KYBERNETIKA — VOLUME 29 (1993), NUMBER 6, PAGES 547–562

ON THE COVER PROBLEMS OF GEOMETRIC THEORY

NICOS KARCANIAS AND DIMITRIS VAFIADIS

For linear systems, a family of cover problems of the geometric theory are introduced as extensions of the standard cover problem and a matrix pencil formulation of such problems is given. It is shown that the solvability of such problems is reduced to a problem of Kronecker Invariant Transformation by Matrix Pencil Augmentation and a Matrix Pencil Realisation Problem. Necessary, as well as sufficient conditions for solvability of both problems are given, which lead to a number of conditions for solvability of the partial, as well as standard cover problem. The special cases of left regular, regular solutions of the cover problem are investigated and a parametrisation of such families of solutions is given.

1. INTRODUCTION

The cover problems arise in the study of several control problems such the observer design [14], the exact model matching, the disturbance decoupling, the identification [2] and the squaring down problem [8]. For a partial realization approach to the cover problem see [1].

The standard cover problem that has been considered so far belongs to a more general class of problems that arise within the general area of selection of input, output schemes for a given system [5]. Althought the formulation of these problems is geometric in nature (find a certain type of invariant subspace that covers a given subspace and is contained in another one), their solvability and parametrisation of solutions is closer in nature to problems of invariant structure assignment. The matrix pencil framework [7, 3] for the characterisation of invariant subspaces of the geometric theory [12, 13] seems to be more suitable for the study of such problems, since it brings together the geometric and Kronecker invariant structure aspects of the problem; furthermore, the constructive nature of the matrix pencil tools allows the computation and parametrisation of solutions in a simple manner. Extending the matrix pencil framework to this new family of geometric problems is essential in the effort to provide unifying matrix pencil tools for the geometric synthesis methods.

The aim of this paper is to provide a classification and a matrix pencil formulation of the family of cover problems of geometric theory, give necessary as well as sufficient conditions for the existence of certain types of solutions and parametrise special families of solution spaces. An integral part of this approach is the splitting of the overall problem into a Kronecker invariant transformation problem by matrix pencil augmentation and a matrix pencil realisation problem. The first deals with the study of the effect of adding matrix pencil columns to a given pencil on the resulting Kronecker structure; the second is equivalent to a problem of generating a given space restricted pencil [7] for a given system. For both problems, we produce necessary, as well as sufficient conditions for their solvability. These conditions in turn, provide criteria for the solvability of the original cover problems. Of special interest are certain families of cover problems, referred to as left regular, regular, families; for these families we provide also some parametrisation of the solution spaces.

2. PRELIMINARY DEFINITIONS AND STATEMENT OF THE PROBLEM

Let $\mathcal{S}(\mathbf{A},\mathbf{B},\mathbf{C})$ be the system characterised by the following state-space equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{2.1a}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{2.1b}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times \ell}$ and $\mathbf{C} \in \mathbb{R}^{m\times n}$. It is assumed that both matrices $\mathbf{B}$ and $\mathbf{C}$ have full rank and that the system is controllable. If $\mathbf{N}$ is a left annihilator of $\mathbf{B}$ (i.e. a basis matrix for the $\text{Ker}_\ell(\mathbf{B})$) and $\mathbf{B}^\dagger$ is a left inverse of $\mathbf{B}$ ($\mathbf{B}^\dagger\mathbf{B} = \mathbf{I}_\ell$), then (2.1a,b) are equivalent to

$$\mathbf{N}\dot{\mathbf{x}} = \mathbf{N}\mathbf{A}\mathbf{x} \tag{2.1c}$$

$$\mathbf{u} = \mathbf{B}^\dagger\,\dot{\mathbf{x}} - \mathbf{B}^\dagger\mathbf{A}\mathbf{x} \tag{2.1d}$$

where (2.1.c) is a "feedback free" system description and the associated pencil $\mathbf{R}(s) = s\mathbf{N} - \mathbf{N}\mathbf{A}$ is known as the restricted input-state pencil [7] of the system. A family of cover problems of the geometric theory are defined below.

Definition 2.1. Let $\mathcal{X}$ be the state-space of the $\mathcal{S}(\mathbf{A},\mathbf{B})$ system and let $\mathcal{J} \subseteq \mathcal{W} \subseteq \mathcal{X}$. Finding all subspaces $\mathcal{V}$ of $\mathcal{X}$ such that

(i) $\mathcal{V}$ is $(\mathbf{A},\mathbf{B})$-invariant, i.e. $\mathbf{A}\mathcal{V} \subseteq \mathcal{V} + \mathcal{B}$ and

$$\mathcal{J} \subseteq \mathcal{V} \subseteq \mathcal{W} \tag{2.2}$$

is known as the standard cover problem [1, 2].

(ii) $\mathcal{V}$ is an almost $(\mathbf{A},\mathbf{B})$-invariant, controllability or almost controllability and (2.2) is also satisfied, will be referred as extended cover problems.

(iii) $\mathcal{V}$ is any of the invariant types of subspaces in (i), (ii) and $\mathcal{W} = \mathcal{X}$, then the problem will be called partial cover problem .

The extended cover problems form an integral part of the investigation of Model Projection Problems (MPP) [5], which arise in the study of selection of control structures. Our approach is based on the matrix pencil characterization of the $(\mathbf{A}, \mathbf{B})$-invariant subspaces [3,4]. If $\mathbf{V}$ is a basis matrix of $\mathcal{V}$, the nature of the subspace $\mathcal{V}$ of the state space of the system as an invariant subspace is characterised by the nature of the set of strict equivalence invariants (cf. [15]) of the $\mathcal{V}$-restricted matrix pencil $\mathbf{R}_{\mathcal{V}}(s) = s\mathbf{NV} - \mathbf{NAV}$ which is referred in short as a $\mathcal{V}$-restriction pencil.

Lemma 2.1. [3] A subspace $\mathcal{V} \subset \mathcal{X}$ is an $(\mathbf{A}, \mathbf{B})$-invariant subspace, if and only if the pencil

$$s\mathbf{NV} - \mathbf{NAV} \tag{2.3}$$

is characterised by c.m.i., f.e.d. and possibly zero r.m.i. In addition if $s\mathbf{NV} - \mathbf{NAV}$ is not characterised by f.e.d., then $\mathcal{V}$ is a controllability subspace (c.s.).

A similar result has been established in [3] for the matrix pencil characterisation of Almost $(\mathbf{A}, \mathbf{B})$-invariant and almost controllability subspaces. In this paper, we shall be mostly concerned with $(\mathbf{A}, \mathbf{B})$-invariant subspaces, whereas almost $(\mathbf{A}, \mathbf{B})$-invariant case is treated in a similar manner.

