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CONTROL OF NONHOLONOMIC SYSTEMS VIA DYNAMIC COMPENSATION[1]

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The problem of controlling nonholonomic systems via dynamic state feedback and its structural aspects are analyzed. Advantages and drawbacks with respect to the use of static state feedback laws are discussed. In particular, nonholonomic constraints are shown to yield possible singularities in the dynamic extension process. Nevertheless, these singularities can be avoided by the proper design of a discontinuous external control law achieving stabilization of the transformed linear system. This is illustrated through simulations for a unicycle.

1. INTRODUCTION

Analysis and synthesis of control strategies for nonlinear systems with nonholonomic constraints are the subject of extensive research. These systems are typical of mechanical applications such as wheeled mobile robots (rolling constraints) [4, 12, 13, 15], free-space manipulators (conservation of angular momentum) [19, 23] and redundant manipulators subject to a given inverse kinematic control [8].

From the theoretical point of view, the control of nonholonomic systems presents interesting aspects. First of all, the control problem is a true nonlinear one since a nonlinear nonholonomic system is not linearly controllable. Moreover, controllability in the nonlinear setting — which is strictly related to the nonholonomic nature of the system — does not imply stabilizability by smooth time-invariant feedback [3]. As a consequence, a combination between feedforward (off-line planning) and feedback laws of a more general kind (e. g. discontinuous [1, 5] or periodic time-varying control [6, 17, 20]) is necessary.

A *nonholonomic* vector constraint for a system with state $x \in X$, an open subset of $\mathbb{R}^n$, is often written in the Pfaffian form

$$c(x)\,\dot{x} = 0, \tag{1}$$

where $c(x)$ is a $p \times n$ matrix of smooth functions, having full row rank for all x. All p constraints in (1) are assumed to be non-integrable; if some of these constraints were integrable (holonomic), then the state-space dimension could be reduced accordingly.

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When the state x coincides with the set of generalized coordinates q of a mechanical system, the problem is considered at a *kinematic* level and $\dot{x}$ are physical velocities. When the full *dynamic* model is considered [2], then $x = (q, \dot{q}) = (x_1, x_2)$ and a differential *first-order* kinematic constraint of the form $J(q)\dot{q} = 0$ can still be written in the form (1) as

$$[J(x_1) \quad 0] \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = 0. \tag{2}$$

However, we note that *second-order* kinematic constraints may fit also in the same framework. For example, in [16] underactuated robot arms with no gravity are considered and their dynamic model is partitioned as

$$\begin{bmatrix} B_a(q) \\ B_u(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_u \end{bmatrix} + \begin{bmatrix} c_a(q, \dot{q}) \\ c_u(q, \dot{q}) \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}, \tag{3}$$

where the p joint variables q_u have no explicit control input. Since the Coriolis and centrifugal terms can always be factored as $c(q, \dot{q}) = S(q, \dot{q})\dot{q}$, the second set of p dynamic equations in (3) takes again the form (1) with

$$[S_u(q, \dot{q}) \quad B_u(q)] \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = 0. \tag{4}$$

Under the full rank assumption, we can always express an admissible $\dot{x}$ as a linear combination of $n - p$ vector fields $n_j(x)$, $j = 1, \cdots, n - p$, which are a basis for the kernel of the matrix c:

$$c(x)\,[\,n_1(x) \quad \cdots \quad n_{n-p}(x)\,] = c(x)n(x) = 0. \tag{5}$$

Depending on the structure of the *control system* and assuming linearity in the input, equations of motion can be derived as

$$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i = g(x)u, \qquad m \leq n - p \tag{6}$$

for systems with *no drift*, and

$$\dot{x} = f(x) + g(x)u, \tag{7}$$

in the more general case of systems *with drift*. In (6), the columns of $g(x)$ have to be specified as linear combinations of the vector fields $n_j(x)$, $j = 1, \ldots, n - p$, with coefficients taken in the space of analytic functions of x. In the following, we set for simplicity $m = n - p$, i.e. full control on the null space of $c(x)$. Moreover, we assume that f and g_i's are analytic, $f(0) = 0$, and rank $g(x) = m$, for all $x \in X$.

