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STRUCTURAL PROPERTIES OF SINGULAR SYSTEMS[1]

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In this paper, we consider the time-invariant singular system

$$\Sigma : Ex' = Ax + Bu;\ y = Cx$$

and carry out a detailed analysis of its structural properties. Duality relations among different structural properties are also established.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we consider the linear, time-invariant system representation

$$\Sigma : Ex' = Ax + Bu;\ y = Cx$$

where $x \in \mathcal{X}(= \Re^n)$, $Ex \in \underline{\mathcal{X}}(= \Re^{\underline{n}})$, $u \in \mathcal{U}(= \Re^m)$ and $y \in \mathcal{Y}(= \Re^p)$. We first recall some of the results of [1]. Let D denote the set of C^∞ functions f from $\Re$ to $\Re$ with bounded support, and let $\mathcal{D}$ denote the space of distributions on D. Let $\mathcal{D}_p$ denote the subspace of $\mathcal{D}$ consisting of piece-wise continuous distributions. That is, $x \in \mathcal{D}$ is in $\mathcal{D}_p$ iff there exist points $\dots, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2, \dots$ in $\Re$ (finitely many in any bounded interval) and a piece-wise continuous function g such that $x = g$ on (τ_{i-1}, τ_i) for every i. Foreseeing the results to follow, we remark that, roughly speaking, $x \in \mathcal{D}_p$ on any finite interval is like a piece-wise continuous function except that at finitely many points in the interval, it is given by a linear combination of the Dirac delta and its derivatives. Let $\mathcal{D}_{[\tau,\infty)}, \mathcal{D}_{(-\infty,\tau]}, \mathcal{D}_{[\tau]}, \mathcal{D}_{[\tau_1,\tau_2]}$ denote the spaces of distributions with support in $[\tau, \infty), (-\infty, \tau]$ at τ and in $[\tau_1, \tau_2]$ respectively. Range spaces of these distributions will be determined by the context.

Now, let $x \in \mathcal{D}_p$ be given. Then, there exists an $\varepsilon > 0$ and a piece-wise continuous function g such that $x = g$ on $(\tau - \varepsilon, \tau)$. Restriction of x to $[\tau, \infty)$, denoted by $x_{[\tau,\infty)}$, is defined as follows:

$$\langle x_{[\tau,\infty)}, \phi \rangle = \begin{cases} 0 & \text{if supp } \phi \subset (-\infty, \tau] \\ \langle x, \phi \rangle - \int_{(\tau-\varepsilon)}^{\tau} g(t)\phi(t)\, dt & \text{if supp } \phi \subset [\tau - \varepsilon, \infty) \end{cases}$$

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This definition determines $x_{[\tau,\infty)}$ uniquely [1]. Similarly, there is a unique restriction of x to $(-\infty,\tau]$, which is defined as follows.

$$\langle x_{(-\infty,\tau]},\phi\rangle = \begin{cases} 0 & \text{if supp } \phi \subset [\tau,\infty) \\ \langle x,\phi\rangle - \int_\tau^{(\tau+\varepsilon)} g(t)\phi(t)\,\mathrm{d}t & \text{if supp } \phi \subset (-\infty,\tau+\varepsilon] \end{cases}$$

Again, $x_{(-\infty,\tau]}$ is uniquely specified by x and τ. Now, given τ_1 and τ_2, there exists a unique restriction of x to $[\tau_1,\tau_2]$ defined by

$$x_{[\tau_1,\tau_2]} = x_{(-\infty,\tau_2]} + x_{[\tau_1,\infty)} - x.$$

In case $\tau_1 = \tau_2$, we shall simply write $x_{[\tau]}$ instead of $x_{[\tau,\tau]}$.

Left and right-hand limits of a piece-wise continuous distribution x are defined as $x(\tau_-)=\lim_{t\uparrow\tau} g(t)$, and $x(\tau_+)= \lim_{t\downarrow\tau} g(t)$ where g is that piece-wise continuous function which agrees with x on $(\tau-\varepsilon,\tau)$ and on $(\tau,\tau+\varepsilon)$ for some $\varepsilon > 0$. In this case, $\Delta_\tau x = x(\tau_+) - x(\tau_-)$ is the jump in x at τ.

Proposition 1.1. For any piece-wise continuous distribution x, we have:

(i) $(x_{[\tau,\infty)})' = (x')_{[\tau,\infty)} + \delta_\tau\, x(\tau_-)$ (δ_τ = the Dirac delta at τ),

(ii) $(x_{(-\infty,\tau]})' = (x')_{(-\infty,\tau]} - \delta_\tau x(\tau_+)$,

(iii) $(x_{[\tau_1,\tau_2]})' = (x')_{[\tau_1,\tau_2]} + \delta_{\tau_1} x(\tau_1^-) - \delta_{\tau_2} x(\tau_2^+)$.

We now define a subspace $\mathcal{B}$ of $\mathcal{D}_p$ as follows. $x \in \mathcal{B}$ iff there exist points $\ldots\tau_{-2},\tau_{-1},\tau_0,\tau_1,\tau_2,\ldots$ (finitely many in any bounded interval) and matrices A_i, B_i, C_i so that on every (τ_{i-1},τ_i), x agrees with the smooth function $g_i(t) = C_i e^{A_i(t-\tau_{i-1})}B_i$. To have a feeling for the way $\mathcal{B}$ is defined, let us recall that a distribution with support on $[0,\infty)$ is said to be of Bohl type if its Laplace transform is a real-rational function of s. That is to say, if $x = \sum_{i=1}^q x_i\delta^{(i)} + Ce^{At}B$ for some A, B, C and q. It is straightforward to show that if x is a Bohl type distribution then its restriction to 0 is given by $x_{[0]} = \sum_{i=1}^q x_i\delta^{(i)}$. Thus, a Bohl-type distribution can be written as $x_{[0]} + x_{(0,\infty)}$ where the latter term is a regular distribution generated by the smooth function $Ce^{At}B$. Now, define the shift operator σ_τ by $\sigma_\tau x=\sum_{i=1}^q x_i\delta_\tau^{(i)}+Ce^{A(t-\tau)}B$.

Given two Bohl-type distributions x, y and a $\tau > 0$, we define the concatenation of x and $\sigma_\tau y$ by $x \circ \sigma_\tau y = x_{[0]} + x_{(0,t)} + (\sigma_\tau y)_{[\tau]} + (\sigma_\tau y)_{(\tau,\infty)}$. If $\tau < 0$, then the definition is modified in the obvious way. At this point, it should be clear that the space $\mathcal{B}$ defined above is the set of all distributions x with the property that on any bounded interval $[T_1,T_2]$ there exist points $\tau_0 = T_1, \tau_1, \ldots, \tau_q = T_2$ and Bohl-type distributions $x_0,\ldots,x_q$ so that the restriction of x to $[T_1,T_2]$ is equal to the restriction to $[T_1,T_2]$ of $\sigma_{\tau_0}x_0 \odot \sigma_{\tau_1}x_1 \ldots \odot \sigma_{\tau_q}x_q$. Thus, roughly speaking, $\mathcal{B}$ is the closure of the set of Bohl-type distributions under the operations of (left and right) translations and of taking finitely many concatenations. In the sequel we shall also make use of restrictions of distributions from $\mathcal{B}$ to $[0,\infty)$ and to $[0,T]$. The classes of these distributions will be denoted by $\mathcal{B}_{[0,\infty)}$ and by $\mathcal{B}_{[0,T]}$.

1.1. Input-trajectory pairs for Σ

In the sequel, the set of admissible inputs for the generalized state-space system Σ will be taken to be $\mathcal{B}$. Also, even when not explicitly mentioned, we shall consider only those solutions of Σ which reside in $\mathcal{B}$. Thus, by an admissible input-trajectory pair (u, x) we shall mean two distributions from $\mathcal{B}$ which satisfy Σ. If Σ is regular, i.e. $\det(sE - A) \not\equiv 0$, then any admissible input generates a unique trajectory from $\mathcal{B}$. However, in case the system is not regular, an input from $\mathcal{B}$ may fail to generate a trajectory or it may generate nonunique trajectories.

1.2. Input-trajectory pairs for $\Sigma_{[0,\infty)}$

Our main interest in Σ is for $t \geq 0$. To talk about it more precisely, let us assume that (u, x) is an admissible input-trajectory pair for Σ. Using part (i) of Proposition 1.1 with the restrictions of x and u to $[0, \infty)$, we get:

$$\Sigma_{[0,\infty)} : E(x_{[0,\infty)})' = Ax_{[0,\infty)} + Bu_{[0,\infty)} + \delta\, Ex(0_-)$$

Thus, any admissible pair (u, x) for Σ generates an input-trajectory pair for $\Sigma_{[0,\infty)}$ compatible with the initial condition $x(0_-)$. In this case, it can be shown that $x(0_-)$ cannot be totally arbitrary, and has to reside in $\mathcal{V}^*$, the supremal (A, E, B)-invariant subspace of $\mathcal{X}$ which is defined as the limit of the subspace recursion

$$\mathcal{V}_{k+1} = A^{-1}(E\mathcal{V}_k + \mathrm{Im}B); \quad \mathcal{V}_0 = \mathcal{X}$$

Indeed, we have:

Proposition 1.2. Let $(u, x) \in \mathcal{B} \times \mathcal{B}$ be an admissible input-trajectory pair for Σ.

(i) $u_{[0,\infty)}$ and $x_{[0,\infty)}$ satisfy $\Sigma_{[0,\infty)}$. In this case we have $x(0_-) \in \mathcal{V}^*$.

(ii) $u_{(-\infty,\tau]}$ and $x_{(-\infty,\tau]}$ satisfy

$$\Sigma_{(-\infty,\tau]} : Ex'_{(-\infty,\tau]} = Ax_{(-\infty,\tau]} + Bu_{(-\infty,\tau]} - Ex(\tau_+)$$

In this case, we have $x(\tau_+) \in \mathcal{V}^*$.

Proof. Let u and x be admissible for Σ. Then, taking (distributional) derivatives, we get $Ex'' = Ax' + Bu'$. Taking restrictions of x'', x' and u' to $[0, \infty)$ and using Proposition 1.1 we conclude that

$$Ex''_{[0,\infty)} - Ax'_{[0,\infty)} - Bu'_{[0,\infty)} - \delta'_0 Ex(0_-) = \delta_0\{Ex'(0_-) - Ax(0_-) - Bu(0_-)\}$$

Noting that the LHS is the derivative of $Ex'_{[0,\infty)} - Ax_{[0,\infty)} - Bu_{[0,\infty)} - \delta_0 Ex(0_-)$ (which equals zero), we conclude that

$$Ex'(0_-) = Ax(0_-) + Bu(0_-)$$

Repeating this procedure, we conclude that

$$Ex^{(j)}(0_-) = Ax^{(j-1)}(0_-) + Bu^{(j-1)}(0_-); \quad j = 1, 2, \ldots$$

Now, $Ex^{(n)}(0_-) = Ax^{(n-1)}(0_-)+Bu^{(n-1)}(0_-)$ implies that $x^{(n-1)}(0_-) \in A^{-1}\{\mathrm{Im}E+\mathrm{Im}B\} = \mathcal{V}_1$. Then, $x^{(n-2)}(0_-) \in A^{-1}\{E\mathcal{V}_1 + \mathrm{Im}B\} = \mathcal{V}_2$. Proceeding, we get $x(0_-) \in \mathcal{V}_n = \mathcal{V}^*$. Proof of (ii) is follows in exactly the same manner and will be left to the reader. □

On the other hand, it is possible to think of a situation where the structure of the system changes at $t = 0$ due to some desired or undesired switching. In this case, such an assumption about the initial condition cannot be made. In what follows,we shall let $x(0_-)$ be free. Thus, we shall also be interested in solutions of $\Sigma_{[0,\infty)}$ which are not necessarily the restrictions of solutions of Σ. However, even in this case, only distributions from $\mathcal{B}_{[0,\infty)}$ will be admitted as control inputs and among all solutions generated by such inputs, we shall be interested in only those which reside in $\mathcal{B}_{[0,\infty)}$.

It follows immediately from the definition that there exists an admissible input-trajectory pair of $\Sigma_{[0,\infty)}$ corresponding to a given initial condition iff there exists a pair of Bohl-type distributions $u_{[0,\infty)}$ and $x_{[0,\infty)}$ satisfying $\Sigma_{[0,\infty)}$ with the same initial condition. Therefore, we shall be interested first in Bohl-type input-trajectory pairs of $\Sigma_{[0,\infty)}$. Given $x(0_-)$, let two Bohl-type distributions $x_{[0,\infty)}$ and $u_{[0,\infty)}$ satisfy $\Sigma_{[0,\infty)}$. Let $X(s)$ and $U(s)$ denote their Laplace transforms. Laurent expansions of $X(s)$ and $U(s)$ around infinity yield

$$\begin{aligned} X(s) &= x_{-q}s^q + \cdots + x_{-1}s + x_0 + x_1 s^{-1} + x_2 s^{-2} + \cdots \\ U(s) &= u_{-q-1}s^{q+1} + \cdots + u_{-1}s + u_0 + u_1 s^{-1} + u_2 s^{-2} + \cdots \end{aligned}$$

Note that the inverse Laplace transform of the polynomial part of $X(s)$ is exactly the restriction $x_{[0]}$ of $x_{[0,\infty)}$ to 0 and the inverse Laplace transform of the strictly proper part of $X(s)$ is $x_{(0,\infty)}$. Substituting the expansions above into the Laplace transformed system equation

$$(sE - A)X(s) - BU(s) = Ex(0_-)$$

we get two sets of equations in terms of the polynomial and strictly proper parts of $X(s)$. The first one reads

$$(sE - A)X_{\mathrm{sp}}(s) - BU_{\mathrm{sp}}(s) = Ex(0_+)$$

(where $x(0_+) = x_1$) and the second one reads

$$(sE - A)X_{\mathrm{poly}}(s) - BU_{\mathrm{poly}}(s) = \Delta_0 Ex.$$

Thus, we conclude that if $x_{[0,\infty)} = x_{[0]} + x_{(0,\infty)}$ and $u_{[0,\infty)} = u_{[0]} + u_{(0,\infty)}$ satisfy $\Sigma_{[0,\infty)}$ with $x(0_-)$ then $x_{[0]}$ and $x_{(0,\infty)}$ satisfy the following equations:

$$\begin{aligned} &\Sigma_{[0]} : E(x_{[0]})' = Ax_{[0]} + Bu_{[0]} - \Delta_0 Ex \\ &\Sigma_{(0,\infty)} : E(x_{(0,\infty)})' = Ax_{(0,\infty)} + Bu_{(0,\infty)} + \delta Ex(0_+) \end{aligned}$$

To investigate some of the properties of Bohl type input-trajectory pairs of $\Sigma_{[0,\infty)}$, let us first denote the limit (which is reached in at most n steps) of the subspace recursion

$$\mathcal{R}_{a,k+1} = E^{-1}(A\mathcal{R}_{a,k} + \mathrm{Im}B); \quad \mathcal{R}_{a,0} = 0$$

by $\mathcal{R}_a^*$. The recursion above is known as the almost controllability subspace algorithm. This terminology will be justified in the sections to follow. Also, let us define

$$\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{R}_a^*.$$

Then, we have the following result:

Proposition 1.3. Let $x(0_-)$ be given.