The main idea underlying the matrix pencil approach to the study of the cover problems is the following: Let $\mathbf{J}$ be the basis matrix of the subspace to be covered. Since $\mathcal{V}$ is the covering subspace, then $\mathcal{V} = \mathcal{J} \oplus \mathcal{T}$ where $\mathcal{T}$ is some appropriate subspace, or in matrix form

$$\mathbf{V} = [\mathbf{J}, \mathbf{T}] \tag{2.4}$$

The restriction pencil of the covering subspace is then

$$\mathbf{R}_{\mathcal{V}}(s) = s\mathbf{NV} - \mathbf{NAV} = (s\mathbf{N} - \mathbf{NA})[\mathbf{J}, \mathbf{T}] \tag{2.5}$$

From the above expression, it is clear that the general family of cover problems are equivalent to problems of Kronecker structure assignment defined below.

Kronecker Structure Assignment Problem (KSAP): Given the $\mathcal{J}$-restriction pencil $\mathbf{R}_{\mathcal{J}}(s) = s\mathbf{NJ} - \mathbf{NAJ}$, find an appropriate $\mathcal{T}$-restriction pencil $\mathbf{R}_{\mathcal{T}}(s) = s\mathbf{NT} - \mathbf{NAT}$ such that the column augmented pencil $\mathbf{R}_{\mathcal{V}}(s)$ in (2.5) has a certain type invariant structure.

The general Kronecker structure assignment problem may be naturally divided to the following two subproblems:

Matrix Pencil Augmentation Problem (MPAP): Given the pencil $s\mathbf{F} - \mathbf{G} \in \mathbb{R}^{m \times k}[s]$, find the conditions for the existence of a pencil $s\bar{\mathbf{F}} - \bar{\mathbf{G}} \in \mathbb{R}^{m \times p}[s]$ such that the pencil

$$\mathbf{P}(s) = [s\mathbf{F} - \mathbf{G}, s\bar{\mathbf{F}} - \bar{\mathbf{G}}] \tag{2.6}$$

has a given set of invariants.

Matrix Pencil Realisation Problem (MPRP): Given the pencil $s\mathbf{N} - \mathbf{NA} \in \mathbb{R}^{(n-\ell) \times n}[s]$, find the conditions under which there exists $\mathbf{T} \in \mathbb{R}^{n \times p}$ such that

$$s\mathbf{NT} - \mathbf{NAT} = s\mathbf{F} - \mathbf{G} \tag{2.7}$$

Note that a special case of MPRP has been recently examined in [6] and corresponds to the case where $s\mathbf{N}-\mathbf{NA}$ is free. The above two problems are integral parts of the KSAP and will be examined here. With reference to the general cover problem, of particular interest are the problems of determining the minimal dimension subspace solutions, when such solution exist.

The above family of structure assignment problems deal with assignment of certain types of invariants, rather than the assignment of exact values of pencil invariants; in this sense they are extensions of the zero assignment problems considered so far [8]. In the following the standard cover problem corresponding to the $(\mathbf{A},\mathbf{B})$-invariant subspace case will be considered. The emphasis will be on the partial cover problem.

3. KRONECKER INVARIANT TRANSFORMATION BY MATRIX PENCIL AUGMENTATION

In this section, we examine a number of results related to the transformation of the types of SE-invariants of a matrix pencil by addition of columns (rows). We consider first an important property established for a general polynomial matrix by [9] and presented for the case of matrix pencils.

Theorem 3.1. Let $\mathbf{P}(s)=s\mathbf{F}-\mathbf{G}$ be a matrix pencil and let $s\mathbf{f}-\mathbf{g}$ be a column pencil and let $\mathbf{P}'(s)=[s\mathbf{F}-\mathbf{G},\ s\mathbf{f}-\mathbf{g}]$. If $\theta_i(s)$, $i=1,\dots,k$, $\zeta_j(s)$, $j=1,\dots,k$ or $k+1$ are the invariant polynomials of $\mathbf{P}(s)$, $\mathbf{P}'(s)$ respectively, then

(a) If $\operatorname{rank}_{\mathbb{R}(s)}\{\mathbf{P}(s)\}<\operatorname{rank}_{\mathbb{R}(s)}\{\mathbf{P}'(s)\}$ then the following interlacing property holds

$$\zeta_1(s)/\theta_1(s)/\zeta_2(s)/\theta_2(s)/\dots/\theta_k(s)/\zeta_{k+1}(s) \tag{3.1}$$

(b) If $\operatorname{rank}_{\mathbb{R}(s)}\{\mathbf{P}(s)\}=\operatorname{rank}_{\mathbb{R}(s)}\{\mathbf{P}'(s)\}$ then the following interlacing property holds

$$\theta_1(s)/\zeta_2(s)/\theta_2(s)/\dots/\theta_k(s)/\zeta_{k+1}(s) \tag{3.2}$$

Note that in the above a/b denotes that a divides b. Some obvious further result is stated below:

Proposition 3.1. Let $s\mathbf{F}-\mathbf{G}$ be a right regular pencil i.e. it is characterized only by r.m.i., i.e.d. and f.e.d. Let $s\mathbf{F}-\mathbf{G}$ be augmented by a single column $s\mathbf{f}-\mathbf{g}$ such that its rank is increased. Then the sets of the i.e.d. and f.e.d. of the original pencil are subsets of the i.e.d. and f.e.d. of the augmented pencil.

Proof. From Theorem 3.1 it follows that the invariant polynomials of the original and the augmented pencils are related by the interlacing inequalities (3.1). The invariant factors ζ_i, $i=1,\dots,\ell+1$ and $\varepsilon_i, i=1,\dots,\ell$ can be factorized as follows:

$$\zeta_i(s)=(s-\alpha_i^1)^{\lambda_{i,1}}(s-\alpha_i^2)^{\lambda_{i,2}}\dots(s-\alpha_i^{\rho_i})^{\lambda_{i,\rho_i}} \tag{3.3}$$
$$\varepsilon_i(s)=(s-\beta_i^1)^{\mu_{i,1}}(s-\beta_i^2)^{\mu_{i,2}}\dots(s-\beta_i^{\phi_i})^{\mu_{i,\phi_i}} \tag{3.4}$$

The factors $(s-\alpha_i^j)^{\lambda_{i,j}}$ and $(s-\beta_k^l)^{\mu_{k,l}}$ are the f.e.d. of the augmented and the original pencil respectively.