In both (6) and (7), the system vector fields induce only motion in the subspace of the tangent space characterized by the columns of the matrix $n(x)$. In particular, for systems with drift,

$$f(x) + \sum_{i=1}^{m} g_i(x)\overline{u}_i \in \operatorname{span} n(x), \tag{8}$$

for any admissible input $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_m)$, although, in this case, state equations are not derived only from the satisfaction of the nonholonomic constraints.

Associated with (6) or (7), an output characterizing the quantities to be controlled is defined as

$$y = h(x), \tag{9}$$

where $y \in \mathbb{R}^m$ and h is assumed to be analytic.

Most contributions on motion planning and control for nonholonomic systems have addressed systems of the form (6) [12, 13, 15]. This is due to two main reasons: first, in view of the above mechanical analogy, nonholonomy usually involves only first-order kinematic constraints so that *velocity inputs* are assumed to be available for control; second, controllability results for this class of nonlinear systems are much stronger than in the general case. More precisely, for system (6), the concepts of *accessibility* and *controllability* coincide and the rank test of [22] for accessibility is satisfied if, and only if, the associated constraint (1) is nonholonomic. Instead, for system (7), only a sufficient test exists for proving small-time local controllability [21], which in turn is only a sufficient condition for controllability.

The interest in analyzing the case in which a drift is present, as in (7), arises from the need of handling the rather common kinematic situation of *acceleration inputs* (as for the Hilare mobile robot [13]) and for including *dynamic* feedback compensation in the planning and control synthesis. Some structural properties and solutions of the control problem for nonholonomic dynamical systems have already been considered in [1, 2, 4]. Some preliminary results on dynamic feedback are given in [7].

In this paper, issues related with the use of *dynamic state feedback* laws (see e.g. [9]) are analyzed for both (6) and (7), in the presence of the nonholonomic constraint (1). First, in Section 2, control objectives which can be achieved by using static state feedback input-output linearization are reviewed. Then, Section 3 illustrates in particular: (i) what are the advantages of using dynamic extension for solving the tracking and stabilization problems; (ii) what are the relations between the nonholonomic nature of the system and the singularities possibly arising in dynamic extension procedures [10]. In Section 4, it is shown that the preliminary use of a dynamic linearization technique allows the design of a singularity-free feedback stabilization scheme, based on the linear equivalent model. Simulation results for a unicycle show the feasibility of the proposed control approach.

2. CONTROL VIA STATIC STATE FEEDBACK

Consider the driftless system (6). Under the full rank hypothesis for g, it is possible to find a change of coordinates $z = \phi(x)$ and an invertible static feedback $u = \beta(x)v$, both globally defined in X, such that (6) reduces to the form

$$\begin{aligned} \dot{z}_1 &= v \\ \dot{z}_2 &= \psi(z)v, \end{aligned} \tag{10}$$

with $z_1 \in \mathbb{R}^m$, $z_2 \in \mathbb{R}^{n-m}$ [14]. Matrix $\psi(z)$ can take special forms, e.g. the *chained* form of [15]. Note that no row of $\psi(z)$ can ever be zero, since this would contradict the assumption that system (6), viz. (10), is subject exactly to $p = n - m$ nonholonomic constraints. Moreover, if some rows of $\psi(z)$ were constant, i.e.

$$\begin{aligned} \dot{z}_1 &= v \\ \dot{z}_{21} &= \psi_{21} v \\ \dot{z}_{22} &= \psi_{22}(z) v, \end{aligned} \tag{11}$$

by replacing the coordinates z_{21} by $\tilde{z}_{21} = z_{21} - \psi_{21} z_1$, the equations would become

$$\begin{aligned} \dot{z}_1 &= v \\ \dot{\tilde{z}}_{21} &= 0 \\ \dot{z}_{22} &= \tilde{\psi}_{22}(z_1, \tilde{z}_{21}, z_{22}) v, \end{aligned} \tag{12}$$

leading again to a contradiction.