(i) There exists a pair $(u_{(0,\infty)}, x_{(0,\infty)})$ of smooth function-distributions satisfying $\Sigma_{(0,\infty)}$ iff $x(0_+) \in \mathcal{V}^*$.

(ii) There exists a pair $(u_{[0]}, x_{[0]})$ of distributions with point support satisfying $\Sigma_{[0]}$ iff $\Delta_0 x \in \mathcal{R}_a^*$.

(iii) There exists a pair $(u_{[0,\infty)}, x_{[0,\infty)})$ of Bohl type distributions satisfying $\Sigma_{[0,\infty)}$ with $x(0_-)$ iff $x(0_-) \in (\mathcal{V}^* + \mathcal{R}_a^*)$.

Proof. Parts (ii) and (iii) have already been proved in [2]. We nevertheless include a sketch of the proof here as its main idea will be used frequently in the sequel. Let $(u_{[0,\infty)}, x_{[0,\infty)})$ be a Bohl-type input-trajectory pair for $\Sigma_{[0,\infty)}$ consistent with the initial condition $x(0_-)$. Let $X(s)$ and $U(s)$ denote their Laplace transforms. Then, equating the coefficients of the like terms in $(sE - A) - BU(s) = Ex(0_-)$, we get:

$$\begin{array}{ll} \Delta_0 x = Ax_0 + Bu_0 & Ex_2 = Ax_1 + Bu_1 \\ Ex_0 = Ax_{-1} + Bu_{-1} & Ex_3 = Ax_2 + Bu_2 \\ \vdots & \vdots \\ Ex_{-q} = Bu_{-q-1} & \vdots \end{array}$$

In the above, $\Delta_0 x = x_1 - x(0_-)$. Now, the last equation in the first column above implies that $x_{-q} \in E^{-1}\text{Im}B = \mathcal{R}_{a,1}$. Then, an easy induction arguement shows that $x_{-q+j} \in \mathcal{R}_{a,j+1}$. Thus, $x_0 \in \mathcal{R}_{a,q+1}$ and $\Delta_0 x \in \mathcal{R}_{q+2} \subset \mathcal{R}_a^*$. On the other hand, if $\Delta_0 x \in \mathcal{R}_a^*$ then there exists an integer l such that $\Delta_0 x \in \mathcal{R}_l$. This implies the existence of an x_0 and a u_0 satisfying $\Delta_0 Ex = Ax_0 + Bu_0$ where $x_0 \in \mathcal{R}_{\ell-1}$. That $x_0 \in \mathcal{R}_{\ell-1}$ implies that there exist $x_{-1} \in \mathcal{R}_{\ell-2}$ and a u_{-1} such that $Ex_0 = Ax_{-1} + Bu_{-1}$. Proceeding in this fashion, we conclude that there exists a finite sequence $\{x_0, x_{-1}, \ldots, x_{-q}\}$ where $x_{-j} \in \mathcal{R}_{\ell-j-1}$ and x_{-q} satisfies $Ex_{-q} = Bu_{-q-1}$ $(q = \ell - 2)$. Note that this completes the proof of (ii).

To prove (i), let $l = n + 1$. Then, $Ex_\ell = Ax_{\ell-1} + Bu_l$ implies that $x_{\ell-1} \in A^{-1}\{E\mathcal{X} + \text{Im}B\} = \mathcal{V}_1$. It then follows by induction that $x_{\ell-j} \in \mathcal{V}_j, j = 1, \ldots, \ell - 1$. Thus, $x_1 \in \mathcal{V}_{\ell-1} = \mathcal{V}_n$. As $\mathcal{V}_n = \mathcal{V}^*$, we have $x_1 \in \mathcal{V}^*$. Conversely, if $x(0_+) = x_1 \in \mathcal{V}^*$ then there exist x_2 and u_1 with $x_2 \in \mathcal{V}^*$ such that $Ex_2 = Ax_1 + Bu_2$. Proceeding, one defines an infinite sequence $\{x_1, x_2, \ldots\}$ satisfying the equations given in the second column above , and this completes the proof of (i). Finally, we note that (iii) follows immediately from (i) and (ii). □

We now present two theorems from [4] which will be crucial to the developement of the section to follow.

Theorem 1.1. Let $(u_{[0,\infty)}, x_{[0,\infty)})$ be a pair of Bohl type distributions admissible for $\Sigma_{[0,\infty)}$ corresponding to some initial condition $x(0_-)$.

(i) For all $t > 0$, $x_{(0,\infty)}(t) \in \mathcal{V}^*$.

(ii) If $x(0_+) \in \mathcal{R}^*$ then for all $t > 0$, $x_{(0,\infty)}(t) \in \mathcal{R}^*$.

(iii) If $x_{(0,\infty)}(T) = 0$ for some $T > 0$ then $x_{(0,\infty)}(t) \in \mathcal{R}^*$ for all $t > 0$.

Remarks. (i) Although it does not make sense to talk about the "value" of a distribution at a point, on any interval $(0, T)$, a Bohl-type distribution can be identified by the smooth function which agrees with it. Similarly, on any interval (τ_1, τ_2) which contains no jump points, a generalized Bohl type distribution can and will be identified by the smooth function which generates it. So the reader should read the statement above and similar ones to be presented below as statements made about the smooth function $x_{(0,\infty)}$.

(ii) A distribution x is said to lie in a subspace $\mathcal{S}$ iff $\langle x, \phi\rangle \in \mathcal{S}$ for any test function ϕ. It can be shown that the distributional equivalents of the statements given in the theorem are: (i)- $x_{[0,\infty)} \in \mathcal{V}^* + \mathcal{R}_a^*$ and for any $\tau > 0$, the restriction of $x_{[0,\infty)}$ to $[\tau, \infty)$ lies in $\mathcal{V}^*$, (ii) If $x(0_+) \in \mathcal{R}^*$ then $x_{[0,\infty)}$ lies in $\mathcal{R}_a^*$ and for any $\tau > 0$ its restriction to $[\tau, \infty)$ lies in $\mathcal{R}^*$, (iii)- If $x(\tau^+) = 0$ for some τ then $x_{[0,\infty)}$ lies in $\mathcal{R}_a^*$ and for any $\tau > 0$ its restriction to $[\tau, \infty)$ lies in $\mathcal{R}^*$.

Proof. To prove (i), we note that if $X(s) = x_1 s^{-1} + x_2 s^{-2} + \cdots$ is the Laplace transform of $x_{(0,\infty)}$ then $x_k \in \mathcal{V}^*, \forall k \in Z^+$. Since $x_{(0,\infty)}(t) = \sum_{k=1}^{\infty} x_k \frac{t^{k-1}}{(k-1)!}$ is obtained via term-by-term inverse transformation of a Laurent series which converges for all $s \in C$ satisfying $|s| > |s_1|$ (for some s_1), it is absolutely convergent for all $t > 0$ (see Section 30 of [5]). This and the fact that $\mathcal{V}^*$ is closed (thanks to finite dimensionality) guarantee that $x_{(0,\infty)}(t)$ lies in $\mathcal{V}^*$ for all $t > 0$.

To prove (ii), we recall that $\Delta_0 x \in \mathcal{R}_a^*$. Then, $\Delta_0 x = x(0_+) - x(0_-)$ and the assumption $x(0_-) = 0$ imply that $x(0_+) \in \mathcal{R}_a^*$. However, $x(0_+)$ is also equal to $\lim_{t\to 0} \sum_{k=1}^{\infty} x_k \frac{t^{k-1}}{(k-1)!} = x_1$. Thus, $x_1 \in \mathcal{V}^* \cap \mathcal{R}_a^*$, i.e. $x_1 \in \mathcal{R}^*$. If $x_k \in \mathcal{R}^*$ is assumed then $Ex_{k+1} = Ax_k + Bu_k$ implies that $x_{k+1} \in E^{-1}\{A\mathcal{R}^* + \mathrm{Im}B\} \subset E^{-1}\{A\mathcal{R}_a^* + \mathrm{Im}B\} = \mathcal{R}_a^*$. As $x_{k+1} \in \mathcal{V}^*$ also, we conclude that $x_{k+1} \in \mathcal{V}^* \cap \mathcal{R}_a^* = \mathcal{R}^*$. Therefore, $x_k \in \mathcal{R}^*, \forall k \in Z^+$. Then, as above, absolute convergence of $x_{(0,\infty)}(t)$ for all $t > 0$ and closedness of $\mathcal{R}^*$ guarantee that $x_{(0,\infty)}(t) \in \mathcal{R}^*, \forall t > 0$.

To prove (iii), suppose $x(T) = 0$ for some given $T > 0$. Clearly, $x(T) \in \mathcal{R}_a^*$. Assume that $x^{(j)}(T) \in \mathcal{R}_a^*$ for some j. Then, smoothness of the solution grants $Ex^{(j+1)}(T) = Ax^j(T) + Bu^j(T)$ which immediately implies that $x^{(j+1)}(T) \in E^{-1}\{A\mathcal{R}_a^* + \mathrm{Im}B\} = \mathcal{R}_a^*$. Thus, $x^j(T) \in \mathcal{R}_a^*, \forall j \in Z^+$. Expanding the solution into a Taylor series around $t = T$, we write $x_{(0,\infty)} = \sum_{j=0}^{\infty} x^j(T) \frac{(t-T)^j}{j!}$. Since $x_{(0,\infty)}$ is absolutely convergent for all $t > 0$, we have $x(0_+) = \sum_{j=1}^{\infty} x^j(T) \frac{(-T)^j}{j!}$. As $x_j \in \mathcal{R}_a^*, \forall j \in Z^+$ and as $\mathcal{R}_a^*$ is closed, we conclude that $x(0_+) \in \mathcal{R}_a^*$. Then it follows that $x(0_+) \in \mathcal{V}^* \cap \mathcal{R}_a^*$. That is to say, $x(0_+) \in \mathcal{R}^*$. As before, let x_i be the coefficient of the s^{-i} term in the Laurent series of $X_{\mathrm{sp}}(s)$. We have shown that

$x_1 = x(0_+)$ is in $\mathcal{R}^*$. This can be used as the first step of an easy induction arguement which establishes the fact $x_j \in \mathcal{R}^*, \forall j \in Z^+$. Then, $x_{(0,\infty)(t)} = \sum_{j=1}^{\infty} x_j \frac{t^j}{j!}$ lies in $\mathcal{R}^*$ for all $t > 0$. □

Theorem 1.2.

(i) If $x(0_+) = 0$ then $x_{(0,\infty)}(t) \in \mathcal{R}^*$ for all $t > 0$. Conversely, if $x_f \in \mathcal{R}^*$ is an arbitrary but given vector then for any $T > 0$ there exists an admissible pair $(u_{(0,\infty)}, x_{(0,\infty)})$ for $\Sigma_{(0,\infty)}$ which is compatible with $x(0_+) = 0$ so that $x_{(0,\infty)}(T) = x_f$.

(ii) If $x_{(0,\infty)}(T) = 0$ for some $T > 0$ then $x(0_+) \in \mathcal{R}^*$. Conversely, if $x(0_+) \in \mathcal{R}^*$ is arbitrary but fixed then given any $T > 0$ there exists a Bohl-type pair $(u_{(0,\infty)}, x_{(0,\infty)})$ for $\Sigma_{(0,\infty)}$ compatible with $x(0_+)$ such that $x_{(0,\infty)}(T) = 0$.

Proof. First, note that the first statements in both parts of the theorem have been already stated and proved in Theorem 1.1. Now, assume that $T > 0$ and $x_f \in \mathcal{R}^*$ are arbitrary but given. To prove (i), we shall construct a Bohl-type input-trajectory pair $(u_{[0,\infty)}, x_{[0,\infty)})$ satisfying $\Sigma_{[0,\infty)}$ with $x(0_-) = 0$ such that $x(0_+) = 0$ and $x_{[0,\infty)}(T) = x_f$. To that end, first define $dim\mathcal{R}^*$ by r. It is established in [14] that there exists a linearly independent chain $\{\xi_1, \ldots, \xi_r\}$ in $\mathcal{R}^*$ so that

$$\begin{aligned} E\xi_1 &= B\omega_0 \\ E\xi_2 &= A\xi_1 + B\omega_1 \\ &\vdots \\ 0 &= A\xi_r + B\omega_r \end{aligned}$$

for some ω_k's. Now, define x_k's by:

$$x_k = \begin{cases} 0 & \text{if } k = 1 \\ \sum_{j=1}^{k-1} \gamma_j \xi_{k-j} & \text{if } k = 2, \ldots, r \\ \sum_{k-r}^{r} \gamma_j \xi_{k-j} & \text{if } k = r+1, \ldots, 2r \end{cases}$$

and u_k's by:

$$u_k = \begin{cases} \sum_{j=1}^{k-1} \gamma_j \omega_{k-j} & \text{if } k = 0, \ldots, r-1 \\ \sum_{j=1}^{r} \gamma_j \omega_{r+1-j} & \text{if } k = r \\ \sum_{k-r}^{r} \gamma_j \omega_{k-j} & \text{if } k = r+1, \ldots, 2r \end{cases}$$

Note that x_k's and u_k's thus defined satisfy

$$\begin{aligned} &Ex_1 = 0 \\ &Ex_{k+1} = Ax_k + Bu_k; \quad k = 1, \ldots, 2r \\ &Ax_{2r} + Bu_{2r} = 0. \end{aligned}$$

Define $x_{[0,\infty)}$ and $u_{[0,\infty)}$ by:

$$x_{[0,\infty)} = \sum_{j=2}^{2r} x_j \frac{t^{j-1}}{(j-1)!}; \quad u_{[0,\infty)} = \sum_{j=1}^{2r} u_j \frac{t^{j-1}}{(j-1)!}$$

Then,$(x_{[0,\infty)}, u_{[0,\infty)})$ is a Bohl-type input-trajectory pair for $\Sigma_{[0,\infty)}$ compatible with $x(0_-) = 0$ such that $x_{[0,\infty)}$ also satifies $x(0_+) = 0$. Given x_f, write $x_f = \alpha_1 x_1 + \ldots + \alpha_r x_r$ for some $\alpha_1, \ldots, \alpha_r$. Then it can easily be checked that $x_{[0,\infty)}(T) = x_f$ iff

$$\begin{bmatrix} T & \frac{T^2}{2!} & \cdots & \frac{T^r}{r!} \\ \frac{T^2}{2!} & \frac{T^3}{3!} & \cdots & \frac{T^{r+1}}{(r+1)!} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{T^r}{r!} & \frac{T^{r+1}}{(r+1)!} & \cdots & \frac{T^{2r-1}}{(2r-1)!} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_r \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{bmatrix}$$

As the matrix above is nonsingular for $T \neq 0$, γ_i can be solved for in terms of α_i. Then, $x_{[0,\infty)}$ defined using these γ_i will satisfy $x_{[0,\infty)}(T) = x_f$.