From the interlacing inequalities (3.1) it is clear that

$$\varepsilon_j(s)/\zeta_{j+1}(s) \tag{3.5}$$

i.e. $\zeta_{j+1}(s)$ can be expressed as

$$\zeta_{j+1}(s)=x_j(s)\varepsilon_j(s) \tag{3.6}$$

or

$$\zeta_{j+1}(s)=x_j[(s-\beta_j^1)^{\mu_{j,1}}(s-\beta_j^2)^{\mu_{j,2}}\dots(s-\beta_j^{\phi_j})^{\mu_{j,\phi_j}}] \tag{3.7}$$

The above yields that all the f.e.d. of $s\mathbf{F}-\mathbf{G}$ are f.e.d. of the augmented pencil $[s\mathbf{F}-\mathbf{G}, s\mathbf{f}-\mathbf{g}]$ and the result follows. □

The case of the i.e.d. may be proved similarly, taking the "dual" pencil $\mathbf{F}-\hat{s}\mathbf{G}$. It should be mentioned that the multiplicities of the common elementary divisors of the two pencils may be different, since the polynomial $x_i(s)$ may have some of its roots equal to the roots of $\varepsilon_i(s)$.

An obvious consequence of the above is the following

Proposition 3.2. Consider the pencil $[s\mathbf{F}-\mathbf{G}, s\mathbf{f}-\mathbf{g}]$.

(i) If the additional column is linearly dependent on the columns of $s\mathbf{F}-\mathbf{G}$, the number of the c.m.i. is increased by one and the number of the r.m.i. remains unchanged.

(ii) If the additional column is linearly independent, then the number of the c.m.i remains unchanged and the number of the r.m.i is reduced by one.

Proof. The number of c.m.i. and r.m.i. of $s\mathbf{F}-\mathbf{G}$ is equal to the dimension of the right and left null space of $s\mathbf{F}-\mathbf{G}$ respectively.

(i) If the additional column $s\mathbf{f}-\mathbf{g}$ is linearly dependent on the columns of $s\mathbf{F}-\mathbf{G}$ then $\text{rank}(s\mathbf{F}-\mathbf{G}) = \text{rank}(s\mathbf{F}-\mathbf{G}, s\mathbf{f}-\mathbf{g})$ and therefore the dimension of the right null space of $s\mathbf{F}-\mathbf{G}$ is increased by one while the dimension of the left null space remains the same. From the above it follows that the number of the c.m.i. is increased by one and the number of the r.m.i. remains unchanged.

(ii) In the case where the additional column is linearly independent from the columns of $s\mathbf{F}-\mathbf{G}$ we have that $\text{rank}(s\mathbf{F}-\mathbf{G}, s\mathbf{f}-\mathbf{g}) = \text{rank}(s\mathbf{F}-\mathbf{G})+1$ and therefore the dimension of the right null space remains unchanged. The dimension of the left null space is reduced by one since it is equal to the number of rows of the augmented pencil minus the rank of that pencil. □

From the above proposition and Theorem 3.1 it follows that when the rank of the pencil $s\mathbf{F}-\mathbf{G}$ is increased by 1 with the addition of a single column, the result is the elimination of one r.m.i. and the possible change of the structure of the f.e.d. / i.e.d.

Thus, when we want to eliminate the r.m.i. of a pencil, it is necessary to augment it by a number of linearly independent columns equal to the number of the r.m.i.

Consider now the general pencil $s\mathbf{F} - \mathbf{G}$ and without loss of generality we may assume to be in the Kronecker canonical form.

$$[s\mathbf{F} - \mathbf{G}, s\bar{\mathbf{F}} - \bar{\mathbf{G}}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{L}_\eta(s) & 0 & 0 & 0 \\ 0 & 0 & \mathbf{L}_\varepsilon(s) & 0 & 0 \\ 0 & 0 & 0 & \mathbf{D}_\infty(s) & 0 \\ 0 & 0 & 0 & 0 & \mathbf{D}_f(s) \end{bmatrix} \tag{3.8}$$

where the blocks $\mathbf{L}_\varepsilon$, $\mathbf{L}_\eta$, $\mathbf{D}_\infty$, $\mathbf{D}_f$ correspond to all the nonzero r.m.i., nonzero c.m.i., i.e.d., f.e.d. respectively.

Proposition 3.3. The number of the zero r.m.i. of the augmented pencil $[s\mathbf{F} - \mathbf{G}, s\bar{\mathbf{F}} - \bar{\mathbf{G}}]$ cannot exceed the number of the zero r.m.i of the pencil $s\mathbf{F} - \mathbf{G}$.

Proof. The number of the z.r.m.i. of $s\mathbf{F} - \mathbf{G}$ is equal to the dimension of the left null space of the matrix $[\mathbf{F}, \mathbf{G}]$ and the number of z.r.m.i of the augmented pencil is the dimension of the left null space of the matrix $[\mathbf{F}, \mathbf{G}, \bar{\mathbf{F}}, \bar{\mathbf{G}}]$. But

$$\mathcal{N}_\ell\{[\mathbf{F}, \mathbf{G}, \bar{\mathbf{F}}, \bar{\mathbf{G}}]\} = \mathcal{N}_\ell\{[\mathbf{F}, \mathbf{G}]\} \cap \mathcal{N}_\ell\{[\bar{\mathbf{F}}, \bar{\mathbf{G}}]\} \subseteq \mathcal{N}_\ell\{[\mathbf{F}, \mathbf{G}]\} \tag{3.9}$$

and therefore

$$\dim \mathcal{N}_\ell\{[\mathbf{F}, \mathbf{G}, \bar{\mathbf{F}}, \bar{\mathbf{G}}]\} \leq \dim \mathcal{N}_\ell\{[\mathbf{F}, \mathbf{G}]\}$$

and the result follows. □

Lemma 3.1. [9] Let $s\mathbf{F} - \mathbf{G}$ be the restriction pencil of the system (2.1) on a subspace $\mathcal{V}$. If $\mathcal{V}$ is an $(\mathbf{A}, \mathbf{B})$-invariant subspace then

$$\mathcal{N}_\ell(\mathbf{F}) \subseteq \mathcal{N}_\ell(\mathbf{G}) \tag{3.10}$$

or equivalently,

$$\text{col} - \text{span}\{\mathbf{F}\} \supseteq \text{col} - \text{span}\{\mathbf{G}\} \tag{3.11}$$

Proposition 3.4. The matrix pencil $[s\mathbf{F} - \mathbf{G}, s\bar{\mathbf{F}} - \bar{\mathbf{G}}]$ is not characterised by i.e.d. and n.z.r.m.i. only if

$$\text{col} - \text{span}\{\mathbf{F}, \bar{\mathbf{F}}\} \supseteq \text{col} - \text{span}\{\mathbf{G}\} \tag{3.12}$$

Proposition 3.5. Necessary condition for the augmented pencil $[s\mathbf{F}-\mathbf{G}, s\bar{\mathbf{F}}-\bar{\mathbf{G}}]$ to have no i.e.d. and no n.z.r.m.i. is that the number of columns of $s\bar{\mathbf{F}}-\bar{\mathbf{G}}$ is greater or equal to the total number of the n.z.r.m.i. and i.e.d. of $s\mathbf{F}-\mathbf{G}$.