Choosing $z_1 = \phi_1(x)$ as the output, the form (10) shows that (6) can be input-output linearized by regular static state feedback so that the control of $y = \phi_1(x)$ is achieved using well-established linear results. Being the diffeomorphism $\phi(x)$ leading to (10) not unique, different sets of m outputs can be easily controlled in this way; however, a *given* output function may still not be compatible with a linearizing coordinate transformation.

Example. Consider the kinematic equations of a *unicycle* [15]:

$$\begin{aligned} \dot{x}_1 &= \cos x_3 \, u_1 \\ \dot{x}_2 &= \sin x_3 \, u_1 \\ \dot{x}_3 &= u_2, \end{aligned} \tag{13}$$

where x_1, x_2 and x_3 denote, respectively, the two position coordinates of the contact point and the orientation of the unicycle. This is a special case of the car-like model, where in addition the turning rate u_2 is upper bounded by a positive function that goes to zero with the rolling rate u_1 [13]. The two vector fields g_1 and g_2 in (13) span the kernel of $c(x)$ in the non-integrable rolling constraint

$$c(x)\dot{x} = \dot{x}_1 \sin x_3 - \dot{x}_2 \cos x_3 = 0. \tag{14}$$

By choosing as new coordinates

$$\begin{aligned} z_1 &= x_3 \\ z_2 &= x_1 \cos x_3 + x_2 \sin x_3 \\ z_3 &= x_1 \sin x_3 - x_2 \cos x_3 \end{aligned} \tag{15}$$

and performing the regular static state feedback

$$u = \begin{bmatrix} z_3 & 1 \\ 1 & 0 \end{bmatrix} v, \tag{16}$$

system (13) takes a *chained* structure, a special form of (10), i.e.

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1. \end{aligned} \tag{17}$$

Hence, control of the orientation (z_1) and of the position in the instantaneous forward direction (z_2) as physical outputs can be achieved independently and with any prescribed linear error dynamics. Moreover, nonlinear *smooth* state feedback can always be used for stabilizing the unicycle to the constant desired equilibrium manifold $\mathcal{M} = \{z_1 = z_{1d}, z_2 = z_{2d}\}$, parameterized by z_3 itself.

However, no smooth feedback control can stabilize simultaneously also z_3 to z_{3d} [3]. Indeed, for controlling z_1, z_2, and z_3 one could design a suitable trajectory $z_{1d}(t)$, $z_{2d}(t)$ such that the unicycle 'lands' close to the desired value z_{3d} on $\mathcal{M}$. Any such trajectory can be tracked in closed loop using appropriate v_1 and v_2 and control (16). Once the manifold $\mathcal{M}$ is reached, open-loop control [15], discontinuous [1] and time-varying feedback control [20], or a combination of the two [18] can be used for adjusting z_3 to the desired value z_{3d}.

Assume that the reference state is $x_d = 0$ (viz. $z_d = 0$), and that a nonlinear static state feedback control has already achieved $z_1 = z_2 = 0$. In terms of the original state variables, we start then at time $t = t_0$ in

$$\begin{aligned} x_1(t_0) &= 0 \\ x_2(t_0) &= -z_3(t_0) \neq 0 \\ x_3(t_0) &= 0. \end{aligned} \tag{18}$$

A possible switching strategy to complete the motion, i.e. to reach $z = 0$ at time $t_0 + \Delta$, is the following:

$$\begin{aligned} v_1(t) &= \{V, 0, -V, 0\} \\ v_2(t) &= \{0, V\,\mathrm{sign}(z_3(t_0)), 0, -V\,\mathrm{sign}(z_3(t_0))\}, \end{aligned} \tag{19}$$

where switchings between different values of the external control v occur at time intervals of $\Delta/4$ and the amplitude V is given by

$$V = \frac{4\sqrt{|z_3(t_0)|}}{\Delta}. \tag{20}$$

When applying (16) with v given by (19), the unicycle will never rotate about itself without rolling at the same time. Hence, this open-loop design of the external input v also applies to the car-like situation [13].