To prove (ii) let the chain $\{\xi_1, \ldots, \xi_r\}$ be as defined above. Write $x_1 = \alpha_1\xi_1 + \cdots + \alpha_r\xi_r$, and define x_k's by:

$$x_k = \begin{cases} \sum_{j=1}^{r-(k-1)} \alpha_j \xi_{j+(k-1)} - \sum_{j=1}^{k-1} \gamma_{k-j}\xi_j & \text{if } k = 2, \ldots, r \\ \sum_{k-r}^{r} \gamma_j \xi_{k-j} & \text{if } k = r+1, \ldots, 2r \end{cases}$$

and u_k's by:

$$u_k = \begin{cases} \sum_{j=1}^{r-(k-1)} \alpha_j \omega_{j+(k-1)} - \sum_{j=1}^{k} \gamma_j \omega_{k-j} & \text{if } k = 1, \ldots, r \\ \sum_{j=k-r}^{r} \gamma_j \omega_{r+1-j} & \text{if } k = r+1, \ldots, 2r \end{cases}$$

Note that x_k's and u_k's thus defined satisfy

$$\begin{aligned} Ex_1 &= Ex(0_+) \\ Ex_{k+1} &= Ax_k + Bu_k; \quad k = 1, \ldots, 2r \\ Ax_{2r} + Bu_{2r} &= 0 \end{aligned}$$

Define $x_{[0,\infty)}$ and $u_{[0,\infty)}$ by:

$$x_{[0,\infty)} = \sum_{j=1}^{2r-1} x_j \frac{t^{j-1}}{(j-1)!}; \quad u_{[0,\infty)} = \sum_{j=1}^{2r} u_j \frac{t^{j-1}}{(j-1)!}$$

Then,$(x_{[0,\infty)}, u_{[0,\infty)})$ is a Bohl-type input-trajectory pair for $\Sigma_{[0,\infty)}$ compatible with any $x(0_-)$ which satisfies $Ex(0_-) = Ex(0_+)$. It can be easily checked that $x_{[0,\infty)}(T) = 0$ iff

$$\begin{bmatrix} T & \frac{T^2}{2!} & \cdots & \frac{T^r}{r!} \\ \frac{T^2}{2!} & \frac{T^3}{3!} & \cdots & \frac{T^{r+1}}{(r+1)!} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{T^r}{r!} & \frac{T^{r+1}}{(r+1)!} & \cdots & \frac{T^{2r-1}}{(2r-1)!} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_r \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_1 T + \alpha_2 \\ \vdots \\ \alpha_1 T^{r-1} + \alpha_2 T^{r-2} + \cdots + \alpha_r \end{bmatrix}$$

As the matrix above is nonsingular for $T \neq 0$, γ_i's can be solved for in terms of T and α_i's. Then, $x_{[0,\infty)}$ defined using these γ_i's will satisfy $x_{[0,\infty)}(T) = 0$. This completes the proof of the theorem. □

After considering $\Sigma_{[0,\infty)}$ with Bohl-type inputs and trajectories, we can now concentrate on the same system with an arbitrary admissible input-trajectory pair $(u_{[0,\infty)}, x_{[0,\infty)})$. Recall that admissible input-trajectory pairs for $\Sigma_{[0,\infty)}$ are of the form

$$u_{[0,\infty)} = u^0 \odot \sigma_{\tau_1} u^1 \odot \sigma_{\tau_2} u^2 \odot \ldots$$
$$x_{[0,\infty)} = x^0 \odot \sigma_{\tau_1} x^1 \odot \sigma_{\tau_2} x^2 \odot \ldots$$

where each pair (u^i, x^i) is of Bohl-type and is admissible for $\Sigma_{[0,\infty)}$ with initial condition $x^{i-1}(\tau_{i-1}^-)$ ($x^0(\tau_0^-)$ is taken to be $x(0_-)$).It should be clear that understanding the behavior of $x_{[0,\infty)}$ is equivalent to understanding its behavior on an interval $[\tau_{i-1}, \tau_i]$ which contains no jump points in its interior. Assume, without loss of generality that $\tau_{i-1} = 0$ and $\tau_i = \tau$. Then, taking the restrictions of $x_{[0,\infty)}$ and $u_{[0,\infty)}$ to $[0,\tau]$ we write $x_{[0,\tau]} = x_{[0]} + x_{(0,\tau)} + x_{[\tau]}$ (where $x_{[0]} = x^0_{[0]}$, $x_{(0,\tau)} = x^0_{(0,\tau)}$, and $x_{[\tau]} = \sigma_\tau(x^1)_{[0]}$), and $u_{[0,\tau]} = u_{[0]} + u_{(0,\tau)} + u_{[\tau]}$ (where $u_{[0]} = u^0_{[0]}$, $u_{(0,\tau)} = u^0_{(0,\tau)}$, and $u_{[\tau]} = \sigma_\tau(u^1)_{[0]}$). Using part (iii) of Proposition 1.1, we get:

$$\Sigma_{[0,\tau]} : E(x_{[0,\tau]})' = Ax_{[0,\tau]} + Bu_{[0,\tau]} + \delta Ex(0_-) - \delta_\tau\, Ex(\tau_+)$$

which, after some simple manipulations yields:

$$\Sigma_{[0]} : E(x_{[0]})' = Ax_{[0]} + Bu_{[0]} - \delta\Delta_0 Ex$$
$$\Sigma_{(0,\tau)} : E(x_{(0,\tau)})' = Ax_{(0,\tau)} + Bu_{(0,\tau)} + \delta Ex(0_+) - \delta_\tau Ex(\tau_-)$$
$$\Sigma_{[\tau]} : E(x_{[\tau]})' = Ax_{[\tau]} + Bu_{[\tau]} - \delta_\tau \Delta_\tau Ex$$

In the light of the equations above, we conclude there exist $x(0_+)$ and $x(\tau_-)$ so that $(u_{[0]}, x_{[0]})$ satisfies $\Sigma_{[0]}$ with jump $x(0_+) - x(0_-)$, $(u_{[\tau]}, x_{[\tau]})$ satisfies $\Sigma_{[\tau]}$ with jump $x(\tau_+) - x(\tau_-)$ and $x(0,\tau)$ solves the two-point boundary value problem with initial and final conditions $x(0_+)$ and $x(\tau_-)$. Our assumption that (u^0, x^0) is a Bohl-type input-trajectory pair for $\Sigma_{[0,\infty)}$ compatible with $x(0_-)$ together with Proposition 1.3 implies that: (i) $x(0_-) \in \mathcal{V}^* + \mathcal{R}_a^*$, and (ii) $x(t) \in \mathcal{V}^*$, $t \in (0,\tau)$. On the other hand, since (u^1, x^1) is assumed to be a Bohl-type input-trajectory for $\Sigma_{[0,\infty)}$ compatible with the initial condition $x(\tau_-)$, we conclude, again by Proposition 1.3, that $x(\tau_+) \in \mathcal{V}^*$. Thus, we have $\Delta_\tau x \in \mathcal{V}^*$. However, that $\Delta_\tau x$ is an admissible jump for $\Sigma_{[\tau]}$ implies, again by Proposition 1.3, that $\Delta_\tau x \in \mathcal{R}_a^*$ also. Therefore, we have $\Delta_\tau x \in \mathcal{V}^* \cap \mathcal{R}_a^* = \mathcal{R}^*$.

Now, consider a general trajectory $x_{[0,\infty)}$ of $\Sigma_{[0,\infty)}$ compatible with some initial condition $x(0_-)$. Let $0 = \tau_0, \tau_1, \tau_2, \ldots$ denote the jump points of the trajectory.The following proposition generalizes our discussion above.

Proposition 1.4.

(i) If $\tau = 0$ then $\Delta_\tau \in \mathcal{R}_a^*$ and if $\tau > 0$ then $\Delta_\tau \in \mathcal{R}^*$.

(ii) For all $t \in (\tau_{i-1}, \tau_i)$, $i = 1, 2, \ldots$, we have $x_{[0,\infty)}(t) \in \mathcal{V}^*$.

Theorem 1.3.

(i) If $x(0_+) \in \mathcal{R}^*$ then for all $t \in (\tau_{i-1}, \tau_i)$, we have $x_{[0,\infty)}(t) \in \mathcal{R}^*$, and for all τ_i's, $i \geq 1$, we have $\Delta_{\tau_i} \in \mathcal{R}^*$.

(ii) If for some $\tau_i > 0$, we have $x_{[0,\infty)}(\tau_i^+) = 0$ then $x_{[0,\infty)}(0_+) \in \mathcal{R}^*$.

Proof. Theorem 1.1 (ii) immediately implies that $x(t) \in \mathcal{R}^*$ for all $t \in (0, \tau_1)$; in particular, we have $x(\tau_1^-) \in \mathcal{R}^*$. It follows from Proposition 1.3(i) that $x(\tau_1^+) \in \mathcal{V}^*$. Thus, $\Delta_{\tau_1} x \in \mathcal{V}^*$. However, Proposition 1.3 (ii) and time invariancy imply that $\Delta_{\tau_1} x \in \mathcal{R}_a^*$ also. Therefore, $\Delta_{\tau_1} x \in \mathcal{V}^* \cap \mathcal{R}_a^* = \mathcal{R}^*$. Then we have $x(\tau_1^+) = x(\tau_1^-) + \Delta_{\tau_1} x \in \mathcal{R}^*$. Now, repeating this arguement on (τ_1, τ_2) proves that $x(\tau_2^+) \in \mathcal{R}^*$, repeating it on (τ_2, τ_3) proves that $x(\tau_3^+) \in \mathcal{R}^*$ etc. This completes the proof of (i).

To prove (ii), assume that $x(\tau_i^+) = 0$. Then, since $\Delta_{\tau_i} x \in \mathcal{R}^*$, we have $x(\tau_i^-) \in \mathcal{R}^*$ also. But then it follows from Theorem 1.1 that $x(\tau_{i-1}^+) \in \mathcal{R}^*$. Again, repeated applications of this arguement proves the result that $x(0_+) \in \mathcal{R}^*$. □

1.3. Input-trajectory pairs for $\Sigma_{[0,T]}$

$\Sigma_{[0,T]}$ arises in a number of different ways. First, if $(u, x) \in \mathcal{B} \times \mathcal{B}$ is an admissible input-trajectory pair for Σ then restrictions of u and x define an admissible input-trajectory pair for $\Sigma_{[0,T]}$ compatible with some boundary values $x(0_-)$ and $x(T_+)$. In this case, it follows immediately from Proposition 1.2 that $x(0_-)$ and $x(T_+)$ cannot be arbitrary but have to reside in $\mathcal{V}^*$.

Secondly, input-trajectory pairs of $\Sigma_{[0,T]}$ may be thought of as the restrictions of admissible pairs for $\Sigma_{[0,\infty)}$ to $[0, T]$. In this case it is not in general true that $x(0_-) \in \mathcal{V}^*$ (although it follows from Proposition 1.3 that $x(0_-) \in \mathcal{V}^* + \mathcal{R}_a^*$). However, the assumption that $x_{[0,T]}$ has an extension to $\Re^+$ implies that $x(T_+) \in \mathcal{V}^*$.

Finally, as we shall be doing in the sequel, $\Sigma_{[0,\tau]}$ can be considered as a separate entity given together with some end conditions $x(0_-)$ and $x(\tau_+)$. To justify this, we note that one may very well conjecture the situation where the system at hand has varying structure, and that the system equations hold only on $[0, \tau]$, with $x(0_-)$ and $x(\tau_+)$ specified by the past and the future of the system (this would, for instance, be the case if controlled or uncontrolled switchings at 0 and at τ result in some changes in the structure of the system). Now, consider $\Sigma_{[0,\tau]}$ with given end conditions $x(0_-)$ and $x(\tau_+)$. Let $(u_{[0,\tau]}, x_{[0,\tau]}) \in \mathcal{B}_{[0,T]} \times \mathcal{B}_{[0,T]}$ be an admissible input-trajectory pair for the system. Clearly, everything that was said for the trajectories of $\Sigma_{[0,\infty)}$ are also valid in this case except for the assertion that $x(T_+) \in \mathcal{V}^*$. Properties of the trajectories which follow immediately from the discussions and the results of Section 1.2 are summarized below.

Proposition 1.5. Let τ_i's denote the jump points in $x_{[0,T]}$.

(i) If $0 < \tau_i < T$ then $\Delta_{\tau_i} x \in \mathcal{R}^*$. If $\tau_i = 0$ or if $\tau_i = T$ then $\Delta_{\tau_i} x \in \mathcal{R}_a^*$.

(ii) For all $t \in (\tau_{i-1}, \tau_i)$, we have $x(t) \in \mathcal{V}^*$.

Theorem 1.4.

(i) If $x(T_+) = 0$ then $x(0_+) \in \mathcal{R}^*$.

(ii) If $x(0_+) = 0$ then $x(T_+) \in \mathcal{R}^*$.

2. CONTROLLABILITY AND REACHABILITY

Before beginning our discussion of structural properties, let us introduce the notion of the order of a distribution. A distribution x is said to be of finite order if it is the distributional derivative of a function f which is continuous on $\Re$, i. e. if $x = f^{(k)}$ for some nonnegative integer k. The least integer for which this equality holds true is known as the order of x. Note that this terminology is not standard and some authors define the order of x as the least integer k for which $x = f^{(k+2)}$ (compare Section 3.4 of [5], Section 12 of [6] and Section II.3 of [7]). Here, we adopt the definition given in [5]. Being the second distributional derivative of the unit ramp function (which is continuous on $\Re$), the Dirac delta has order 2, and its jth-order derivative $\delta^{(j)}$ has order $j+2$. Note that if x is a Bohl type distribution then order of x is $k+2$ where k is the degree of the polynomial part of $X(s)$. It is of order 1 if $X(s)$ is strictly proper. For a trajectory of the form $x_{[0,\tau]} = x_{[\tau_0]} + x_{(\tau_0,\tau_1)} + x_{[\tau_1]} + x_{(\tau_1,\tau_2)} + \cdots + x_{(\tau_{q1},\tau_q)} + x_{[\tau_q]}$ we define its order as $\max\{\text{ord}(x_{[\tau_i]}) : i = 0, 1, \ldots, q\}$ if not all of $\text{ord}(x_{[\tau_i]})$'s are zero, and otherwise we define its order to be one. Note that an order one trajectory $x_{[0,T]}$ is piece-wise continuous on any open set which contains $[0, T]$.