Proof. From Proposition 3.3 it follows that in order to eliminate the n.r.m.i., we need at least equal number of linearly independent columns. Obviously, the minimal number of the additional columns is obtained when the composite pencil $[s\mathbf{F}-\mathbf{G}, s\bar{\mathbf{F}}-\bar{\mathbf{G}}]$ has equal number of zero r.m.i., to the number of the z.r.m.i. of the original pencil $s\mathbf{F}-\mathbf{G}$. From Proposition 3.1 it follows that as long as we augment the pencil by linearly independent columns, the resulting pencil is characterised by i.e.d. Since we keep the number of the z.r.m.i. unchanged, we can assume that the composite pencil has the form

$$[s\mathbf{F}-\mathbf{G}, s\bar{\mathbf{F}}-\bar{\mathbf{G}}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{L}_\eta(s) & 0 & 0 & 0 & s\mathbf{K}_2-\mathbf{M}_2 \\ 0 & 0 & \mathbf{L}_\varepsilon(s) & 0 & 0 & s\mathbf{K}_3-\mathbf{M}_3 \\ 0 & 0 & 0 & \mathbf{D}_\infty(s) & 0 & s\mathbf{K}_4-\mathbf{M}_4 \\ 0 & 0 & 0 & 0 & \mathbf{D}_f(s) & s\mathbf{K}_5-\mathbf{M}_5 \end{bmatrix}. \tag{3.13}$$

where $\mathbf{L}_\eta, \mathbf{L}_\varepsilon, \mathbf{D}_\infty, \mathbf{D}_f$ are the nonzero r.m.i., nonzero c.m.i., i.e.d. and f.e.d. blocks respectively.

The structure of that pencil as far as the n.z.r.m.i. and the i.e.d. are concerned, is identical to the structure of the pencil

$$\left[s\tilde{\mathbf{F}}-\tilde{\mathbf{G}}\right] = \begin{bmatrix} 0 & \mathbf{L}_\eta(s) & 0 & 0 & 0 & s\mathbf{K}_2-\mathbf{M}_2 \\ 0 & 0 & \mathbf{L}_\varepsilon(s) & 0 & 0 & s\mathbf{K}_3-\mathbf{M}_3 \\ 0 & 0 & 0 & \mathbf{D}_\infty(s) & 0 & s\mathbf{K}_4-\mathbf{M}_4 \\ 0 & 0 & 0 & 0 & \mathbf{D}_f(s) & s\mathbf{K}_5-\mathbf{M}_5 \end{bmatrix}. \tag{3.14}$$

This matrix pencil cannot be characterised by zero r.m.i. since $\mathcal{N}_\ell\{[\tilde{\mathbf{F}}, \tilde{\mathbf{G}}]\} = \{0\}$. Therefore pencil (3.14) is not characterised by i.e.d. and n.r.m.i. only if the matrix $\tilde{\mathbf{F}}$ is left regular. From the form of the pencil (3.14) we can see that the matrix $\tilde{\mathbf{F}}$ can have full rank only if the matrix that consists of the rows of the pencil $s\bar{\mathbf{F}}-\bar{\mathbf{G}}$ that correspond to the bottom rows of the blocks of the n.r.m.i. and i.e.d. has full rank. Since the number of the rows of that matrix is equal to the total number of i.e.d. and n.r.m.i. of the pencil $s\mathbf{F}-\mathbf{G}$, the result follows. □

One of the major issues in characterising the solvability of the extended cover problems is the investigation of the conditions under which the resulting pencil after augmentation has no n.z.r.m.i. By assuming the pencil in the canonical form we have:

$$[s\mathbf{F}-\mathbf{G}, s\bar{\mathbf{F}}-\bar{\mathbf{G}}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & s\mathbf{K}_1-\mathbf{M}_1 \\ 0 & \mathbf{L}_\eta(s) & 0 & 0 & 0 & s\mathbf{K}_2-\mathbf{M}_2 \\ 0 & 0 & \mathbf{L}_\varepsilon(s) & 0 & 0 & s\mathbf{K}_3-\mathbf{M}_3 \\ 0 & 0 & 0 & \mathbf{D}_\infty(s) & 0 & s\mathbf{K}_4-\mathbf{M}_4 \\ 0 & 0 & 0 & 0 & \mathbf{D}_f(s) & s\mathbf{K}_5-\mathbf{M}_5 \end{bmatrix} \tag{3.15}$$

Now it is obvious that necessary and sufficient condition for $\mathbf{P}'(s)$ to have any type of r.m.i., is that the subpencil

$$\mathbf{P}''(s) = \begin{bmatrix} \mathbf{0} & s\mathbf{K}_1 - \mathbf{M}_1 \\ \mathbf{L}_\eta(s) & s\mathbf{K}_2 - \mathbf{M}_2 \end{bmatrix} \tag{3.16}$$

to provide this type of r.m.i. since the rest of the blocks are left regular. We may summarise as follows:

Proposition 3.6. Necessary and sufficient conditions for $\mathbf{P}''(s)$ to have all its r.m.i. with values strictly less than those in the $\mathbf{L}_\eta$ block, or $\mathbf{P}''(s)$ has no r.m.i. are:

(i) If $\mathcal{R}(\mathbf{K}_1, \mathbf{M}_1), \mathcal{R}(\mathbf{K}_2, \mathbf{M}_2)$ are the $\mathbb{R}(s)$-row spaces of the pencils $s\mathbf{K}_1 - \mathbf{M}_1, s\mathbf{K}_2 - \mathbf{M}_2$ respectively, then

$$\mathcal{R}(\mathbf{K}_1, \mathbf{M}_1) \cap \mathcal{R}(\mathbf{K}_2, \mathbf{M}_2) = \{0\} \tag{3.17}$$

(ii) The pencil $[s\mathbf{K}_2 - \mathbf{M}_2, \mathbf{L}_\eta]$ is left regular.

(iii) All r.m.i. of $s\mathbf{K}_1 - \mathbf{M}_1$ are strictly less than those of $\mathbf{L}_\eta$, or the pencil $s\mathbf{K}_1 - \mathbf{M}_1$ is left regular if $\mathbf{P}''(s)$ has no r.m.i.