In the above example, we note that any perturbation (disturbance) occurring during the open-loop control phase will lead to a final error in z_3 (and, in general, on both z_1 and z_2). Therefore, we would like to pursue alternative control designs where the maximum amount of feedback information is used. On the other hand, the recently proposed design of time-varying (periodic) controllers, although of appealing theoretical relevance, is still rather difficult to be carried out even in this simple case. Furthermore, the above example shows the limit of static control laws. If the position of the unicycle is chosen as the output to be controlled, then $y = (x_1, x_2)$ and from (13) it follows that the input-output behavior cannot be decoupled nor linearized by *static* state feedback[1]. In this case, *dynamic* feedback may be useful for obtaining the desired linear input-output behavior as shown in the following section.

[1] If this were possible, we could reach the desired position (x_{1d}, x_{2d}) in the plane with linear error dynamics and then simply rotate to reach the desired orientation. This strategy, however, is not allowed in the car-like case.

3. CONTROL VIA DYNAMIC STATE FEEDBACK

Extending the class of control laws to *dynamic* state feedback of the form

$$\begin{aligned} \dot{\xi} &= \gamma(x,\xi) + \delta(x,\xi)v, \qquad \xi \in \mathbb{R}^{\nu} \\ u &= \alpha(x,\xi) + \beta(x,\xi)v, \end{aligned} \tag{21}$$

has been shown to help in designing controllers based on feedback equivalence to linear systems [11]. Indeed, the class of systems that can be input-output linearized via dynamic feedback includes all those systems (with or without drift) which are right-invertible [9]. Moreover, if (6) or (7) are controllable, their dynamic extensions obtained by adding an integrator on each of the input channels also are controllable [22].

In this section, we consider systems of the general form (7). Note that, even if the original system is without drift, the application of a dynamic extension leads to an extended system having a drift term, provided that the functions $\alpha(x,\xi)$ and $\gamma(x,\xi)$ are not zero. A drift term can also be introduced by a static state feedback of the form $u = \alpha(x) + \beta(x)v$, if the function $\alpha(x)$ is not identically zero. It is shown next that a major problem in using dynamic feedback for linearizing nonholonomic systems consists in the occurrence of *singularities* in the dynamic extension process.

Proposition 1. Suppose that system (7), (9) has no zero-dynamics (in the sense of [11]), locally around the origin. Then, either

(i) system (7), (9) is input-output decoupable around the origin by dynamic extension;

or

(ii) system (7) is associated with a nonholonomic constraint.

Proof. The proof is by contradiction. Suppose that (i) holds and let Σ^e be the extended input-output decoupled system obtained from (7), (9) by the application of one of the existing dynamic extension algorithms. Let n^e be the dimension of its state space and $\{r_1^e, r_2^e, \cdots, r_m^e\}$ its vector relative degree. Since the zero-dynamics is left unchanged under the addition of integrators on the input channels and invertible static state feedback [11], the absence of zero-dynamics for (7), (9) implies the absence of zero-dynamics for the extended system Σ^e. Hence,

$$r_1^e + r_2^e + \ldots + r_m^e = n^e. \tag{22}$$

Then, after dynamic compensation, the system becomes globally diffeomorphic to a controllable and observable linear system and thus is stabilizable by smooth state feedback. Suppose now that (ii) also holds. Then, the nonholonomic system (7) would be smoothly stabilizable, contradicting the result of [3]. □

Most of the existing dynamic decoupling algorithms (e.g. [9]) are based on the application of an invertible static feedback and the addition of integrators on selected inputs, i.e. on the dynamic extension procedure referred to in assumption (i) of

Proposition 1. Regularity conditions [10] are associated with each dynamic extension algorithm: these are sufficient conditions for (i) to hold.

The proof of the above proposition can be easily extended to *minimum phase* systems, that is, systems which possess an asymptotically stable zero-dynamics.