We can now start our investigation of controllability/reachability properties of three different types of systems introduced in the previous section. We first consider the system defined over some finite interval $[0, T]$.

2.1. Controllability and Reachability of $\Sigma_{[0,T]}$

In this subsection, we let $T \geq 0$ be an arbitrary but fixed time. We resume our discussion by first presenting a number of definitions for the system defined over $[0, T]$.

Definition 2.1.

(i) An initial condition ξ of $\Sigma_{[0,T]}$ is said to be *controllable of order* k ($k \geq 1$) if there exists an admissible input-trajectory pair $(u_{[0,T]}, x_{[0,T]})$ which is compatible with $Ex(0_-) = \xi$ and $Ex(T_+) = 0$, and satisfies $\text{ord}(x_{[0,T]}) = k$.

(ii) A final condition is said to be *reachable of order* k ($k \geq 1$) if there exists an admissible input-trajectory pair $(u_{[0,T]}, x_{[0,T]})$ for $\Sigma_{[0,T]}$ which is compatible with $Ex(0_-) = 0$ and $Ex(T_+) = \zeta$, and satisfies $\text{ord}(x_{[0,T]}) = k$.

Definition 2.2. If $T = 0$, then ξ is said to be *instantaneously controllable of order* k and ζ is said to be *instantaneously reachable of order* k.

Definition 2.3.

(i) $\Sigma_{[0,T]}$ is *controllable (instantaneously controllable) of order k* if every initial condition is controllable (instantaneously controllable) of order k ($k \geq 1$).

(ii) $\Sigma_{[0,T]}$ is *reachable (instantaneously reachable) of order k* if every final condition is reachable (instantaneously reachable) of order k ($k \geq 1$).

After defining them precisely, we can now characterize these structural properties. We first present a more detailed version of Proposition 1.3 (ii).

Proposition 2.1. Let $\tau \geq 0$ and a $\Delta_\tau x$ be given. Let $\mathcal{R}_{a,k}$ denote the kth step of the almost reachability subspace algorithm. There exists a pair of Bohl-type distributions $(u_{[\tau]}, x_{[\tau]})$ with $\mathrm{ord}(x_{[\tau]}) = k$ ($k \geq 1$) satisfying $Ex'_{[\tau]} = Ax_{[\tau]} + Bu_{[\tau]} - \delta_\tau E\Delta_\tau x \iff E\Delta_\tau x \in A\mathcal{R}_{a,k-1} + \mathrm{Im}B \iff \Delta_\tau x \in \mathcal{R}_{a,k}$.

P r o o f has already been given for the case $k \geq n$ (see the proof of Proposition 1.3). One only needs the do some bookkeeping to show the result when $k < n$. □

The following result, that we present in the form of a theorem, is indeed nothing more than an immediate corollary to the result above when τ is taken to be 0.

Theorem 2.1.

(i) An initial condition $Ex(0_-)$ is instantaneously controllable of order $k \iff x(0_-) \in \mathcal{R}_{a,k} \iff Ex(0_-) \in A\mathcal{R}_{a,k-1} + \mathrm{Im}B$.Therefore $\Sigma_{[0,T]}$ is instantaneously controllable of order $k \iff \mathcal{R}_{a,k} = \mathcal{X} \iff A\mathcal{R}_{a,k-1} + \mathrm{Im}B \supset \mathrm{Im}E + \mathrm{Im}B$.

(ii) A final condition $Ex(T_+)$ is instantaneously reachable of order $k \iff x(T_+) \in \mathcal{R}_{a,k} \iff Ex(T_+) \in A\mathcal{R}_{a,k-1} + \mathrm{Im}B$. Therefore $\Sigma_{[0,T]}$ is instantaneously reachable of order $k \iff \mathcal{R}_{a,k} = \mathcal{X} \iff A\mathcal{R}_{a,k-1} + \mathrm{Im}B \supset \mathrm{Im}E + \mathrm{Im}B$.

Having characterized controllability/reachability of $\Sigma_{[0,T]}$ for $T = 0$, we can now consider the same system with $T > 0$. In this case, we have the following results.

Theorem 2.2. Let $k \geq 1$ and $T > 0$. Then:

(i) An initial condition $Ex(0_-)$ is controllable of order $k \iff x(0_-) \in \mathcal{R}^* + \mathcal{R}_{a,k} \iff Ex(0_-) \in E\mathcal{R}^* + A\mathcal{R}_{a,k-1} + \mathrm{Im}B$. Therefore $\Sigma_{[0,T]}$ is controllable of order $k \iff \mathcal{R}^* + \mathcal{R}_{a,k} = \mathcal{X} \iff E\mathcal{R}^* + A\mathcal{R}_{a,k-1} + \mathrm{Im}B \supset \mathrm{Im}E + \mathrm{Im}B$.

(ii) A final condition $Ex(T_+)$ is reachable of order $k \iff x(T_+) \in \mathcal{R}^* + \mathcal{R}_{a,k} \iff Ex(T_+) \in E\mathcal{R}^* + A\mathcal{R}_{a,k-1} + \mathrm{Im}B$. Therefore $\Sigma_{[0,T]}$ is reachable of order $k \iff \mathcal{R}^* + \mathcal{R}_{a,k} = \mathcal{X} \iff E\mathcal{R}^* + A\mathcal{R}_{a,k-1} + \mathrm{Im}B \supset \mathrm{Im}E + \mathrm{Im}B$.

P r o o f. We shall only prove part (*i*). Part (*ii*) is proved similarly. Now, let $x(0_-)$ be a controllable initial condition of order k for $\Sigma_{[0,T]}$. Then, there exists an input-trajectory pair $(u_{[0,T]}, x_{[0,T]}) \in \mathcal{B}_{[0,T]} \times \mathcal{B}_{[0,T]}$ satisfying $\Sigma_{[0,T]}$with the end conditions $Ex(0_-)$ and $Ex(T_+) = 0$. It should be clear from the proof below that there is no loss of generality in assuming that $x_{[0,T]}$ is of the form $x_{[0,T]} = x_0 \odot \sigma_\tau x_1 \odot (\sigma_T x_2)_{[0]}$ where x_0, x_1, x_2 are Bohl-type distributions. By taking restrictions of u and x to t for $t \in (0,\tau)$ and for $t \in (\tau, T)$, it can be easily deduced that the input u has to be of the form $u = u^0 \odot \sigma_\tau u^1 \odot (\sigma_T u^2)_{[0]}$ where u^0, u^1, u^2 are Bohl-type distributions.

Now, it follows from Proposition 1.5 that $\Delta_T x \in \mathcal{R}_a^*$. Since, by assumption of controllability, we have $Ex(T_+) = 0$, it follows that $x(T_-) \in \mathcal{R}_a^* + \mathrm{Ker} E = \mathcal{R}_a^*$. As (u_1, x_1) is a Bohl-type input-trajectory pair we also have, again by Proposition 1.5, $x(T_-) \in \mathcal{V}^*$. Thus, $x(T_-) \in \mathcal{R}^*$. Then, a very minor change in the proof of Theorem 1.1(iii) shows that $x(\tau_+) \in \mathcal{R}^*$ also.

Now, $Ex'_{[\tau]} = Ax_{[\tau]} + Bu_{[\tau]} - \delta_\tau \Delta_\tau Ex$ implies that $\Delta_\tau x \in \mathcal{R}_a^*$ (see Proposition 1.3). Then, $x(\tau_+) - x(\tau_-) \in \mathcal{R}_a^*$, $\mathcal{R}^* \subset \mathcal{R}_a^*$ and $x(\tau_+) \in \mathcal{R}^*$ imply that $x(\tau_-) \in \mathcal{R}_a^*$. On the other hand, since (x_0, u_0) is a Bohl-type input-trajectory pair we conclude from Theorem 1.1(i) that $x(\tau_-) \in \mathcal{V}^*$ also. Then, $x(\tau_-) \in \mathcal{V}^* \cap \mathcal{R}_a^* = \mathcal{R}^*$, and it follows that $x(0_+) \in \mathcal{R}^*$. (Note that this is indeed the proof of Theorem 1.4(i) which was not given when it was stated.) Thus, $x(0_-) \in \mathcal{R}^* + \mathcal{R}_a^*$ because $Ex'_{[0]} = Ax_{[0]} + Bu_{[0]} - \delta_0 \Delta_0 Ex$ implies $\Delta_0 x \in \mathcal{R}_a^*$. Now, let p be the least integer for which $\Delta_0 x \in \mathcal{R}_{a,p}$. Then, $x(0_-) \in \mathcal{R}_a^* + \mathcal{R}_{a,p}$. Finally, we note that the way order of $x_{[0,T]}$ is defined guarantees that $p \leq \mathrm{ord}(x_{[0,T]})$. Let k denote $\mathrm{ord}(x_{[0,T]})$. Then, since $\{\mathcal{R}_{a,j}\}$ is monotone nondecreasing, we have $\mathcal{R}_{a,p} \subset \mathcal{R}_{a,k}$ and consequently $x(0_-) \in \mathcal{R}_a^* + \mathcal{R}_{a,k}$.

To prove the converse, let $x(0_-) \in \mathcal{R}_a^* + \mathcal{R}_{a,k}$. Choose an $x(0_+) \in \mathcal{R}^*$ such that $\Delta_0 x = x(0_+) - x(0_-)$ is in $\mathcal{R}_{a,k}$. Then there exists a sequence $\{x_0, x_{-1}, \ldots, x_{-k+2}\}$ of vectors which satisfy

$$\begin{array}{ccccc} \Delta_0 Ex & = & Ax_0 & + & Bu_0 \\ Ex_0 & = & Ax_{-1} & + & Bu_{-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Ex_{-k+2} & = & 0 & + & Bu_{-k+1} \end{array}$$

for some u_i, $i = 0, -1, \ldots, -k+1$. Clearly, if $u_{[0]}$ and $x_{[0]}$ are defined by

$$u_{[0]} = u_0\delta + u_{-1}\delta^{(1)} + \cdots u_{-k+1}\delta^{(k-1)}$$

$$x_{[0]} = x_0\delta + x_{-1}\delta^{(1)} + \cdots x_{-k+2}\delta^{(k-2)}$$

then $(u_{[0]}, x_{[0]})$ is an admissible pair for $\Sigma_{[0]}$ compatible with $x(0_-)$ and $x(0_+)$. Now, given $x(0_+)$ from $\mathcal{R}^*$ and $T > 0$ there exists a smooth input-trajectory pair $(u_{(0,\infty)}, x_{(0,\infty)})$ of $\Sigma_{(0,\infty)}$ compatible with $x(0_+)$ such that x satisfies $x(T_+) = 0$. Then, $u = u_{[0]} + u_{(0,\infty)}$ and $x = x_{[0]} + x_{(0,\infty)}$ yield an input-trajectory pair for $\Sigma_{[0,\infty)}$ compatible with the given initial condition. Restricting this pair to $[0,T]$ yields an input-trajectory pair for $\Sigma_{[0,T]}$ compatible with the end condition $Ex(0_-)$ and with $x(T_+) = 0$. Clearly, order of $x_{[0,T]}$ is equal to the order of $x_{[0]}$ which is k. Hence the proof. □

Remarks. (i) The results above make it clear that controllability (instantaneous controllability) of order k is the same property as reachability (instantaneous reachability) of order k for all $k \geq 1$.

(ii) Note that $\mathcal{R}^* \subset \mathcal{R}_{a,n}$ and $\{\mathcal{R}_{a,k}\}$ is monotone nondecreasing. Thus, if k is large enough ($k \geq n$) then the distinction between controllability of order k and instantaneous controllability of order k also disappears (and all four properties collapse down to one single condition which is $\mathcal{R}_a^* = \mathcal{X}$). Indeed, this result is in tune with intuition which states that whatever can be done by impulses can be done instantaneously. The only difference between these properties is in the highest order derivative of the Dirac delta required to drive the initial condition down to zero. If the control effort is distributed over a finite interval then the highest order derivative of δ which is required may turn out to be less than its counterpart when the control effort is concantrated at one point.
(iii) Again, in the limit, we have indeed one property characterized by the condition $\mathcal{R}_a^* = \mathcal{X}$. Although it would be quite natural to call this property simply controllability/reachability in our context, it has already been introduced and called "almost controllability" in the pioneering work of J.C. Willems [8] for proper state-space systems (the adjective "almost" is used to differentiate the concept from the usual controllability where an initial condition can be taken down to the origin in finite time along a smooth trajectory generated by a smooth input). Thus, adopting the terminology of Willems, we shall also use the phrase "almost controllability (reachability)". To emphasize this choice of terminology, we repeat the result above one more time.

Definition 2.4. $\Sigma_{[0,T]}$ is almost controllable (almost reachable) $\stackrel{\Delta}{\Longleftrightarrow} \mathcal{R}_a^* = \mathcal{X} \stackrel{\Delta}{\Longleftrightarrow} A\mathcal{R}_a^* + \mathrm{Im}B \supset \mathrm{Im}E + \mathrm{Im}A + \mathrm{Im}B$.

The following corollary follows immediately from the results established above.

Corollary 2.1. Given any $x(0_-) \in \mathcal{R}_a^*$, $x(T_+) \in \mathcal{R}_a^*$ and $T \geq 0$ there exists an input-trajectory pair admissible for $\Sigma_{[0,T]}$ and compatible with $Ex(0_-)$ and $Ex(T_+)$.

One would naturally try to avoid impulsive trajectories when driving an initial condition down to zero or in reaching a finite condition from the origin. Therefore, first order controllable (reachable) initial (final) conditions are of special significance because they involve piece-wise continuous trajectories. An order-one (i.e., a piece-wise continuous) trajectory is generated by inputs of order ≤ 2. Impracticality of generating impulsive inputs render it more important to have a separate classification of those first-order controllable (reachable) initial conditions (final conditions) which can be driven down to the origin (which can be reached from the origin) by using first-order (i.e., piece-wise continuous) inputs.

Definition 2.5. A first order controllable initial condition $Ex(0_-)$ (reachable final condition $Ex(T_+)$) will be referred to as simply a *controllable initial condition (reachable final condition)* if there exists an input-trajectory pair $(u_{[0,T]}, x_{[0,T]})$ satisfying $\Sigma_{[0,T]}$ with $Ex(0_-)$ and $Ex(T_+) = 0$ (with $Ex(0_-)$ and $Ex(T_+) = 0$) and such that $\mathrm{ord}(u_{[0,T]}) = 1$. $\Sigma_{[0,T]}$ is said to be controllable (reachable) if every initial (final) condition is controllable.