Proof. Let $\mathbf{y}^t(s) = [\mathbf{y}_1^t(s), \mathbf{y}_2^t(s)]$ be an $\mathbb{R}[s]$ vector in $\mathcal{N}_\ell(\mathbf{P}''(s))$. Then we have

$$[\mathbf{y}_1^t(s), \mathbf{y}_2^t(s)]\,[\,s\mathbf{K}_1 - \mathbf{M}_1 s\mathbf{K}_2 - \mathbf{M}_2\,] = \mathbf{0}, \qquad \mathbf{y}_2^t(s)\mathbf{L}_\eta(s) = 0$$

or equivalently

$$\mathbf{y}_2^t(s)\mathbf{L}_\eta(s) = 0 \tag{3.18}$$

$$\mathbf{y}_1^t(s)(s\mathbf{K}_1 - \mathbf{M}_1) = -\mathbf{y}_2^t(s)(s\mathbf{K}_2 - \mathbf{M}_2) \tag{3.19}$$

From condition (3.19) we see that either $\mathbf{y}_2^t(s) \neq 0$, or $\mathbf{y}_2^t(s) = 0$. We distinguish the following cases:

(i) $\mathbf{y}_2^t(s) \neq 0$. In this case, if $\bar{n}$ is the minimal of the degrees in $\mathbf{L}_\eta(s)$ block, then $\partial\{\mathbf{y}_2^t(s)\} \geq \bar{n}$. It is thus a necessary condition that $\mathbf{y}_2^t(s) = 0$ for the degree of $\mathbf{y}(s)$ to be less than $\bar{n}$.

(ii) If $\mathbf{y}_2^t(s) = 0$, then (3.19) is reduced to

$$\mathbf{y}_1^t(s)(s\mathbf{K}_1 - \mathbf{M}_1) = 0 \tag{3.20}$$

and it is necessary that $\mathcal{N}_\ell(s\mathbf{K}_1 - \mathbf{M}_1)$ is either $\{0\}$, or if it is nonzero, then its r.m.i. are strictly less than $\bar{n}$. Thus necessary conditions are

$$\mathbf{y}_2^t(s) = 0 \quad \text{and} \quad \mathcal{N}_\ell(s\mathbf{K}_1 - \mathbf{M}_1) = \{0\}$$

or the r.m.i. of $s\mathbf{K}_1 - \mathbf{M}_1$ are strictly less than $\bar{n}$.

For $\mathbf{y}_2^t(s) = 0$ we must determine the necessary conditions for this to happen. From equation (3.19) we have that:

(a) If $\mathbf{y}_2^t(s) \neq 0$ and $\mathbf{y}_1^t(s) \neq 0$ then

$$\mathcal{R}(\mathbf{K}_1, \mathbf{M}_1) \cap \mathcal{R}(\mathbf{K}_2, \mathbf{M}_2) \neq \{0\} \tag{3.21}$$

(b) If $\mathbf{y}_2^t(s) \neq 0$ and $\mathbf{y}_1^t(s) = 0$, then by (3.18) and $\mathbf{y}_1^t(s) = 0$ in (3.19) we have

$$\mathbf{y}_2^t(s)[s\mathbf{K}_2 - \mathbf{M}_2, \mathbf{L}_\eta(s)] = 0 \tag{3.22}$$

It is clear that from (a) and (b) above that for $\mathbf{y}_2^t(s) = 0$ it is necessary that both (3.21) and (3.22) conditions to be true, which proves the necessity.

To prove the sufficiency we argue as follows:

$$\mathcal{R}(\mathbf{K}_1, \mathbf{M}_1) \cap \mathcal{R}(\mathbf{K}_2, \mathbf{M}_2) = \{0\} \tag{3.23}$$

implies that condition (3.19) yields

$$\mathbf{y}_1^t(s)(s\mathbf{K}_1 - \mathbf{M}_1) = 0 \tag{3.24}$$

$$\mathbf{y}_2^t(s)(s\mathbf{K}_2 - \mathbf{M}_2) = 0 \tag{3.25}$$

and from (3.25) and (3.18) we have

$$\mathbf{y}_2^t(s)[s\mathbf{K}_2 - \mathbf{M}_2, \mathbf{L}_n(s)] = 0 \tag{3.26}$$

which since $[s\mathbf{K}_2 - \mathbf{M}_2, \mathbf{L}_n(s)]$ is left regular implies $\mathbf{y}_2^t(s) = 0$. Since $s\mathbf{K}_1 - \mathbf{M}_1$ is either left regular, or has r.m.i. with values strictly less than $\bar{n}$ the sufficiency is established.

4. THE MATRIX PENCIL REALISATION PROBLEM

The analysis of the previous section has assumed that the pencil used in the augmentation process, $s\bar{\mathbf{F}} - \bar{\mathbf{G}}$, is arbitrary; however, this pencil is generated from the input-state pencil of the system as

$$(s\mathbf{N} - \mathbf{NA})\mathbf{T} = s\bar{\mathbf{F}} - \bar{\mathbf{G}} \tag{4.1}$$

or equivalently as a solution of the system

$$\begin{bmatrix} \bar{\mathbf{F}} \\ \bar{\mathbf{G}} \end{bmatrix} = \begin{bmatrix} \mathbf{N} \\ \mathbf{NA} \end{bmatrix} \mathbf{T} \tag{4.2}$$

The problem of matrix pencil realisation is equivalent to finding a $\mathbf{T}$, when $(\mathbf{N}, \mathbf{A})$, $(\mathbf{F}, \mathbf{G})$ are given such that (4.2) is satisfied. A more general form of this problem is the "Invariant Realisation Problem", [4], where the pair $(\mathbf{N}, \mathbf{A})$ is also free. Our present version of the problem is equivalent to generating an appropriate $\mathcal{T}$-restriction pencil for the given system. Clearly, this problem, does not always have a solution i.e. not any pair $(\bar{\mathbf{F}}, \bar{\mathbf{G}})$ may be created as a $\mathcal{T}$-restriction of a pair $(\mathbf{N}, \mathbf{NA})$; this problem is a generalisation of the zero assignment problems [6]. Clearly, the family of pairs $(\bar{\mathbf{F}}, \bar{\mathbf{G}})$ provide the necessary input to the Matrix Pencil Augmentation Problem.

In the case of the cover problem the matrices $\mathbf{F}$, $\mathbf{G}$, $\mathbf{N}$, $\mathbf{A}$ are given and the problem is to find $\mathbf{T}$ such that (4.1) is satisfied. An obvious result for solvability of this problem is:

Remark 4.1. The matrix pencil realisation problem is solvable if and only if

$$\text{col}-\text{span}\left\{\begin{bmatrix}\bar{\mathbf{F}}\\ \bar{\mathbf{G}}\end{bmatrix}\right\} \subseteq \text{col}-\text{span}\left\{\begin{bmatrix}\mathbf{N}\\ \mathbf{NA}\end{bmatrix}\right\}\mathbf{T} \tag{4.3}$$

Proposition 4.1. If $n < 2\ell$ the matrix pencil realisation problem is always solvable.