Corollary 2. Suppose that system (7), (9) is minimum phase. Then, (i) and (ii) of Proposition 1 are mutually exclusive.

As a consequence of the above results, the straightforward application of a dynamic linearizing control is not feasible. The previous unicycle example can be used to highlight the typical nature of dynamic singularities arising in the control of nonholonomic systems.

Example (reprise). Consider again equations (13) with

$$y_1 = x_1, \qquad y_2 = x_2. \tag{23}$$

The application of the algorithm of [9] yields a one-dimensional dynamic decoupling controller of the form

$$\begin{aligned} \dot{\xi} &= \cos x_3 v_1 + \sin x_3 v_2 \\ u_1 &= \xi \\ u_2 &= \frac{1}{\xi}(-\sin x_3 v_1 + \cos x_3 v_2). \end{aligned} \tag{24}$$

The extended system has $r_1^e = r_2^e = 2$ and $n^e = 4$, so that, in the new coordinates

$$\begin{aligned} z_1 &= x_1, & z_2 &= x_2, \\ z_3 &= \xi \cos x_3, & z_4 &= \xi \sin x_3, \end{aligned} \tag{25}$$

it is fully described by two chains of two input-output integrators. For the original system (13), no zero-dynamics can be defined in the sense of [11] and Proposition 1 cannot be used directly. Nevertheless, any motion of x_3, obtained with $u_1 = 0$ and an arbitrary $u_2(t)$, is compatible with the output (23) being constantly zero. Since the extended system is completely linear and controllable, in appearance smooth feedback stabilization would be allowed. However, the linearizing control law (24) has a singularity in $\xi = 0$, i.e. when the unicycle is not rolling because $u_1 = 0$. Intuitively, in this situation an infinite amount of energy is required on the second input u_2 for moving the output y without using the first input u_1.

From the example above, two observations follow: (i) when the output reference trajectory is persistently non-zero and sufficiently smooth, then the dynamic compensator (24) (properly initialized) can be effectively used as a tracking control law, as pointed out also in [7]; (ii) for a typical point-to-point motion, the use of (24) as it stands would fail.

This recognized critical flaw of the dynamic linearization approach may be avoided by using the additional degree of freedom left in the design, namely the free initialization (at any time) of the dynamic compensator, together with a suitable synthesis of the control input v. This will be illustrated in the next section.

4. STABILIZATION DESIGN

In this section, we illustrate how to construct a feedback stabilizing controller for (13) in a point-to-point motion. The proposed approach is based on the preliminary use of (24) and takes advantage of the linear equivalent structure of the closed-loop system. The unicycle is used here to point out this possibility, although, in general, each situation would require a case-by-case solution. The final control law is discontinuous, but contains as much feedback information as possible.

Without loss of generality, the desired state is chosen as the origin of the state space, $x_d = 0$. The reference frame in which the motion is described is determined accordingly. We assume first that the initial conditions of the unicycle are such that $x_1(0) \neq 0$ and that $\cos x_3(0) > 0$. In a second step of the design procedure, we will show how the more general situation can be reduced to this one.

Consider system (13) with outputs (23) and apply the dynamic compensator (24). Then, the extended system is diffeomorphic to the linear controllable system

$$\begin{aligned} \dot{z}_1 &= z_3, & \dot{z}_3 &= v_1, \\ \dot{z}_2 &= z_4, & \dot{z}_4 &= v_2, \end{aligned} \tag{26}$$

where the new coordinates z are defined by (25) in terms of x and of the compensator state ξ. The validity of this representation is restricted by the assumption that $\xi(t) \neq 0$ at all times t. Thus, the control design will have to ensure that this constraint is never violated. In terms of the new states, the control objective can be restated as

$$\begin{aligned} \lim_{t\to\infty} z_1(t) &= 0, && \text{(first position component)} \\ \lim_{t\to\infty} z_2(t) &= 0, && \text{(second position component)} \\ \lim_{t\to\infty} \mathrm{ATAN2}\{z_4(t), z_3(t)\} &= 0, && \text{(orientation)} \\ z_3^2(t) + z_4^2(t) &\neq 0,\ \forall t, && \text{(singularity avoidance).} \end{aligned} \tag{27}$$

The function ATAN2 is the well-known two arguments inverse tangent function, using the signs of both arguments to locate the solution in the proper quadrant.