Theorem 2.3. $\Sigma_{[0,T]}$ is controllable (reachable) $\Longleftrightarrow \mathcal{R}^* + \mathrm{Ker}E = \mathcal{X} \Longleftrightarrow E\mathcal{R}^* + A\mathrm{Ker}E + \mathrm{Im}B \supset \mathrm{Im}E + \mathrm{Im}A + \mathrm{Im}B$.

Proof. Note that at any jump point τ (including $\tau = 0$ and $\tau = T$), we have $x_{[\tau]} = 0$ because the trajectory has order one as implied by the first order controllability of the initial (final) condition. Then the restriction of the equation to τ yields $0 = Bu_{[\tau]} - \delta_\tau \Delta_\tau Ex$. If it is further required that $u_{[\tau]} = 0$ then the equation becomes $\delta_\tau \Delta_\tau Ex = 0$. Thus, we have $\Delta_\tau x \in \mathrm{Ker}E$. Then taking $\tau = 0$ and noticing that $Ex(T_+) = 0$ implies $x(0_+) \in \mathcal{R}^*$ proves that $x(0_-) \in \mathcal{R}^* + \mathrm{Ker}E$. Similarly, taking $\tau = T$ and noticing that $Ex(0_-) = 0$ implies $x(T_+) \in \mathcal{R}^*$ proves that $x(T_+) \in \mathcal{R}^* + \mathrm{Ker}E$. □

Definition 2.6. If the trajectory in Definition 2.4, is restricted to have no jumps at $t \in (0, T)$ and also satisfies $x(0_-) = x(0_+)$ and $x(T_-) = x(T_+) = 0$ then $x(0_-)$ is said to be *completely controllable*. If the trajectory in Definition 2.4, is restricted to have no jumps at $t \in (0, T)$ and also satisfies $x(0_-) = x(0_+) = 0$ and $x(T_-) = x(T_+)$ then $x(T_+)$ is said to be *completely reachable*. We say that $\Sigma_{[0,T]}$ is *completely controllable (reachable)* if every initial condition (final condition) is completely controllable (completely reachable).

Using the conditions $x(0_-) = x(0_+) = 0$ and $x(T_-) = x(T_+)$ in the proof of Theorem 2.3 shows that:

Theorem 2.4. $x(0_-)$ is completely controllable $\Longleftrightarrow x(0_-) \in \mathcal{R}^*$. $x(T_+)$ is completely reachable $\Longleftrightarrow x(T_+) \in \mathcal{R}^*$. Therefore $\Sigma_{[0,T]}$ is completely controllable (completely reachable) $\Longleftrightarrow \mathcal{R}^* = \mathcal{X}$.

The way it is defined together with the result above shows that complete controllability is a property of $x(0_-)$ rather than of $Ex(0_-)$. Therefore, the condition for complete controllability (complete reachability) was given in the domain. Indeed, if the order of the trajectory mentioned in the definition of complete controllability is defined to be zero, then the result expressed in the domain on controllability of order k can be re-written to state "$x(0_-)$ is controllable of order k for $k \geq 0$ iff $x(0_-) \in \mathcal{R}^* + \mathcal{R}_{a,k}$".

Note that $\mathcal{R}^* = \mathcal{X}$ iff $\mathcal{V}^* = \mathcal{X}$ and $\mathcal{R}_a^* = \mathcal{X}$. We have already stated that although any trajectory of $\Sigma_{[0,\infty)}$, when restricted to [0,T], yields a trajectory of $\Sigma_{[0,T]}$, the converse statement is not true in general in the sense that given a trajectory of $\Sigma_{[0,T]}$ compatible with some end condition $x(T_+)$ there does not always exist a trajectory z of $\Sigma_{[0,\infty)}$ such that $z_{[0,T]} = x_{[0,T]}$ and $\lim_{t \downarrow T} z = x(T_+)$. In case the converse statement is also true we shall say that the system is *complete* on $\Re^+$. A trajectory $x_{[0,T]}$ can be extended to a trajectory $x_{[0,\infty)}$ in the sense mentioned above iff $x(T_+)$ lies in $\mathcal{V}^*$. Thus, the system is complete iff $\mathcal{V}^* = \mathcal{X}$. Thus, complete controllability is equivalent to completeness and almost controllability. Consequently, a more appropriate choice for terminology would be to call almost controllablity simply controllability in which case complete controllability would be the joint property of completeness and controllability. As the theory of singular systems has already suffered a lot from nonhomogenous terminology, we have adopted the terminology that we introduced because, as mentioned above, the condition $\mathcal{R}_a^* = \mathcal{X}$ has already been called almost controllability by J. C. Willems [8].

If the reachability/controllability properties defined above above hold for a system with the property $\mathrm{Im}E + \mathrm{Im}A + \mathrm{Im}B = \underline{\mathcal{X}}$ then they are prefixed by the word "strong", e. g., strong almost controllability, strong complete reachability *etc.* The adjective strong is justified by the observation that if $\mathrm{Im}E + \mathrm{Im}A + \mathrm{Im}B = \underline{\mathcal{X}}$ then all the dynamical properties introduced hold true even in the presence of deterministic (generalized Bohl-type) disturbances. Here, we shall not discuss this aspect. For interpretations of strong properties in terms of a discrete-time system with disturbances, see [9]. See also [12]. We also note that a condition expressed in the codomain, e. g., $E\mathcal{R}^* + A\mathcal{R}^* + \mathrm{Im}B = \underline{\mathcal{X}}$ implies $\mathcal{R}^* = \mathcal{X}$ although the converse statement is true only if $\mathrm{Im}E + \mathrm{Im}A + \mathrm{Im}B = \underline{\mathcal{X}}$. This fact provides extra justification for the adjective "strong". Finally, we note that restricting the system operators E, A and B in their codomains to $\mathrm{Im}E + \mathrm{Im}A + \mathrm{Im}B$ makes the distinction between the plain and stronger versions superfluous.

Definition 2.7.

(i) $\Sigma_{[0,T]}$ is strongly instantaneously controllable of order k ($k \geq 1$) $\stackrel{\Delta}{\Longleftrightarrow} A\mathcal{R}_{a,k-1} + \mathrm{Im}B = \underline{\mathcal{X}}$

(ii) $\Sigma_{[0,T]}$ is strongly controllable of order k ($k \geq 1$) $\stackrel{\Delta}{\Longleftrightarrow} E\mathcal{R}^* + A\mathcal{R}_{a,k-1} + \mathrm{Im}B = \underline{\mathcal{X}}$

(iii) $\Sigma_{[0,T]}$ is stronly almost controllable (reachable) $\stackrel{\Delta}{\Longleftrightarrow} A\mathcal{R}_a^* + \mathrm{Im}B = \underline{\mathcal{X}}$.

(iv) $\Sigma_{[0,T]}$ is strongly controllable (reachable) $\stackrel{\Delta}{\Longleftrightarrow} E\mathcal{R}^* + A\mathrm{Ker}E + \mathrm{Im}B = \underline{\mathcal{X}}$.

(v) $\Sigma_{[0,T]}$ is strongly completely controllable (strongly completely reachable) $\stackrel{\Delta}{\Longleftrightarrow} E\mathcal{R}^* + A\mathcal{R}^* + \mathrm{Im}B = \underline{\mathcal{X}}$.

2.2. Controllability and Reachability of $\Sigma_{[0,\infty)}$

Having defined and characterized controllability/reachability properties of $\Sigma_{[0,T]}$ we now return to $\Sigma_{[0,\infty)}$. As controllability/reachability properties for $\Sigma_{[0,\infty)}$ are also defined for finite time, we naturally adopt all the definitions introduced for $\Sigma_{[0,T]}$. For instance, we shall say that $Ex(0_-)$ is an order-k controllable initial condition of $\Sigma_{[0,\infty)}$ if it is an order-k controllable initial condition of $\Sigma_{[0,T]}$ for some $T \geq 0$. However, we shall require that all the inputs and trajectories involved in these definitions have extensions to $\Re^+$. That is to say, it will be assumed that $x_{[0,T]} = (z)_{[0,T]}$ and $u_{[0,T]} = (\omega)_{[0,T]}$ for some $z \in \mathcal{B}_{[0,\infty)}$ and $\omega \in \mathcal{B}_{[0,\infty)}$ where z also satisfies $\lim_{t\downarrow T} z = x(T_+)$. It is easy to realize that this requirement does not impose any restrictions on any one of the controllability definitions introduced for $\Sigma_{[0,T]}$. Indeed, if $(u_{[0,T]}, x_{[0,T]})$ is an admissible input-trajectory pair for $\Sigma_{[0,T]}$, $T \geq 0$, which drives $Ex(0_-)$ down to the origin at $t = T_+$ then $(u_{[0,T]} \odot \sigma_T\theta, x_{[0,T]} \odot \sigma_T\theta)$ (where θ denotes the null distribution on $\Re^+$) is an admissible input-trajectory pair for $\Sigma_{[0,\infty)}$ compatible with initial condition $Ex(0_-)$ and with final condition $x(T_+) = 0$. Furthermore, a careful reading of the proofs of the results of the previous subsection reveals that there is absolutely no loss of generality in assuming that in definitions of all types of controllability properties, admissible input-trajectory pairs can be taken to be of Bohl-type (rather than of generalized Bohl-type).

Theorem 2.5. Let $\mathcal{I}$ denote $\text{Im}E + \text{Im}A + \text{Im}B$. Then:

(i) $\Sigma_{[0,\infty)}$ is kth-order instantaneously controllable $\iff \mathcal{R}_{a,k} = \mathcal{X} \iff A\mathcal{R}_{a,k-1} + \text{Im}B \supset \text{Im}E + \text{Im}B$

(ii) $\Sigma_{[0,\infty)}$ is kth-order controllable $\iff \mathcal{R}^* + \mathcal{R}_{a,k} = \mathcal{X} \iff E\mathcal{R}^* + A\mathcal{R}_{a,k-1} + \text{Im}B \supset \text{Im}E + \text{Im}B$

(iii) $\Sigma_{[0,\infty)}$ is almost controllable $\iff \mathcal{R}_a^* = \mathcal{X} \iff A\mathcal{R}_a^* + \text{Im}B = \mathcal{I}$

(iv) $\Sigma_{[0,\infty)}$ is controllable $\iff \mathcal{R}^* + \text{Ker}E = \mathcal{X} \iff E\mathcal{R}^* + A\text{Ker}E + \text{Im}B = \mathcal{I}$

(v) $\Sigma_{[0,\infty)}$ is completely controllable $\iff \mathcal{R}^* = \mathcal{X} \iff E\mathcal{R}^* + A\mathcal{R}^* + \text{Im}B = \mathcal{I}$

We also have the stronger versions of these properties.

Theorem 2.6.

(i) $\Sigma_{[0,\infty)}$ is kth-order strongly instantaneously controllable $\iff A\mathcal{R}_{a,k-1} + \text{Im}B = \underline{\mathcal{X}}$

(ii) $\Sigma_{[0,\infty)}$ is kth-order strongly controllable $\iff E\mathcal{R}^* + A\mathcal{R}_{a,k-1} + \text{Im}B = \underline{\mathcal{X}}$

(iii) $\Sigma_{[0,\infty)}$ is strongly almost controllable $\iff A\mathcal{R}_a^* + \text{Im}B = \underline{\mathcal{X}}$

(iv) $\Sigma_{[0,\infty)}$ is strongly controllable $\iff E\mathcal{R}^* + A\text{Ker}E + \text{Im}B = \underline{\mathcal{X}}$

(v) $\Sigma_{[0,\infty)}$ is strongly completely controllable $\iff E\mathcal{R}^* + A\mathcal{R}^* + \text{Im}B = \underline{\mathcal{X}}$

Unlike controllability, reachability properties of $\Sigma_{[0,\infty)}$ do get affected by the requirement that the input-trajectory pair defined on $[0,T]$ to reach a final condition should have an extension to $[0,\infty)$. Now, kth-order reachability of some final condition $x(T_+)$ can be defined in the same way as it is done for $\Sigma_{[0,T]}$ except we now have to assume that the final condition $x(T_+)$ is reached along a trajectory which is of the form $\{x_{[0,\infty)}\}_{[0,T]}$. Proposition 1.2 (ii) implies that in this case $x(T_+)$ lies in $\mathcal{V}^*$. This and $x(T_+) \in \mathcal{R}_a^*$ (implied by the reachability of the final condition) together imply that $x(T_+) \in \mathcal{R}^*$. Thus, in this case all definitions of reachability reduce to that of what we called complete reachability.

Theorem 2.7.

(i) $\Sigma_{[0,\infty)}$ is completely reachable $\iff \mathcal{R}^* = \mathcal{X} \iff E\mathcal{R}^* + A\mathcal{R}^* + \text{Im}B = \mathcal{I}$.

(ii) $\Sigma_{[0,\infty)}$ is strongly completely reachable $\iff E\mathcal{R}^* + A\mathcal{R}^* + \text{Im}B = \underline{\mathcal{X}}$.

Note that, the proof of Theorem 1.2 shows that if a final condition for $\Sigma_{[0,\infty)}$ can ever be reached in some finite time T then it can be reached along a trajectory which is smooth on $(0,T)$ and has no jumps at $t = 0$ or at $t = T$.

The relation between the results of [10,11] and the definitions introduced here is worth considering. In [10,11], only inputs and trajectories of order 1 were considered.

Thus, what is defined to be controllability in [10] is the controllability of this paper also. However, in [10,11] the trajectory on [0,T] along which a given final condition is reached has an extension to $[0,\infty)$. Thus, reachability of [10,11] is the same as complete reachability of this paper. As far as other definitions introduced in the literature are concerned, we note that our controllability is impulse controllabilty of Cobb, or equivalently, controllability as defined by Verghese and our complete controllability is controllability of Cobb, or equivalently, controllability of Rosenbrock (see [1]) when the system is regular.

