathbf{R}\}$. □

7.2. Proposition. The different gap topologies are related according to the diagram below, where the arrows point from weaker to stronger topologies.

$$\begin{array}{ccccc} & & \delta_L & & \\ & \swarrow & & \searrow & \\ \delta_{L^+} \equiv \delta_{H^-} & & & & \delta_{L^-} \equiv \delta_{H^+} \\ & \searrow & & \swarrow & \\ & & \delta_{\mathbf{C}_\infty} & & \end{array}$$

Proof. The statement follows from the interpretation of all the topologies in the diagram as pointwise gap topologies. To prove that the inclusions are strict, it is sufficient to observe that perturbations of the form $G_\varepsilon = G + \varepsilon/(s-1)$ for G stable are continuous in δ_{H^-}, but certainly not in the graph topology, since stability is a robust property in this topology. □

It can be shown that the topology induced by $\delta_{\mathbf{C}_\infty}$ is not connected and falls apart into components according to the McMillan degrees of the transfer functions, on which components it is equivalent the parameter topology of minimal realizations modulo state space isomorphism [2]. Part of this result can easily be obtained as a corollary to some of the observations in this paper.

7.3. Proposition. Equipped with the topology induced by $\delta_{\mathbf{C}_\infty}$, $\mathcal{S}_{\text{fd}}$ is not connected. In particular, systems with different McMillan degree are in different components.

Proof. By the same argument as in the previous section we know that δ_{L^+} is equivalent to δ_{H^-}, and from the equivalence of the graph topology and the pointwise gap topology we know $\delta_{\mathbf{C}_\infty}$ is stronger than both δ_{H^-} and δ_{L_+}. So $\delta_{\mathbf{C}_\infty}$ is stronger than δ_{H_+} and δ_{L^+}. Hence, for g in a sufficiently small neighborhood of any curve f on the Grassmannian, by continuity of orthogonal complementation and intersection, the minimal state spaces $\mathcal{L}(\mathcal{B}_+(g))/\mathcal{G}(g) \cong \mathcal{B}_+(g) \ominus \mathcal{B}_+^0(g) = \mathcal{B}_+(g) \cap \mathcal{B}_+^0(g)^\perp$ will be close to the minimal state space of f. It follows that they have the same dimension. □

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REFERENCES

[1] E. Berkson: Some metrics on the subspaces of a Banach space. Pacific J. Math. *13* (1963), 7–22.

[2] C. I. Byrnes and T. E. Duncan: On certain topological invariants arising in system theory. In: New Directions in Applied Mathematics, Springer, New York 1982, pp. 29–71.

[3] J. de Does, H. Glüsing-Lüerßen, and J. M. Schumacher: Connectedness properties of spaces of linear systems. In: Proc. SINS '92 International Symposium on Implicit and Nonlinear Systems (Ft. Worth, Texas, Dec. 14–15, 1992), (F. L. Lewis, ed.), ARRI, Univ. Texas at Arlington, pp. 210–215.

[4] I. C. Gohberg and M. G. Krein: Introduction to the Theory of Linear Nonselfadjoint Operators. Transl. Math. Monogr. 18, Amer. Math. Soc., Providence, RI 1969.

[5] C. Foias, T. T. Georgiou and M. C. Smith: Robust stabilization in the gap metric: a geometric approach. Proc. 1991 International Symposium on the Mathematical Theory of Networks and Systems, Kobe, Japan, June 1991.

[6] C. Martin and R. Hermann: Applications of algebraic geometry to systems theory: The McMillan degree and Kronecker indices of transfer functions as topological and holomorphic system invariants. SIAM J. Control Optim. *16* (1978), 743–755.

[7] T. Kato: Perturbation Theory for Linear Operators. Springer, New York 1982.

[8] P. D. Lax: Translation invariant subspaces. Acta Math. *101* (1959), 163–178.

[9] N. K. Nikol'skiĭ: Treatise on the Shift Operator. Springer, Berlin 1986.

[10] R. J. Ober and J. A. Sefton: Stability of control systems and graphs of linear systems. Systems Control Lett. *17* (1991), 265–280.

[11] L. Qiu and E. J. Davison: Pointwise gap metrics on transfer matrices. IEEE Trans. Automat. Control *37* (1992), 741–758.

[12] L. Qiu and E. J. Davison: Feedback stability under simultaneous gap metric uncertainties in plant and controller. Systems Control Lett. *18* (1992), 9–22.

[13] J. M. Schumacher: A pointwise criterion for controller robustness. Systems Control Lett. *18* (1992), 1–8.

[14] M. Vidyasagar: Control System Synthesis: a Factorization Approach. MIT Press, Cambridge, MA 1985.

[15] S. Weiland: Theory of Approximation and Disturbance Attenuation for Linear Systems. Doctoral Dissertation, University of Groningen 1991.

[16] J. C. Willems: Input-output and state-space representations of finite-dimensional linear time-invariant systems. Linear Algebra Appl. *50* (1983), 581–608.

[17] J. C. Willems: Paradigms and puzzles in the theory of dynamical systems. IEEE Trans. Automat. Control *36* (1991), 259–294.

[18] G. Zames and A. K. El-Sakkary: Unstable systems and feedback: The gap metric. In: Proc. Allerton Conf. (1980), pp. 380–385.

[19] S. Q. Zhu: Graph topology and gap topology for unstable systems. IEEE Trans. Automat. Control *34* (1989), 848–855.

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