The control law that we propose will achieve the point-to-point motion of the unicycle in a finite time, denoted $\bar{t}$. The following external control $v(t)$ is chosen for $t \in [0, \bar{t}]$:

$$\begin{aligned} v_1(t) &= 0, \\ v_2(t) &= -k\,\mathrm{sign}\big(z_2(t) + \tfrac{z_4(t)|z_4(t)|}{2k}\big), \end{aligned} \tag{28}$$

where $\bar{t}$ is yet to be defined, and $k > 0$ can be chosen such that, at time $\bar{t}$; the two variables z_2 and z_4 have reached zero (see e. g. [19]). To complete the design of the dynamic controller, we need to specify the initial condition $\xi(0)$. This is chosen as:

$$\xi(0) = -\frac{\mathrm{sign}(x_1(0))\bar{\xi}}{\cos x_3(0)} \neq 0, \tag{29}$$

where $\bar{\xi}$ is an arbitrary *positive* constant. As a consequence of the previous choices,

$$z_3(t) = z_3(0) = -\bar{\xi}\,\mathrm{sign}(z_1(0)), \tag{30}$$

and hence $\text{sign}(z_3(t)) = -\text{sign}(z_1(0))$. Since $\xi^2(t) = z_3^2(t) + z_4^2(t)$, we note that $\xi(t)$ is bounded away from zero, thus keeping the same sign for all t (the one chosen at $t = 0$). Moreover,

$$z_1(t) = z_1(0) - \bar{\xi}\,\text{sign}(z_1(0))t. \tag{31}$$

Therefore, imposing $z_1(\bar{t}) = 0$ gives

$$\bar{t} = \frac{z_1(0)}{\bar{\xi}\,\text{sign}(z_1(0))} = \frac{|x_1(0)|}{\bar{\xi}} > 0. \tag{32}$$

To verify that all the objectives have been satisfied, it remains to show that $\text{sign}(\cos x_3(\bar{t})) > 0$ so that $\text{ATAN2}\{z_4(\bar{t}), z_3(\bar{t})\} = 0$. Since $\cos x_3(t) = z_3(t)/\xi(t)$, then

$$\text{sign}(\cos x_3(\bar{t})) = \frac{-\text{sign}(z_1(0))}{-\text{sign}(\xi(0))} > 0, \tag{33}$$

giving $x_3(\bar{t}) = 0$ (not π!).

A couple of remarks are now in order.

- If a disturbance occurs, feedback is used to counteract the perturbation; time $\bar{t}$ will not be anymore the final instant, but this does not affect the overall behavior. A robust controller is designed so that the variables z_2 and z_4 converge to the origin faster than z_1. The way z_1 converges to the origin is governed by the choice of the constant $\bar{\xi}$, as shown by (32).
- In a practical approach, the stabilizing terminal controller v_2 in (28) is replaced by a continuous version yielding arbitrary exponential rate of convergence for the variables z_2 and z_4:

$$v_2(t) = -k_p z_2(t) - k_d z_4(t), \qquad k_p, k_d > 0. \tag{34}$$

  The gains k_p and k_d are selected so that the error on $z_2 = x_2$ becomes less than a small tolerance error ϵ, before time $\bar{t}$.