2.3. Controllability and Reachability of Σ

Finally, we consider controllability/reachability properties of Σ. In this case also, we basically use the definitions introduced for $\Sigma_{[0,T]}$. However, we require that the inputs and the trajectories used in the definitions have extensions to $\Re$. In this case, it follows from Proposition 1.2 that any initial condition has to reside in $\mathcal{V}^*$ and so does any final condition. Among these initial conditions only those from $\mathcal{R}_a^*$ can be driven down to the origin in some finite time. Therefore, the set of all controllable initial conditions is $\mathcal{V}^* \cap \mathcal{R}_a^* = \mathcal{R}^*$. Among these final conditions only those from $\mathcal{R}_a^*$ can be reached from the trivial initial condition in finite time. Thus, the set of all reachable final conditions is $\mathcal{V}^* \cap \mathcal{R}_a^* = \mathcal{R}^*$. Therefore, for Σ, the only meaningful definition of controllability is that of complete controllability, and the only meaningful definition of reachability is that of complete reachability (of course, we also have the stronger versions of these properties). Thus, for Σ, just as it was the case for $\Sigma_{[0,T]}$, there is no need to make a distinction between controllability and reachability. However, unlike the case for $\Sigma_{[0,T]}$ or for $\Sigma_{[0,\infty)}$, we now have only one meaningful property rather than a set of distinct properties.

Theorem 2.8.

(i) Σ is completely controllable (completely reachable) $\iff \mathcal{R}^* = \mathcal{X} \iff E\mathcal{R}^* + A\mathcal{R}^* + \mathrm{Im}B = \mathcal{I}$.

(ii) Σ is strongly completely controllable (strongly completely reachable) $\iff E\mathcal{R}^* + A\mathcal{R}^* + \mathrm{Im}B = \underline{\mathcal{X}}$.

3. OBSERVABILITY AND RECONSTRUCTIBILITY

In this section, we shall consider Σ, $\Sigma_{[0,\infty)}$ and $\Sigma_{[0,T]}$, all with $B = 0$, and consider observability properties of these representations. We define the following algorithms:

$$\mathcal{V}_{\mathcal{K}}^{k+1} = \mathrm{Ker}C \cap A^{-1}E\mathcal{V}_{\mathcal{K}}^{k}; \quad \mathcal{V}_{\mathcal{K}}^{0} = \mathrm{Ker}C$$
$$\mathcal{R}_{\mathcal{K}}^{a,k+1} = \mathrm{Ker}C \cap E^{-1}A\mathcal{R}_{\mathcal{K}}^{a,k}; \quad \mathcal{R}_{\mathcal{K}}^{a,0} = 0.$$

Limits of these algorithms are reached in at most n steps and they will be denoted by $\mathcal{V}_{\mathcal{K}}^*$ and by $\mathcal{R}_{a,\mathcal{K}}^*$ respectively. The first limit denotes the supremal (A,E)-invariant subspace contained in Ker$C$ and the second one is the supremal almost controllability subspace of the homogenous system $Ex' = Ax$ contained in Ker$C$.

3.1. Observability and reconstructibility of Σ

We resume our discussion by first considering the restriction of Σ to some finite interval [0,T]. That is,we consider

$$\Sigma_{[0,T]} : E(x_{[0,T]})' = Ax_{[0,T]} + \delta Ex(0_-) - \delta_T Ex(T_+); \quad y_{[0,T]} = Cx_{[0,T]}$$

where $x_{[0,T]} \in \mathcal{B}_{[0,T]}$ is the restriction to the interval of some $\mathcal{B}$ solution of the homogenous equation with end conditions $Ex(0_-)$ and $Ex(T_+)$. We consider the problem of identifying the initial condition (final condition) from the knowledge of the output and its derivatives on $[0,T]$. We shall assume that $(y^j)_{[0,T]}$ is given for $j = 0,1,2,\ldots,k$. Note that, by Proposition 1.1, this is equivalent to saying that $y_{[0,T]}$, $y(0_-)$, $y'(0_-)$, $\ldots$ $y^{(k-1)}(0_-)$ and $y(T_+)$, $y'(T_+),\ldots$ $y^{(k-1)}(T_+)$ are given. Note that the proof of Proposition 1.2 shows that a compatible initial condition satisfies $Ex^{(j)}(0_-) = Ax^{(j-1)}(0_-)$ for $j = 1,2\ldots$. Therefore, the given data cannot uniquely determine the initial condition uniquely if there exists a nontrivial $Ex(0_-)$ which is compatible with a trajectory $x_{[0,T]} \in \mathrm{Ker}C$ and which satisfies $Ex^j(0_-) = Ax^{j-1}(0_-)$ $(j = 1,\ldots,k-1)$ for some sequence $\{x'(0_-),\ldots,x^{(k-1)}(0_-)\}$ from Ker$C$. This observation motivates the definition to follow.

Definition 3.1. Let $\{Ex(0_-), Ex(T_+)\}$ be compatible with some $x_{[0,T]}$ which yields $y_{[0,T]} = Cx_{[0,T]} = 0$. $Ex(0_-)$ is said to be *unobservable with k differentiations* ($k \geq 1$) if there exists a sequence $\{x'(0_-),\ldots,x^{(k-1)}(0_-)\}$ from Ker$C$ satisfying $Ex^j(0_-) = Ax^{j-1}(0_-)$ $(j = 1,\ldots,k-1)$. $Ex(T_+)$ is said to be *unconstructible with k differentiations* if there exists a sequence $\{x'(T_+),\ldots,x^{(k-1)}(T_+)\}$ from Ker$C$ satisfying $Ex^j(T_+) = Ax^{j-1}(T_+)$ $(j = 1,\ldots,k-1)$.

This definition is slightly stronger than necessary in the sense that it does not take into account any further restrictions imposed on the initial condition by the assumption that the trajectory involved is the restriction of some trajectory x of Σ. (For instance, it follows immediately from Proposition 1.2 that $x(0_-)$ has to lie in $\mathcal{V}^*$ of the homogenous system). We nevertheless adopt it because of two reasons. First, it admits a nice duality interpretation. Secondly, and more importantly, we shall mainly be interested in the limiting case of $k \geq n$ when the above mentioned restriction is satisfied anyway.

Definition 3.2. Σ is *observable with k differentiations* iff there exists no nontrivial $Ex(0_-)$ which is unobservable with k differentiations. Σ is *reconstructible with k differentiations* iff there exists no nontrivial $Ex(T_+) \neq 0$ which is unconstructible with k differentiations.

Definition 3.3. In case T=0, then the properties given in Definitions 2.1 and 2.2 are said to hold instantaneously.

We can now proceed to characterize these properties geometrically.

Theorem 3.1.

(i) $Ex(0_-)$ is instantaneously unobservable with k differentiations iff $Ex(0_-) \in E\mathcal{V}_{\mathcal{K}}^{k-1}$.

(ii) $Ex(T_+)$ is instantaneously unconstructible with k differentiations iff $Ex(T_+) \in E\mathcal{V}_{\mathcal{K}}^{k-1}$.

Proof. Now suppose that there exists a sequence $\{x'(0_-), \ldots, x^{(k-1)}(0_-)\}$ from Ker$C$ satisfying $Ex^j(0_-) = Ax^{j-1}(0_-)$ $(j = 1, \ldots, k)$. Then, $Ex^{(k-1)} = Ax^{(k-2)}$ implies $x^{(k-2)} \in A^{-1}E\mathrm{Ker}C = \mathcal{V}_{\mathcal{K}}^1$. Then, this and $Ex^{(k-2)} = Ax^{(k-3)}$implies that $x^{(k-3)} \in A^{-1}E\mathrm{Ker}C = \mathcal{V}_{\mathcal{K}}^2$. Proceeding, it follows that $x(0_-) \in \mathcal{V}_{\mathcal{K}}^{k-1}$. Conversely, if $x(0_-) \in \mathcal{V}_{\mathcal{K}}^{k-1}$ then there exists an $x'(0_-) \in \mathcal{V}_{\mathcal{K}}^{(k-2)}$ such that $Ax(0_-) = Ex'(0_-)$. Given $x'(0_-) \in \mathcal{V}_{\mathcal{K}}^{(k-2)}$ there exists an $x''(0_-) \in \mathcal{V}_{\mathcal{K}}^{(k-3)}$ such that $Ax'(0_-) = Ex''(0_-)$ *etc.* Thus, such a sequence from Ker$C$ exists for $x(0_-)$ iff $x(0_-) \in \mathcal{V}_{\mathcal{K}}^{(k-1)}$ and therefore such a sequence exists for $Ex(0_-)$ iff $Ex(0_-) \in E\mathcal{V}_{\mathcal{K}}^{(k-1)}$. Then, finally, taking $x^j(0_+) = x^j(0_-)$ and taking $x_{[0]} = 0$ completes the proof of (i). (ii) is proved similarly. □

Theorem 3.2.

(i) $Ex(0_-)$ is unobservable with k differentiations iff $Ex(0_-) \in E\mathcal{V}_{\mathcal{K}}^{k-1} \cap (E\mathcal{V}_{\mathcal{K}}^* + A\mathcal{R}_{a,\mathcal{K}}^*)$. Therefore, Σ is observable with k differentiations iff $E\mathcal{V}_{\mathcal{K}}^{k-1} \cap (E\mathcal{V}_{\mathcal{K}}^* + A\mathcal{R}_{a,\mathcal{K}}^*) = 0$.

(ii) $Ex(T_+)$ is unconstructible with k differentiations iff $Ex(T_+) \in E\mathcal{V}_{k-1} \cap (E\mathcal{V}_{\mathcal{K}}^* + A\mathcal{R}_{a,\mathcal{K}}^*)$. Therefore, Σ is reconstructible with k differentiations iff $E\mathcal{V}_{\mathcal{K}}^{k-1} \cap (E\mathcal{V}_{\mathcal{K}}^* + A\mathcal{R}_{a,\mathcal{K}}^*) = 0$

Proof. Again, only (i) will be proved. Symmetrical arguements will prove (ii). There is no loss of generality in assuming that $x_{[0,T]}$ is the restriction of some Bohl-type trajectory to $[0, T]$. It has been established in [15,16] that $Ex(0_-)$ is compatible with a Bohl-type trajectory iff $Ex(0_-) \in (E\mathcal{V}_{\mathcal{K}}^* + A\mathcal{R}_{a,\mathcal{K}}^*) \cap \mathrm{Im}E$. On the other hand, the restrictions at 0_- are satisfied iff $Ex(0_-) \in E\mathcal{V}_{\mathcal{K}}^{k-1}$. Thus, $Ex(0_-)$ is unobservable with k differentiations iff $Ex(0_-) \in E\mathcal{V}_{\mathcal{K}}^{k-1} \cap (E\mathcal{V}_{\mathcal{K}}^* + A\mathcal{R}_{a,\mathcal{K}}^*)$. □

It should be clear from the results above that unobservability (resp. instantaneous unobservability) and unconstructibility (resp. instantaneous unconstructibility) are the one and the same property. That in at most n steps the distinction between these properties disappears should be clear from the results above and the fact that the sequence $\mathcal{V}_k$ is monotone nonincreasing and reaches its limit in at most n steps. Thus, if k is large enough then all of these four properties reduce down to the condition $E\mathcal{V}_{\mathcal{K}}^* = 0$.

Now, let us consider the condition $E\mathcal{V}_{\mathcal{K}}^* = 0$, or equivalently, $\mathcal{V}_{\mathcal{K}}^* \in \mathrm{Ker}E \cap \mathrm{Ker}A \cap \cap \mathrm{Ker}C$. It follows from the analysis of [15,16] that $E\mathcal{V}_{\mathcal{K}}^* = 0$ is a necessary and sufficient condition for a Bohl-type output to determine $Ex(0_+)$ uniquely. On the other hand, it follows from the way instantaneous unobservability is defined and characterized that it may also be called jump-observability. Indeed, if x is a trajectory of Σ and if $x_{[\tau]}$ is its restriction to τ then it can be shown that $\Delta_\tau x^{(i)}$'s satisfy

$E\Delta_\tau x^{(i)} = A\Delta_\tau x^{(i-1)}$ for $i = 0, 1, 2, \ldots$. Then $\Delta_\tau y^{(i)} = C\Delta_\tau x^{(i)}$'s uniquely determine $E\Delta_\tau x$ iff $E\mathcal{V}^*_{\mathcal{K}}=0$. One is interested in the jump $E\Delta_\tau x$ because, together with $y_{[\tau]}$, it may specify $x_{[\tau]}$. However, a necessary condition for this to happen turns out to be Ker$E\cap$Ker$A\cap$Ker$C = 0$, and this condition holds true for a system satisfying $E\mathcal{V}^*_{\mathcal{K}} = 0$ iff $\mathcal{V}^*_{\mathcal{K}} = 0$. Thus, it seems that a more meaningful condition is $\mathcal{V}^*_{\mathcal{K}} = 0$, which could possibly be called strong jump observability. Note that in such a system, the knowledge of $\Delta_\tau y^{(i)}$'s is sufficient to determine $\Delta_\tau x$ uniquely. In this case, if a segment of the output is smooth on some $(0, T)$ then it follows that x is also smooth on the same interval. Then, on $(0, T)$, x may be identified by the smooth function generating the distribution , and the relations $Ex^{(i)}(t) = Ax^{(i-1)}$; $y^{(i)}(t) = Cx^{(i)}(t)$ may be employed to determine $x(t)$ uniquely for all $t \in (0, T)$.

These interpretations of the conditions $E\mathcal{V}^*_{\mathcal{K}} = 0$ and $\mathcal{V}^*_{\mathcal{K}} = 0$ notwithstanding, we note that the first one is necesssary and sufficient for the knowledge of $(y^{(j)})_{[0,T]}$ for $j = 0, 1, \ldots, n$ to determine $Ex(0_-)$ $(Ex(T_+))$ uniquely, and the second one is necessary and sufficient for the same data to determine $x(0_-)$ $(x(T_+))$ uniquely. Forseeing the duality results to be presented in the sequel, we call them almost observability and strong almost observability.

Definition 3.4.

(i) Σ is said to be *almost observable (almost reconstructible)* iff the knowledge of $(y^{(j)})_{[0,T]}$ for $j = 0, 1, \ldots, n$ determines $Ex(0_-)$ $(Ex(T_+))$ uniquely.

(ii) $\Sigma$is said to be *strongly almost observable (strongly almost reconstructible)* iff the knowledge of $(y^{(j)})_{[0,T]}$ for $j = 0, 1, \ldots, n$ determines $x(0_-)$ $(x(T_+))$ uniquely.

Geometric characterizations of these properties are summarized below.

Theorem 3.3.

(i) Σ is almost observable$\Longleftrightarrow E\mathcal{V}^*_{\mathcal{K}} = 0 \Longleftrightarrow \mathcal{V}^*_{\mathcal{K}} = \text{Ker}E \cap \text{Ker}A \cap \text{Ker}C$.

(ii) Σ is strongly almost observable$\Longleftrightarrow \mathcal{V}^*_{\mathcal{K}} = 0$.