Proof. Since the system $\mathcal{S}(\mathbf{A},\mathbf{B})$ is controllable, the pencil $s\mathbf{N}-\mathbf{NA}$ is characterised only by c.m.i. and has the following canonical form.

$$s\mathbf{N}-\mathbf{NA} = \text{block}-\text{diag}\left\{\dots\begin{bmatrix} s & -1 & 0 & \dots & 0\\ 0 & \vdots & \ddots & \ddots & \vdots\\ 0 & \dots & 0 & s & -1\end{bmatrix}\dots\right\} \tag{4.4}$$

where the dimensions of the blocks are $(\sigma_i - 1)\times\sigma_i$ and σ_i are the controllability indices of the pair $(\mathbf{A},\mathbf{B})$. From the form of the above pencil we can easily see that the matrix $[\mathbf{N}^t,\mathbf{A}^t\mathbf{N}^t]^t$ has always full rank. The dimensions of $[\mathbf{N}^t,\mathbf{A}^t\mathbf{N}^t]^t$ are $(2n-2\ell)\times$n. Then if $n < 2\ell$ the equation

$$\begin{bmatrix}\bar{\mathbf{F}}\\ \bar{\mathbf{G}}\end{bmatrix} = \begin{bmatrix}\mathbf{N}\\ \mathbf{NA}\end{bmatrix}\mathbf{T} \tag{4.5}$$

is always solvable with respect to $\mathbf{T}$ and the result follows. □

Remark 4.2. For controllable systems with $n \le 2\ell$, any particular cover problem is equivalent to a matrix pencil augmentation problem as discussed in the previous section; otherwise, the Matrix Pencil Realisation Problem becomes an essential part of the overall cover problem.

5. LEFT REGULAR SOLUTIONS AND THE OVERALL COVER PROBLEM

In this section some special cases of the cover problem are investigated and some sufficient conditions for the solvability of the general case of the cover problem are given. The left regular cover problem is defined as that where the resulting augmented pencil has no left null space. For such cases a parametrisation of the solution spaces is also given. Note that a special case of the left regular case is when the resulting pencil is square and regular. This is defined as the regular case.

First we tackle the cover problem corresponding to the case where the subspaces are $(\mathbf{A},\mathbf{B})$-invariant subspaces and the restriction pencil has no r.m.i. at all. Some preliminary results are given below:

Proposition 5.1. If the restriction pencil $s\mathbf{NJ} - \mathbf{NAJ}$ of the given subspace $\mathcal{J}$ has no zero r.m.i., then the restriction pencil of any solution of the cover problem is not characterised by r.m.i. at all.

Proof. From Proposition 3.4 it follows that, since the number of the z.r.m.i. if $s\mathbf{NJ} - \mathbf{NAJ}$ is zero, then any augmentation of that pencil is not characterised by z.r.m.i. □

Proposition 5.2. Let $\mathcal{L} \subset \mathbb{R}^n, \dim\{\mathcal{L}\} = n - \ell$, $\mathbf{L}$ be a basis matrix of $\mathcal{L}$. If the restriction pencil has full rank (over $\mathbb{R}(s)$) and has no i.e.d., then:

(i) $\mathcal{L} + \mathcal{J}$ is a solution of the partial cover problem.

(ii) Any subspace defined as

$$\mathcal{L}' = \mathcal{L} + \hat{\mathcal{L}} + \mathcal{J} \tag{5.1}$$

where $\hat{\mathcal{L}}$ is arbitrary is also a solution of the partial cover problem.

Proof. Let $\mathbf{L} \in \mathbb{R}^{n\times(n-\ell)}$, such that $s\mathbf{NL} - \mathbf{NAL}$ is regular and has no i.e.d.; clearly the restriction pencil $[s\mathbf{NL} - \mathbf{NAL}, s\mathbf{NJ} - \mathbf{NAJ}]$ has no r.m.i. and thus $\mathcal{L}$ is a solution of the partial cover problem which proves (i).

For any $\hat{\mathcal{L}} \in \mathbb{R}^{n\times k}$ matrix the augmented pencil

$$(s\mathbf{N} - \mathbf{NA})[\mathbf{L}, \hat{\mathbf{L}}, \mathbf{J}] = [s\mathbf{NL} - \mathbf{NAL}, s\mathbf{N}\hat{\mathbf{L}} - \mathbf{NA}\hat{\mathbf{L}}, s\mathbf{NJ} - \mathbf{NAJ}] \tag{5.2}$$

has an $(n-\ell) \times (n-\ell)$ subpencil, which is regular and thus, the pencil $(s\mathbf{N} - \mathbf{NA})[\mathbf{L}, \hat{\mathbf{L}}, \mathbf{J}]$ has no r.m.i. Given that $(s\mathbf{N} - \mathbf{NA})\mathbf{L}$ is regular and has no i.e.d., we have that $\mathbf{NL}$ has full rank and thus also $\mathbf{N}[\mathbf{L}, \hat{\mathbf{L}}, \mathbf{J}]$; the latter shows that $(s\mathbf{N} - \mathbf{NA})[\mathbf{L}, \hat{\mathbf{L}}, \mathbf{J}]$ has also no i.e.d. The space $\mathcal{L}' = \mathcal{L} + \hat{\mathcal{L}} + \mathcal{J}$ is thus a solution to the partial cover problem. □

The specific solution defined by the space $\mathcal{L}$ for which the pencil $s\mathbf{NL} - \mathbf{NAL}$ is regular and has no i.e.d. will be referred to as a "squaring" solution and conditions for it existence will be examined next.

Remark 5.1. The family $\mathcal{L}' = \mathcal{L} + \hat{\mathcal{L}} + \mathcal{J}$, where $\mathcal{L}$ is a squaring solution does not necessarily cover the hole set of solutions of the partial cover problem; even for the squaring partial cover problem, different $\mathcal{L}$ squaring solutions, in general lead to different families. The squaring partial cover problem mentioned above may be formally stated as follows: Given the pencil $s\mathbf{N} - \mathbf{NA}$, find $\mathbf{L}$ such that

$$\det(s\mathbf{N} - \mathbf{NA})\mathbf{L} \neq 0, \qquad \det(\mathbf{NL}) \neq 0 \tag{5.3}$$

The above conditions combined yield that the squaring problem is solvable if and only if $\mathbf{L}$ is such that

$$\deg\det\{(s\mathbf{N} - \mathbf{NA})\mathbf{L}\} = n - \ell \tag{5.4}$$

or equivalently

$$\det(\mathbf{NL}) \neq 0 \tag{5.5}$$

Lemma 5.1. The matrix **NL** has full rank if and only if

$$\mathcal{L} \cap \mathcal{B} = \{0\} \tag{5.6}$$

Proposition 5.3. Necessary condition for (5.5) to be true is that

$$\dim\{\mathcal{L}\} \leq n - \ell \tag{5.7}$$

Theorem 5.1. The squaring partial cover problem is always solvable.