If the initial state of the unicycle does not satisfy the conditions $x_1(0) \neq 0$ and $\cos x_3(0) > 0$, it is still possible to use the above stabilizing control by the preliminary application of the following feedback law for both inputs v, for $t \in [-t_0, 0]$:

$$\begin{aligned} v_1(t) &= -\sin x_3(t), \\ v_2(t) &= \cos x_3(t). \end{aligned} \tag{35}$$

The duration t_0 of this first control phase will be such that $x_1(0) \neq 0$ and $\cos x_3(0) > 0$, so that the previously described control phase can be successfully started. In fact, from (24), it is seen that the control law (35) yields a constant $\xi = \xi(-t_0)$ and that the original inputs become

$$u_1 = \xi(-t_0), \qquad u_2 = \frac{1}{\xi(-t_0)}. \tag{36}$$

It follows that $z_3^2(t) + z_4^2(t) = \xi^2(-t_0)$ and the two variables z_3 and z_4 will vary on a circle of radius $|\xi(-t_0)|$. Therefore, there exists a finite time t_0 such that z_3

will become of the same sign required for the initial condition of the compensator at time zero, i. e. $\text{sign}(z_3(0)) = \text{sign}(\xi(0))$ and $\cos x_3(0)$ will be thus positive. From (25) and (26), it follows that the same control (35) leads to a value for $x_1(0)$ which is different from zero.

Intuitively, control (35) makes the unicycle rotate so that the angle formed with the x-axis will belong to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In the second phase, the control law (28) makes the unicycle move forwards (or backwards) to the origin, whenever its initial position is in the left (respectively, right) half-plane. As a result, the two control phases can be patched together *without* changing structure, but just resetting the compensator state ξ and the external inputs v_1 and v_2. We note that this automatically leads to a discontinuous control input u_1 and u_2.

Simulation results with the previous stabilizing dynamic control law are reported next for two different point-to-point motions. In both cases, the exponential stabilizing controller (34) is used, with gains chosen as

$$k_p = 25, \quad k_d = 10, \tag{37}$$

corresponding to a double real pole at -5 for the error dynamics of the second component of the output. The final error norm tolerance is set to $\epsilon = 0.0005$.

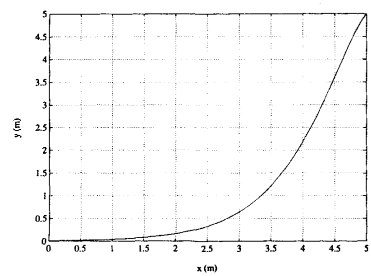

Fig. 1. Output evolution (case 1).

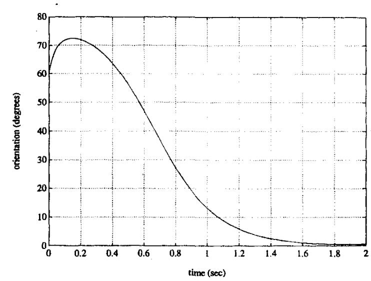

Fig. 2. Unicycle orientation x_3 (case 1).

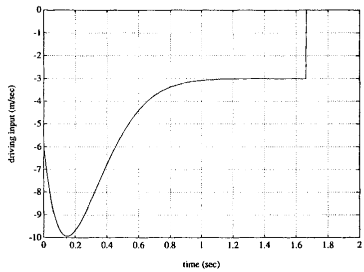

Fig. 3. Driving input u_1 (case I).

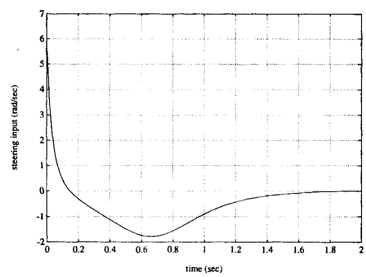

Fig. 4. Steering input u_2 (case I).

In the first simulation (case I), the initial state is

$$x_1(0) = 5, \quad x_2(0) = 5, \quad x_3(0) = \pi/3, \tag{38}$$

thus with the unicyle in the positive quadrant. The compensator state is initially set to $\xi(0) = -6$. Figures 1 - 5 show the motion of the unicycle in the (x, y)-plane, its orientation, the original control input u, and the compensator state. The overall motion is very smooth, with the driving input being always negative (the unicycle rolls backwards). The sudden change in Fig. 3 corresponds to the reaching of the origin. Note that the compensator state in Fig. 5 is bounded away from zero, ensuring in turn bounded control effort.