Proof. To prove (i), note that sufficiency of the condition $E\mathcal{V}^*_{\mathcal{K}} = 0$ is immediate in the light of the theorem above. To prove necessity, assume that $E\mathcal{V}^*_{\mathcal{K}} \neq 0$. Let $Ex(0_-) \in E\mathcal{V}^*_{\mathcal{K}}$. Then, there exists a Bohl-type trajectory x (see [15, 16]) compatible with $Ex(0_-)$ and which lies in Ker$C$. Let $\tau > 0$ be arbitrary. Define $\xi = \sigma_{-\tau}x$. Clearly, $\xi(0_-) \in$ Ker$C$, satisfies the restrictions $E\xi^{(j)}(0_-) = A\xi^{(j-1)}(0_-)$ and $\xi_{[0,T]}$ lies in Ker$C$ for any $T \geq 0$. Therefore, if $E\mathcal{V}^*_{\mathcal{K}} \neq 0$ then the initial condition cannot be determined from the knowledge of $(y^j)_{[0,T]}$. This proves (i) as the equivalance of the conditions $E\mathcal{V}^*_{\mathcal{K}} = 0$ and $\mathcal{V}^*_{\mathcal{K}} =$ Ker$E\cap$Ker$A\cap$Ker$C$ is trivial. Note that with almost no change this also proves (ii). □

The more interesting (and the more realistic) problem is that of observing the initial condition and reconstructing the final condition from the knowledge of $y_{[0,T]}$ only. Even more significant is the problem of reconstructing the trajectory $x_{[0,T]}$ from the knowledge of $y_{[0,T]}$.

Definition 3.5.

(i) An almost observable system Σ is said to be *observable* (respectively *reconstructible*) if $y_{[0,T]}$ uniquely determines $Ex(0_-)$ (respectively $Ex(T_+)$).

(ii) A strongly almost observable system Σ is said to be *strongly observable* (respectively *strongly reconstructible*) if $y_{[0,T]}$ uniquely determines $x_{[0,T]}$ and $Ex(0_-)$ (respectively $x_{[0,T]}$ and $Ex(T_+)$).

Theorem 3.4.

(i) Σ is simply observable (reconstructible)$\Longleftrightarrow$ $\mathrm{Im}E \cap (E\mathcal{V}_{\mathcal{K}}^* + A\mathcal{R}_{a,\mathcal{K}}^*) = 0 \Longleftrightarrow$ $(\mathcal{V}_{\mathcal{K}}^* + \mathcal{R}_{a,\mathcal{K}}^*) \cap A^{-1}\mathrm{Im}E = \mathrm{Ker}E \cap \mathrm{Ker}A \cap \mathrm{Ker}C$.

(ii) Σ is strongly observable (strongly reconstructible) $\Longleftrightarrow$ $(\mathcal{V}_{\mathcal{K}}^* + \mathcal{R}_{a,\mathcal{K}}^*) \cap A^{-1}\mathrm{Im}E =$ $= 0$.

P r o o f. Since the system is assumed to be almost observable, without loss of generality we may assume that $Ex(0_+)$ is known. Then, $Ex(0_-)$ can be determined iff $\Delta_0 Ex$ can be determined from $y_{[0]}$. Clearly, this can be done iff $\Delta_0 Ex = 0$ is the only jump corresponding to $y_{[0]} = 0$. Now, if $x_{[0]} = x_0\delta + \ldots + x_{-q}\delta^{(q)}$ lies in $\mathrm{Ker}C$ and satisfies $\Sigma_{[0]}$ with some jump $\Delta_0 Ex$ then it follows immediately that (i) $\Delta_0 Ex = Ax_0$, and (ii) $x_{[0]} \subset \mathcal{R}_{a,\mathcal{K}}^*$. Thus, $\Delta_0 Ex \in A\mathcal{R}_{a,\mathcal{K}}^* \cap \mathrm{Im}E$. On the other hand, if $\Delta_0 Ex \in A\mathcal{R}_{a,\mathcal{K}}^* \cap \mathrm{Im}E$ then there exists an $x_0 \in \mathcal{R}_{a,\mathcal{K}}^*$ satisfying $\Delta_0 Ex = Ax_0$. Since $x_0 \in \mathcal{R}_{a,\mathcal{K}}^*$ one can easily construct a sequence $\{x_{-1}, \ldots, x_{-q}\}$ in $\mathcal{R}_{a,\mathcal{K}}^*$ such that $x_{[0]} = x_0\delta + \ldots + x_{-q}\delta^{(q)}$ satisfies $\Sigma_{[0]}$ with $\Delta_0 Ex$. As $x_{[0]}$ lies in $\mathcal{R}_{a,\mathcal{K}}^* \subset \mathrm{Ker}C$, we have $y_{[0]} = 0$. To recap, $y_{[0]}$ determines $\Delta_0 Ex$ uniquely iff $A\mathcal{R}_{a,\mathcal{K}}^* \cap \mathrm{Im}E = 0$. Therefore, in an almost observable system, $Ex(0_-)$ can be determined uniquely iff $A\mathcal{R}_{a,\mathcal{K}}^* \cap \mathrm{Im}E = 0$. Then, $E\mathcal{V}_{\mathcal{K}}^* = 0$ and $A\mathcal{R}_{a,\mathcal{K}}^* \cap \mathrm{Im}E = 0$ can be compactly written as $\mathrm{Im}E \cap (E\mathcal{V}_{\mathcal{K}}^* + A\mathcal{R}_{a,\mathcal{K}}^*) = 0$. As the symmetrical arguement for reconstructibility will be left to the reader, this completes the proof of (i).

To prove (ii), we note that we may safely assume that y is smooth on $(0,T)$ and that the corresponding smooth function-distribution $x_{(0,T)}$ has already been determined uniquely. It remains to consider the problem of determining $x_{[0]}$ and $x_{[T]}$ from $y_{[0]}$ and from $y_{[T]}$. Only the first problem will be tackled and the second one will be left to the reader. Now, $y_{[0]}$ determines $x_{[0]}$ uniquely iff the only $x_{[0]} \in \mathrm{Ker}C$ which satisfies $\Sigma_{[0]}$ for some $\Delta_0 Ex$ is the trivial one, i.e., $x_{[0]} = 0$. Clearly, a necessary condition for this to happen is that no $x_0 \in \mathcal{R}_{a,\mathcal{K}}^*$ satisfies $Ax_0 = \Delta_0 Ex$ for any possible $\Delta_0 Ex$. That is to say, a necessary condition is $A\mathcal{R}_{a,\mathcal{K}}^* \cap \mathrm{Im}E = 0$. In case the system is strongly almost controllable, i.e., $\mathcal{V}_{\mathcal{K}}^* = \mathrm{Ker}E \cap \mathrm{Ker}A \cap \mathrm{Ker}C =$ 0, this condition is also sufficient. Indeed, if this condition is satisfied then the coefficients of the only possible $x_{[0]}$ satisfy $Ax_0 = 0$; $Ex_0 = Ax_{-1}$; ...; $Ex_{-q+1} = Ax_{-q}$; $Ex_{-q} = 0$. If a subspace recursion is defined by $\mathcal{L}_{k+1} = \mathrm{Ker}C \cap A^{-1}E\mathcal{L}_k$ with $\mathcal{L}_0 = \mathrm{Ker}A \cap \mathrm{Ker}C$, then it immediately follows that $x_j \in \mathcal{L}_j$. Note that the sequence $\{\mathcal{L}_j\}$ is monotone nondecreasing and therefore we have $\mathcal{L}_j \subset \mathcal{L}^*$ for all j where $\mathcal{L}^*$ denotes the limit of the recursion. However, it is immediate that $\mathcal{L}^* \subset \mathcal{V}_{\mathcal{K}}^* = 0$. Thus, we have $\mathcal{L}^* = 0$. Thus, in a strongly almost observable system, $y_{[0]}$ uniquely determines $x_{[0]}$ iff $A\mathcal{R}_{a,\mathcal{K}}^* \cap \mathrm{Im}E = 0$. Again, this is a necessary and sufficient condition for identification of $x_{[T]}$ also. Thus, it is necessary and sufficient for determining $x_{[0,T]}$ from $y_{[0,T]}$ in a strongly almost observable system. Hence the proof. □

We now present the strongest observability/reconstructibility definition.

Definition 3.6. A strongly observable system Σ is said to be *strongly completely observable* (respectively *strongly completely reconstructible*) if $y_{[0,T]}$ and $y(0_-)$ (respectively $y_{[0,T]}$ and $y(T_+)$) uniquely determine $x(0_-)$ (respectively $x(T_+)$).

Theorem 3.5. Σ is strongly completely observable (strongly completely reconstructible) $\iff \mathcal{V}_{\mathcal{K}}^* + \mathcal{R}_{a,\mathcal{K}}^* = 0$.

Proof. Note that strong observability grants that $x(0_+)$ can be identified from the output data. Thus, $x(0_-)$ can be determined iff $\Delta_0 x$ can be determined from $y_{[0,T]}$ and from $y(0_-)$. Clearly, since $y(0_+)$ is also given, the problem is that of identifying $\Delta_0 x$ using the knowledge of $y_{[0]}$ and $\Delta_0 y$. $\Delta_0 x$ can be determined uniquely from the knowledge of $\Delta_0 x$ and $y_{[0]}$ iff the only jump $\Delta_0 x$ which is compatible with a trajectory $x_{[0]} = x_0\delta + \ldots + x_{-q}\delta^q$ in Ker$C$, and which satisfies $C\Delta_0 x = 0$ is the trivial one. Now, it is trivial to show that a jump $\Delta_0 x$ satisfies $C\Delta_0 y = 0$ and is compatible with some $x_{[0]} \in$ Ker$C$ iff $\Delta_0 x \in \mathcal{R}_{a,\mathcal{K}}^*$. Hence the proof. □

Note that in any one of the properties defined above the stronger version is nothing but the plain version together with the condition Ker$E\cap$Ker$A\cap$Ker$C = 0$. This fact and the observation that the geometric condition $E\mathcal{V}_{\mathcal{K}}^* = A\mathcal{R}_{a,\mathcal{K}}^* = 0$, when augmented by the condition Ker$E\cap$Ker$A\cap$Ker$C = 0$, is equivalent to strong complete observability motivates the following *ad hoc* definition. We would like to emphasize the fact that it is a formal rather than a dynamical definition.

Definition 3.7. An observable (respectively reconstructible) system is said to be completely observable (respectively completely constructible) iff $E\mathcal{V}_{\mathcal{K}}^* + A\mathcal{R}_{a,\mathcal{K}}^* = 0$.

3.2. Observability and reconstructibility of $\Sigma_{[0,\infty)}$

Having finished our treatment of the system $\Sigma_{[0,T]}$ which was assumed to be the restriction of Σ to $[0,T]$, we now consider the same system as the restriction of $\Sigma_{[0,\infty)}$ to the same interval. This is equivalent to saying that no information whatsoever may be assumed about its past. Thus, no information about the "values" of the output at 0_- can be assumed to be given. However, information at T_+ may be assumed to be given (the reader should keep in mind the problem of smoothing). In this case, all definitions of reconstructibility apply. However, the only type of observability which makes sense is that of observability (together with its stronger version) as it is the only definition which does not assume the knowledge of $y(0_-)$.

Theorem 3.6.

(i) $\Sigma_{[0,\infty)}$ is instantaneously reconstructible with k differentiations $\iff E\mathcal{V}_K^{k-1} = 0 \iff \mathcal{V}_K^{k-1} \subset$ Ker$E \cap$ Ker$C$.

(ii) $\Sigma_{[0,\infty)}$ is reconstructible with k differentiations $\iff E\mathcal{V}_K^{k-1} \cap (E\mathcal{V}_\mathcal{K}^* + A\mathcal{R}_{a,\mathcal{K}}^*) = 0 \iff \mathcal{V}_K^{k-1} \cap (\mathcal{V}_\mathcal{K}^* + \mathcal{R}_{a,\mathcal{K}}^*) \subset \mathrm{Ker}E \cap \mathrm{Ker}C$.

(iii) $\Sigma_{[0,\infty)}$ is almost reconstructible $\iff E\mathcal{V}_\mathcal{K}^* = 0 \iff \mathcal{V}_\mathcal{K}^* \subset \mathrm{Ker}E \cap \mathrm{Ker}C$.

(iv) $\Sigma_{[0,\infty)}$ is reconstructible $\iff \mathrm{Im}E \cap (E\mathcal{V}_\mathcal{K}^* + A\mathcal{R}_{a,\mathcal{K}}^*) = 0 \iff A^{-1}\mathrm{Im}E \cap (\mathcal{V}_\mathcal{K}^* + \mathcal{R}_{a,\mathcal{K}}^*) \subset \mathrm{Ker}E \cap \mathrm{Ker}A \cap \mathrm{Ker}C$.

(v) $\Sigma_{[0,\infty)}$ is completely reconstructible $\iff E\mathcal{V}_\mathcal{K}^* + A\mathcal{R}_{a,\mathcal{K}}^* = 0 \iff \mathcal{V}_\mathcal{K}^* + \mathcal{R}_{a,\mathcal{K}}^* \subset \mathrm{Ker}E \cap \mathrm{Ker}A \cap \mathrm{Ker}C$.

Theorem 3.7.

(i) $\Sigma_{[0,\infty)}$ is strongly instantaneously reconstructible with k differentiations $\iff \mathcal{V}_K^{k-1} = 0$.

(ii) $\Sigma_{[0,\infty)}$ is strongly reconstructible with k differentiations $\iff \mathcal{V}_K^{k-1} \cap (\mathcal{V}_\mathcal{K}^* + \mathcal{R}_{a,\mathcal{K}}^*) = 0$.

(iii) $\Sigma_{[0,\infty)}$ is strongly almost reconstructible $\iff \mathcal{V}_\mathcal{K}^* = 0$.

(iv) $\Sigma_{[0,\infty)}$ is strongly reconstructible $\iff A^{-1}\mathrm{Im}E \cap (\mathcal{V}_\mathcal{K}^* + \mathcal{R}_{a,\mathcal{K}}^*) = 0$.

(v) $\Sigma_{[0,\infty)}$ is strongly completely reconstructible $\iff \mathcal{V}_\mathcal{K}^* + \mathcal{R}_{a,\mathcal{K}}^* = 0$.

Theorem 3.8.

(i) $\Sigma_{[0,\infty)}$ is observable $\iff \mathrm{Im}E \cap (E\mathcal{V}_\mathcal{K}^* + A\mathcal{R}_{a,\mathcal{K}}^*) = 0 \iff A^{-1}\mathrm{Im}E \cap (\mathcal{V}_\mathcal{K}^* + \mathcal{R}_{a,\mathcal{K}}^*) \subset \mathrm{Ker}E \cap \mathrm{Ker}A \cap \mathrm{Ker}C$.