Proof. We can always find **L** such that (5.6) is satisfied. □

The solution of the squaring cover problem is considered next. Condition (5.6) is equivalent to

$$\det[\mathbf{B}, \mathbf{L}] \neq 0 \tag{5.8}$$

where **L** is the basis-matrix of $\mathcal{L}$. The above is equivalent to

$$\det\{\mathbf{Q}[\mathbf{B}, \mathbf{L}]\} \neq 0 \tag{5.9}$$

where **Q** is any invertible matrix. Since $\text{rank}(\mathbf{B}) = \ell$ we can always choose **Q** such that

$$\mathbf{QB} = \begin{bmatrix} \mathbf{B}_1^* \\ 0 \end{bmatrix} = \mathbf{B}^* \tag{5.10}$$

where $\mathbf{B}^*$ is an $\ell \times \ell$ invertible matrix. Then (5.8) is equivalent to

$$\det\left(\begin{bmatrix} \mathbf{B}_1^* & \mathbf{L}_1^* \\ \mathbf{0} & \mathbf{L}_2^* \end{bmatrix}\right) \neq 0 \tag{5.11}$$

where

$$\begin{bmatrix} \mathbf{L}_1^* \\ \mathbf{L}_2^* \end{bmatrix} = \mathbf{QL} = \mathbf{L}^* \tag{5.12}$$

Relation (5.11) is equivalent to

$$\det(\mathbf{B}_1^*) \cdot \det(\mathbf{L}_2^*) \tag{5.13}$$

$$\det(\mathbf{L}_2^*) \neq 0 \tag{5.14}$$

since $\mathbf{B}^*$ is invertible. Note that $\mathbf{L}^*$ is an arbitrary $\ell \times (n-\ell)$, $\mathbf{L}_2^*$ is an $(n-\ell) \times (n-\ell)$ matrix. Let now, **W** be the basis matrix of $\mathcal{W}$ and $w = \dim(\mathcal{W})$. Then, since $\mathcal{L} \subset W$

$$\text{rank}[\mathbf{W}, \mathbf{L}] = \text{rank}[\mathbf{W}] \tag{5.15}$$

$$\text{rank}[\mathbf{QW}, \mathbf{QL}] = \text{rank}[\mathbf{QW}] \tag{5.16}$$

$$\text{rank}[\mathbf{W}^*, \mathbf{L}^*] = \text{rank}[\mathbf{W}^*] \tag{5.17}$$

where

$$\mathbf{W}^* = \mathbf{QW}, \ \mathbf{L}^* = \mathbf{QL} \tag{5.18}$$

From (5.18)

$$\text{rank}\left(\begin{bmatrix} \mathbf{W}_1^* & \mathbf{L}_1^* \\ \mathbf{W}_2^* & \mathbf{L}_2^* \end{bmatrix}) = \text{rank}(\mathbf{W}^*\right) \tag{5.19}$$

The above is equivalent to the existence of a matrix $\mathbf{K}$ of dimensions $w \times (n - \ell)$ such that

$$\mathbf{W}^*\mathbf{K} = \mathbf{L}^* \tag{5.20}$$

or

$$\begin{bmatrix} \mathbf{W}_1^* \\ \mathbf{W}_2^* \end{bmatrix} \mathbf{K} = \begin{bmatrix} \mathbf{L}_1^* \\ \mathbf{L}_2^* \end{bmatrix} \tag{5.21}$$

or

$$\mathbf{W}_1^*\mathbf{K} = \mathbf{L}_1^*, \ \mathbf{W}_2^*\mathbf{K} = \mathbf{L}_2^* \tag{5.22}$$

where $\mathbf{L}_2^*$ must be invertible.

Proposition 5.4. Necessary and sufficient condition for the invertibility of $\mathbf{L}_2^*$ is that

$$\mathrm{rank}(\mathbf{W}_2^*) = n - \ell \tag{5.23}$$

P r o o f. The necessity is obvious. For the sufficiency, if we assume that (5.23) holds true, we can choose

$$\mathbf{K} = (\mathbf{W}_2^*)^t \tag{5.24}$$

and the result follows. □

The matrices $\mathbf{K}$ that satisfy the requirement of the invertibility of $\mathbf{L}_2^*$ can be found as follows. From (5.22) we have that $\mathbf{K}$ must be such that the intersection of its column space with the null space of $\mathbf{W}_2^*$ must be the zero space or, in matrix form

$$\det[\hat{\mathbf{W}}, \mathbf{K}] \neq 0 \tag{5.25}$$

where $\hat{\mathbf{W}}$ is the basis matrix of the null space of $\mathbf{W}_2^*$ and has dimensions $w \times (w-n+\ell)$. From (5.23) we have that $\mathrm{rank}(\hat{\mathbf{W}}) = w - n + \ell$. Then there exists a nonsingular matrix $\mathbf{P}$ such that

$$\mathbf{P}\hat{\mathbf{W}} = \begin{bmatrix} \hat{\mathbf{W}}_1^* \\ \mathbf{0} \end{bmatrix} = \hat{\mathbf{W}}^* \tag{5.26}$$

where $\hat{\mathbf{W}}^*$ is an $(w-n+\ell) \times (w-n+\ell)$ invertible matrix. Now, (5.25) is equivalent to

$$\det\left(\begin{bmatrix} \hat{\mathbf{W}}_1^* & \mathbf{K}_1^* \\ \mathbf{0} & \mathbf{K}_2^* \end{bmatrix}\right) \neq 0 \tag{5.27}$$

or

$$\det(\hat{\mathbf{W}}_1^*) \det(\mathbf{K}_2^*) \neq 0 \tag{5.28}$$

where

$$\begin{bmatrix} \mathbf{K}_1^* \\ \mathbf{K}_2^* \end{bmatrix} = \mathbf{P}\mathbf{K} \tag{5.29}$$

Provided that (5.23) holds true, we can always find $\mathbf{K}$ such that $\mathbf{L}_2^*$ is invertible, by choosing $\mathbf{K}_2^*$ to be invertible. The expression for the matrix $\mathbf{L}$ that satisfies (5.5) and (5.15) simultaneously is

$$\mathbf{L} = \mathbf{W}\mathbf{P}^{-1} \begin{bmatrix} \mathbf{K}_1^* \\ \mathbf{K}_2^* \end{bmatrix} \tag{5.30}$$