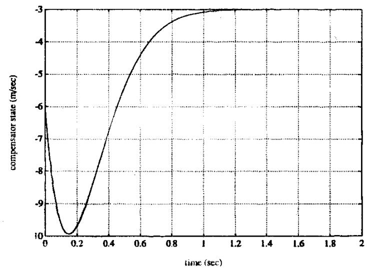

Fig. 5. Compensator state ξ (case I).

In the second simulation (case II), the initial state is

$$x_1(0) = -5, \quad x_2(0) = 0, \quad x_3(0) = -9\pi/20, \tag{39}$$

while the initial state of the compensator is set to $\xi(0) = -18$. Figures 6 – 10 refer to the associated motion, which is now performed always in the forward rolling direction. Since the initial position is on the negative x-axis and the unicycle points almost at $-90°$, the control will not immediately force the system towards the origin, allowing simultaneous reorientation during motion. After 0.2 seconds, the unicycle is in a more convenient state to be pushed to its final destination with reduced effort.

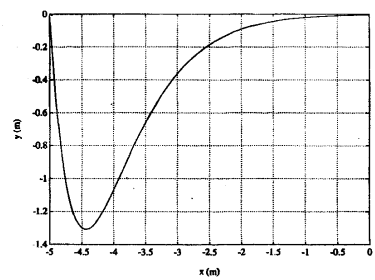

Fig. 6. Output evolution (case II).

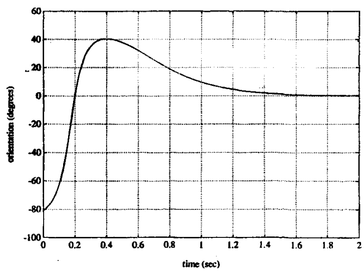

Fig. 7. Unicycle orientation x_3 (case II).

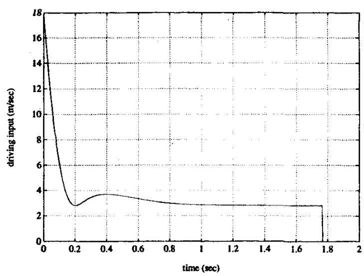

Fig. 8. Driving input u_1 (case II).

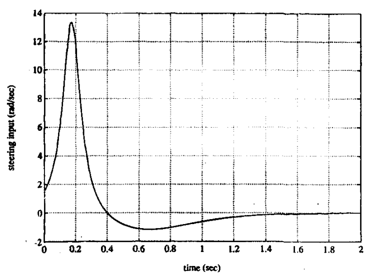

Fig. 9. Steering input u_2 (case II).

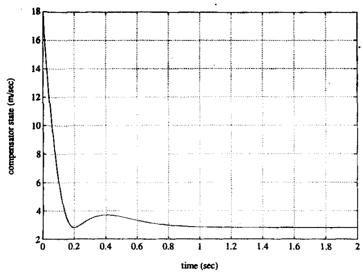

Fig. 10. Compensator state ξ (case II).

5. CONCLUDING REMARKS

In this paper, the question whether it is possible to use dynamic compensation for achieving tracking and stabilization of nonholonomic nonlinear systems has been investigated. First, the relation between the presence of nonholonomic constraints and that of singularities in the dynamic extension process has been established. Then, for a unicycle example, it has been shown that dynamic compensation can nevertheless be used by fully exploiting the degrees of freedom of the control scheme, in particular the proper initialization of the dynamic part of the controller.

The proposed stabilizing control law shows discontinuities due to the re-initialization of the dynamic compensator state "on the fly", with the same control structure being preserved. The overall stabilization design is simpler than that of time-varying feedback control laws since it takes full advantage of the linear equivalent system under feedback linearizing dynamic control. Moreover, the proposed control scheme contains as much feedback information as possible, thereby presenting obvious benefits of robustness with respect to open-loop solutions.

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