(ii) $\Sigma_{[0,\infty)}$ is strongly observable $\iff A^{-1}\mathrm{Im}E \cap (\mathcal{V}_\mathcal{K}^* + \mathcal{R}_{a,\mathcal{K}}^*) = 0$.

3.3. Observability and reconstructibility of $\Sigma_{[0,T]}$

Finally, we consider $\Sigma_{[0,T]}$ not as a restriction of some system to a finite interval but as a separate entitiy of its own. In this case, the output is a distribution with support $[0, T]$. Thus, no information about the "values" of the output at 0_- or at T_+ may be assumed to be given. Then, being the only property which does not require the "values" of the output at the end points, plain observability and its strong version are the only relevant observability properties and they are equivalent to the corresponding reconstructibility properties.

Theorem 3.8.

(i) $\Sigma_{[0,T]}$ is observable (equivalently, reconstructible) $\iff \mathrm{Im}E \cap (E\mathcal{V}_\mathcal{K}^* + A\mathcal{R}_{a,\mathcal{K}}^*) = 0 \iff A^{-1}\mathrm{Im}E \cap (\mathcal{V}_\mathcal{K}^* + \mathcal{R}_{a,\mathcal{K}}^*) \subset \mathrm{Ker}E \cap \mathrm{Ker}A \cap \mathrm{Ker}C$.

(ii) $\Sigma_{[0,T]}$ is strongly observable (equivalently, strongly reconstructible) $\iff A^{-1}\mathrm{Im}E \cap (\mathcal{V}_\mathcal{K}^* + \mathcal{R}_{a,\mathcal{K}}^*) \subset \mathrm{Ker}E \cap \mathrm{Ker}A \cap \mathrm{Ker}C$.

4. DUALITY

Although only a few of them are used to prove the duality relations among different structural properties , we nevertheless present a number of duality results hoping that they will be of independent interest. To that end, let us define the following recursions:

$$
\begin{array}{ll}
\mathcal{V}_{k+1} = \mathrm{Ker}C \cap A^{-1}\{E\mathcal{V}_k + \mathrm{Im}B\} & ; \ \mathcal{V}_0 = \mathcal{X} \\
\tilde{\mathcal{V}}_{k+1} = \mathrm{Ker}C \cap A^{-1}\{E\tilde{\mathcal{V}}_k + \mathrm{Im}B\} & ; \ \tilde{\mathcal{V}}_0 = \mathrm{Ker}C \\
\mathcal{R}_{a,k+1} = \mathrm{Ker}C \cap E^{-1}\{A\mathcal{R}_{a,k} + \mathrm{Im}B\} & ; \ \mathcal{R}_{a,0} = 0 \\
\tilde{\mathcal{R}}_{a,k+1} = \mathrm{Ker}C \cap E^{-1}\{A\tilde{\mathcal{R}}_{a,k} + \mathrm{Im}B\} & ; \ \tilde{\mathcal{R}}_{a,0} = \mathrm{Ker}C \cap \mathrm{Ker}E \\
\mathcal{S}_{k+1} = \mathrm{Im}B + A\{E^{-1}\mathcal{S}_k \cap \mathrm{Ker}C\} & ; \ \mathcal{S}_0 = 0 \\
\tilde{\mathcal{S}}_{k+1} = \mathrm{Im}B + A\{E^{-1}\tilde{\mathcal{S}}_k \cap \mathrm{Ker}C\} & ; \ \tilde{\mathcal{S}}_0 = \mathrm{Im}B \\
\mathcal{N}_{a,k+1} = \mathrm{Im}B + E\{A^{-1}\mathcal{N}_{a,k} \cap \mathrm{Ker}C\} & ; \ \mathcal{N}_{a,0} = \underline{\mathcal{X}} \\
\tilde{\mathcal{N}}_{a,k+1} = \mathrm{Im}B + E\{A^{-1}\tilde{\mathcal{N}}_{a,k} \cap \mathrm{Ker}C\} & ; \ \tilde{\mathcal{N}}_{a,0} = \mathrm{Im}E + \mathrm{Im}B
\end{array}
$$

The relations between these subspaces are summarized by the next proposition. Its proof is a trivial exercise in mathematical induction and will be left to the reader.

Proposition 4.1.

(i) $\tilde{\mathcal{N}}_{a,k} = E\mathcal{V}_k + \mathrm{Im}B$ and $\mathcal{N}_{a,k+1} = E\tilde{\mathcal{V}}_k + \mathrm{Im}B$.

(ii) $\mathcal{V}_{k+1} = A^{-1}\tilde{\mathcal{N}}_{a,k} \cap \mathrm{Ker}C$ and $\tilde{\mathcal{V}}_k = A^{-1}\mathcal{N}_{a,k} \cap \mathrm{Ker}C$.

(iii)- $\tilde{\mathcal{S}}_k = A\mathcal{R}_{a,k} + \mathrm{Im}B$ and $\mathcal{S}_{k+1} = A\tilde{\mathcal{R}}_{a,k} + \mathrm{Im}B$.

(iv)- $\mathcal{R}_{a,k+1} = E^{-1}\tilde{\mathcal{S}}_k \cap \mathrm{Ker}C$ and $\tilde{\mathcal{R}}_{a,k} = E^{-1}\mathcal{S}_k \cap \mathrm{Ker}C$.

Now, we let $\perp$ denote denote the orthogonal complement and note the following duality relations. The proof is very standard and will be omitted.

Proposition 4.1.

(i) $\mathcal{V}_k^{\perp}(C,E,A,B) = \mathcal{S}_k(B',E',A',C')$

(ii) $\tilde{\mathcal{V}}_k^{\perp}(C,E,A,B) = \tilde{\mathcal{S}}_k(B',E',A',C')$

(iii) $\mathcal{R}_{a,k}^{\perp}(C,E,A,B) = \mathcal{N}_{a,k}(B',E',A',C')$

(iv) $\tilde{\mathcal{R}}_{a,k}^{\perp}(C,E,A,B) = \tilde{\mathcal{N}}_{a,k}(B',E',A',C')$

4.1. Duality for Σ

We can now start our discussion of the duality relations between different structural properties defined in the previous sections. We first note that for the system Σ defined over the real line by $(0, E, A, B)$, the only dynamically significant controllability/reachability property was that of complete controllability which was characterized by the condition $\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{R}_a^* = \mathcal{X}$ and its stronger version which was characterized by $E\mathcal{R}^* + A\mathcal{R}^* + \mathrm{Im}B = \underline{\mathcal{X}}$. Now, let Σ' denote the dual system defined over the real-line by the quadruple $(B', E', A', 0)$.

Theorem 4.1.

(i) Σ is completely reachable (or, equivalently, completely controllable) iff Σ' is completely observable (or, equivalently, completely reconstructible).

(ii) Σ is completely strongly reachable (or, equivalently, completely strongly controllable) iff Σ' is completely strongly observable (or, equivalently, completely strongly reconstructible).

Remark. Note that the definition of complete observability was not motivated by dynamical reasoning and was put forward in a somewhat *ad hoc* manner. Thus, the dynamically meaningful duality is between strong complete reachability and strong complete observability. Also note that in case $\mathrm{Im}E + \mathrm{Im}A + \mathrm{Im}B = \underline{\mathcal{X}}$ then the distinction between the plain and the strong versions of these properties disappear and one is left with only one and meaningful duality result.

Proof. Σ is completely controllable iff $\mathcal{R}^* = \mathcal{X}$, or equivalently, iff $(\mathcal{R}^*)^\perp = 0$. Note that $(\mathcal{R}^*)^\perp = (\mathcal{V}^* \cap \mathcal{R}_a^*)^\perp = (\mathcal{V}^*)^\perp + (\mathcal{R}_a^*)^\perp = E'(\mathcal{V}^*)' + A'(\mathcal{R}_a^*)'$ where $(\mathcal{V}^*)'$ and $(\mathcal{R}_a^*)'$ denote the $\mathcal{V}^*$ and $\mathcal{R}_a^*$ of the dual system Σ' defined by $(B', E', A', 0)$. Therefore, $(\mathcal{R}^*)^\perp = 0$ iff $E'(\mathcal{V}^*)' + A'(\mathcal{R}_a^*)' = 0$. However, this is exactly the condition for complete observability/reconstructibility of the dual system. This proves (i).

To prove (ii), we note that Σ is strongly completely controllable iff $E\mathcal{R}^* + \mathrm{Im}B = A\mathcal{R}^* + \mathrm{Im}B = \underline{\mathcal{X}}$, or equivalently iff $(E')^{-1}(\mathcal{R}^*)^\perp \cap \mathrm{Ker}B' = 0$. Note that $(E')^{-1}(\mathcal{R}^*)^\perp \cap \mathrm{Ker}B' = (E')^{-1}\{E'(\mathcal{V}^*)' + A'(\mathcal{R}_a^*)'\} \cap \mathrm{Ker}B' = \{(\mathcal{V}^*)' + (E')^{-1}A'(\mathcal{R}_a^*)'\} \cap \mathrm{Ker}B' = (\mathcal{V}^*)' + (\mathcal{R}_a^*)'$. Thus, $E\mathcal{R}^* + \mathrm{Im}B = \underline{\mathcal{X}}$ iff $(\mathcal{V}^*)' + (\mathcal{R}_a^*)' = 0$. But, this is exactly the condition for strong complete observability/reconstructibility of Σ'. □

Note that other definitions of observability like almost observability, plain observability etc. have also been shown to have dynamical significance; however, they do not enter the duality picture for Σ.

4.2. Duality for $\Sigma_{[0,\infty)}$

The reader should recall that three different controllability properties (almost controllability, controllability and complete controllability) and their stronger versions have been introduced and characterized for $\Sigma_{[0,\infty)}$. On the other hand, it was discussed that only one of these properties (namely, complete controllability) and its stronger version would admit reachability interpretations. Similarly, for the homogenous system with observations, three distinct reconstructibility definitions (almost reconstructibility, reconstructibility and complete reconstructibility) and their stronger versions were shown to have dynamical significance; however, only one of these definitions (namely plain reconstructibility) and its stronger version would admit observability interpretations. Thus, it should be expected that controllability and reconstructibility are duals, and so are reachability and observability. As shown below, the first conjecture is true. That is to say, controllability and reconstructibility are indeed dual properties. However, duality between reachability and observability is problematic in the sense that the dual of the only meaningful reachability property turns out to be stronger than the only meaningful observability property for the dual system.

We summarize the duality results for $\Sigma_{[0,\infty)}$ below. The proofs follow trivially from the characterizations of the properties and from the duality results of the relevant subspaces given above.

Theorem 4.2.

(i) $\Sigma_{[0,\infty)}$ is kth-order instantaneously controllable (kth-order instantaneously strongly controllable) iff $\Sigma'_{[0,\infty)}$ is instantaneously reconstructible with k differentiations (strongly instantaneously reconstructible with k differentiations).

(ii) $\Sigma_{[0,\infty)}$ is kth-order controllable (kth-order strongly controllable) iff $\Sigma'_{[0,\infty)}$ is reconstructible with k differentiations (strongly reconstructible with k differentiations).

(iii) $\Sigma_{[0,\infty)}$ is almost controllable (strongly almost controllable) iff $\Sigma'_{[0,\infty)}$ is almost reconstructible (strongly almost reconstructible).

(iv) $\Sigma_{[0,\infty)}$ is controllable (strongly controllable) iff $\Sigma'_{[0,\infty)}$ is reconstructible (strongly reconstructible).

(v) $\Sigma_{[0,\infty)}$ is completely controllable (strongly completely controllable) iff $\Sigma'_{[0,\infty)}$ is completely reconstructible (strongly completely reconstructible).

4.3. Duality for $\Sigma_{[0,T]}$

For the system defined over only some finite interval, it was shown that different definitions of controllability and reachability were possible and the corresponding controllability and reachability properties were equivalent. That is to say, the need to make a distinction between controllability and reachability disappeared for $\Sigma_{[0,T]}$. On the other hand, for the homogenous system with observations defined over a finite interval, it was shown that there was only one meaningful way to define reconstructibility (namely, plain reconstructibility) which also happened to be equivalent to the only meaningful definition of observability (which was that of plain observability). Stronger versions of these definitions were also given. Thus, there is basically one duality result which is presented next.

Theorem 4.3.

(i) $\Sigma_{[0,T]}$ is controllable (equivalently, reachable) iff $\Sigma'_{[0,T]}$ is reconstructible (equivalently, observable).

(ii) $\Sigma_{[0,T]}$ is strongly controllable (equivalently, strongly reachable) iff $\Sigma'_{[0,T]}$ is strongly reconstructible (equivelently, strongly observable).

5. DISCUSSIONS

In this paper, we have discussed structural properties for singular systems. It was shown that the structural properties depend not only on C, E, A, B but also on the interval over which the system equations are defined. An interesting exercise, yet undone, is to compare the properties introduced in here with the various properties introduced for discrete-time systems (see [9] and the references therein). For a review of existing definitions of structural properties, see [12]. We remark that the need for introducing the strong versions of various properties is because the condition $\mathrm{Ker}E \cap \mathrm{Ker}A \cap \mathrm{Ker}C = 0$ is not satisfied in general, or equivalently, the state-output pencil $[sE' - A', C']'$ may have zero column minimal indices. Clearly, existence of zero c.m.i's reflects the existence of redundant variables (see [13]). However, since no assumption about the squareness of the system is made, there is no loss of generality in working with the system defined over the restricted domain $\mathcal{X}/(\mathrm{Ker}E \cap \mathrm{Ker}A \cap \mathrm{Ker}C)$ in case the kernels intersect nontrivially. This would reduce the number of properties involved by half. The dual situation is also valid in the controlled system. Existence of zero row minimal indices of the input-state pencil $[sE - A, B]$ results in proper containment of $\mathrm{Im}E + \mathrm{Im}A + \mathrm{Im}B$ in $\underline{\mathcal{X}}$. This reflects the fact some of the equations are redundant. Again, since no assumption about the squareness of the system is made, there is no loss of generality in working with the system restricted in the codomain to $\mathrm{Im}E + \mathrm{Im}A + \mathrm{Im}B$, in which case the number of different controllability/reachability properties are reduced by half.

Duality between different properties introduced in the paper have also been established. It is interesting to note that for Σ not all of the controllability propeties have meaningful duals, and for $\Sigma_{[0,T]}$ not all of the meaningful reconstructibility properties have meaningful duals. The most complete duality picture exists for $\Sigma_{[0,\infty)}$. However, in this case, it is somewhat disturbing to note that the only dynamically significant reachability definition does not admit a dual observability definition, and the dual of the only dynamically significant observability condition does not admit a reachability interpretation. It seems to be possible to remedy this situation by providing a reachability interpretation of what was called plain controllability by relaxing the way the extension of the trajectory $x_{[0,T]}$ is defined. However, this point will not be elaborated here.

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