Next we are going to investigate (5.23) further and obtain an equivalent condition in terms of the matrices $\mathbf{B}$ and $\mathbf{W}$. Consider the matrix

$$[\mathbf{B}, \mathbf{W}] \tag{5.31}$$

Then

$$\mathbf{Q}[\mathbf{B}, \mathbf{W}] = \begin{bmatrix} \mathbf{B}_1^* & \mathbf{W}_1^* \\ 0 & \mathbf{W}_2^* \end{bmatrix} \in \mathbf{R}^{n\times(\ell+w)} \tag{5.32}$$

and $\mathbf{B}_1^*$ is invertible. Obviously, $\operatorname{rank}[\mathbf{B}_1^*, \mathbf{W}_1^*] = l$ and all the nonzero rows of $\mathbf{W}_2^*$ are linearly independent of the rows of $[\mathbf{B}_1^*, \mathbf{W}_1^*]$. Thus,

$$\operatorname{rank}\begin{bmatrix} \mathbf{B}_1^* & \mathbf{W}_1^* \\ 0 & \mathbf{W}_2^* \end{bmatrix} = \operatorname{rank}[\mathbf{B}_1^*, \mathbf{W}_1^*] + \operatorname{rank}[\mathbf{0}, \mathbf{W}_2^*] \tag{5.33}$$

and since $\mathbf{B}_1^*$ is invertible

$$\operatorname{rank}\begin{bmatrix} \mathbf{B}_1^* & \mathbf{W}_1^* \\ 0 & \mathbf{W}_2^* \end{bmatrix} = \operatorname{rank}[\mathbf{B}_1^*] + \operatorname{rank}[\mathbf{W}_2^*] \tag{5.34}$$

We may now state the following theorem

Theorem 5.2. Necessary and sufficient condition for the solvability of the squaring cover problem is the following

$$\dim\{\mathcal{B}\} \cap \{\mathcal{W}\} = \ell + w - n \tag{5.35}$$

and the general solution is (5.30) where $\mathbf{K}_1^*$ is completely arbitrary and $\mathbf{K}_2^*$ is an arbitrary nonsingular matrix.

Proof. From (5.34) we get that (5.23) holds true if and only if $\operatorname{rank}\{[\mathbf{B}, \mathbf{W}]\} = n$ or equivalently if and only if (5.35) holds true. □

Theorem 5.3. The left regular cover problem is solvable if and only if the subspace $\mathcal{W}$ is an $(\mathbf{A}, \mathbf{B})$-invariant subspace and the $\mathcal{W}$-restricted pencil is not characterised by z.r.m.i. If the problem is solvable, then the solutions have the following form

$$\mathcal{T} = \mathcal{L} + \mathcal{J} + \hat{\mathcal{L}} \tag{5.36}$$

where $\mathcal{L}$ has a basis matrix given in (5.9) and $\hat{\mathcal{L}}$ is an arbitrary subspace of $\mathcal{W}$.

Proof. Let the left regular cover problem to be solvable. Then from Proposition 5.2 we have that the squaring problem is solvable. Let $\mathcal{L}$ be a solution of the squaring problem. Then there exists a subspace $\hat{\mathcal{L}} \subseteq \mathcal{W}$ such that $\mathcal{W} = \mathcal{L} \oplus \hat{\mathcal{L}}$. Since the $\mathcal{L}$-restricted pencil is characterised by i.e.d. and r.m.i., it follows that the $\mathcal{W}$-restricted pencil does not have i.e.d. and r.m.i. and therefore $\mathcal{W}$ is an $(\mathbf{A}, \mathbf{B})$-invariant subspace not characterised by r.m.i.

Conversely let $\mathcal{W}$ be a subspace such that the $\mathcal{W}$-restricted pencil has neither i.e.d. nor r.m.i. Then $\mathcal{W}$ is a solution to the problem and the result follows. □

6. CONCLUSIONS

The results in this paper were mostly concerned with the partial realisation of the standard cover problem. Extension of the results to the more general cases, where the subspace $\mathcal{V}$ is almost $(\mathbf{A},\mathbf{B})$-invariant, controllability, almost controllability is quite natural, using the present formulation of the problem and their treatment is given in a forthcoming report. The present paper considers the case of proper (regular) systems, which are also assumed to be controllable. The extension of the results to the singular systems case is still under investigation. The matrix pencil framework provides the appropriate tools for the study of the cover problems; the present results are of a preliminary nature and current work is also directed towards conditions which take into account the specific algebraic characteristics of the space to be covered.

(Received March 23, 1993.)

REFERENCES

[1] A. C. Antoulas: New results on the algebraic theory of linear systems: the solution of cover problems. Linear Algebra Appl. *50* (1983), 1–43.

[2] E. Emre, L. M. Silverman and K. Glover: Generalised dynamic covers for linear systems with applications to deterministic identification problems. IEEE Trans. Automat. Control *AC-22* (1977), 26–35.

[3] S. Jaffe and N. Karcanias: Matrix pencil characterisation of almost (A,B)-invariant subspaces: a classification of geometric concepts. Internat. J. Control *33* (1981), 51–93.

[4] N. Karcanias: Matrix pencil approach to geometric system theory. Proc. IEE *126* (1990), 585–590.

[5] N. Karcanias: The global role of instrumentation in systems design and control. The concise Encyclopedia of Measurement and Instrumentation, Pergamon Press, to appear.

[6] N. Karcanias: Proper invariant realisations of singular system problems. IEEE Trans. Automat. Control *AC-35* (1990), 230–233.

[7] N. Karcanias and B. Kouvaritakis: The output zeroing problem and its relationship to the invariant zero structure: a matrix pencil approach. Internat. J. Control *30* (1979), 395–415.

[8] N. Karcanias and C. Giannacopoulos: Necessary and sufficient conditions for zero assignment by constant squaring down. Linear Algebra Appl., Special issue on control theory *122/123/124* (1989), 415–446.

[9] N. Karcanias and G. Kalogeropoulos: Geometric theory and feedback invariants of generalized linear systems: a matrix pencil approach. Circuits Systems Signal Process. *8* (1989), 3, 375–397.

[10] A. S. Morse: Minimal solutions to transfer matrix equations. IEEE Trans. Automat. Control *AC-18* (1973), 346–354.

[11] R. C. Thompson: Interlacing inequalities for invariant factors. Linear Algebra Appl. *24* (1979), 1–31.

[12] J. C. Willems: Almost invariant subspaces: an approach to high gain feedback design – Part I, almost controlled invariant subspaces. IEEE Trans. Automat. Control *AC-26* (1981), 235–252.

[13] W. M. Wonham: Linear Multivariable Control: A Geometric Approach. Springer-Verlag, New York 1979.

[14] W. M. Wonham and A. S. Morse: Feedback invariants for linear multivariable systems. Automatica *8* (1972), 93–100.
[15] F. R. Gantmacher: The Theory of Matrices. Volume I, II. Chelsea, New York 1959.

Prof. Dr. Nicos Karcanias and Dr. Dimitris Vafiadis, Control Engineering Research Centre, City University, Northampton Square, London EC1V 0HB. United Kingdom.