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MA REPRESENTATION OF ℓ_2 2D SYSTEMS

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In this paper we study the representation of 2D systems with ℓ_2 signals. Starting from a (deterministic) 2D AR model, we investigate under which conditions there exists an alternative description of the MA type. Such a description is further used in order to obtain 2D state space model for the given system.

1. INTRODUCTION

In the behavioral approach a system is characterized by the way that it interacts with the environment through its, so-called, external variables. These variables are all considered to be at a same level, since there is no a priori division into inputs and outputs. The system laws can then be expressed by means of relationships between the external variables; this yields a set of admissible external signals known as the system behavior. A system for which all the admissible signals are square summable sequences over $\mathbb{Z}^2$ is called an ℓ_2 2D system.

An interesting class of 2D systems is associated with the class $\mathbb{B}^q$ of linear, shift-invariant, closed 2D behaviors in q variables. Representation results of such behaviors have been derived in [5] and [6]. Particularly, $\mathbb{B}^q$ coincides with the family of 2D AR behaviors (that can be described as the kernel of a polynomial operator $R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$ in the 2D shifts and their inverses).

In this paper we consider ℓ_2 systems obtained by imposing a square summability condition to the trajectories of the behaviors in $\mathbb{B}^q$. These systems will be called ℓ_2 AR systems. We are concerned with the existence of suitable descriptions for such systems. Namely, we investigate whether or not it is possible to represent an ℓ_2 AR behavior $\mathcal{B}$ as the image of a polynomial operator $M(\sigma_1, \sigma_1, \sigma_1^{-1}, \sigma_2^{-1})$ acting on an ℓ_2 space, instead of representing it as a kernel. (Such an image representation is also called an MA description). In this case $\mathcal{B}$ can be generated as the output behavior of a 2D quarter-plane causal FIR filter driven by free ℓ_2 inputs. Such a description is of particular interest for the construction of state space realizations.

We will show by means of an example that ℓ_2 MA representations cannot always be obtained. However, it turns out that a broad class of ℓ_2 AR systems allows for such representations.

2. PRELIMINARIES

We start by introducing some basic definitions and results that will be useful in the sequel.

We consider discrete 2D systems $\Sigma = (T, W, \mathcal{B})$ in q variables, with trajectories defined over the domain $T = \mathbb{Z}^2$ and taking their values on $W = \mathbb{R}^q$. The set $\mathcal{B} \subseteq \{w : \mathbb{Z}^2 \to \mathbb{R}^q\} =: (\mathbb{R}^q)^{\mathbb{Z}^2}$ specifies which are the admissible system signals, and constitutes the system behavior. We remark that in this characterization of Σ the system variables are stacked together in a q-dimensional vector w instead of being split into inputs and outputs. Thus we do not impose an input-output structure in the signal components.

The behavior $\mathcal{B}$ is said to be shift-invariant if it is invariant under the 2D shift-operators and their inverses. These are, as usual, given by $\sigma_1 w(i,j) = w(i+1,j)$, $\sigma_2 w(i,j) = w(i,j+1)$, with the obvious definitions for σ_1^{-1} and σ_2^{-1}. Here we consider the class $\mathbb{B}^q$ of linear, shift-invariant behaviors in q variables which are closed subsets of $(\mathbb{R}^q)^{\mathbb{Z}^2}$ in the topology of pointwise convergence. For this class of systems the following representation result holds.

Proposition 1. [4]: The behavior $\mathcal{B}$ belongs to $\mathbb{B}^q$ if and only if there exists a polynomial matrix $R(s_1, s_2, s_1^{-1}, s_2^{-1})$ such that $\mathcal{B} = \{w : \mathbb{Z}^2 \to \mathbb{R}^q \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})\, w = 0\} =: \ker R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$.

We refer to the equation $R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})\, w = 0$ as a (deterministic) autoregressive (AR) equation, and to the elements of $\mathbb{B}^q$ as AR behaviors.

If the polynomial matrix $R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$ is (factor) left-prime the corresponding behavior $\mathcal{B} := \ker R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$ can alternatively be represented as the image of a polynomial operator $M(\sigma_1^{-1}, \sigma_2^{-1})$ acting on $(\mathbb{R}^p)^{\mathbb{Z}}$ (cf. [6]). Thus $\mathcal{B} = \{w : \mathbb{Z}^2 \to \mathbb{R}^q \mid \exists\, v : \mathbb{Z}^2 \to \mathbb{R}^p \text{ s.t. } w = M(\sigma_1^{-1}, \sigma_2^{-1})\, v\}$, meaning that the trajectories in $\mathcal{B}$ can be obtained as the outputs of the 2D quarter-plane causal FIR filter M driven by the input v.

Based on such a representation the following state space model for $\mathcal{B}$ is easily derived.

$$\begin{aligned} \sigma_1 x_1 &= A_{11} x_1 + B_1 v \\ \sigma_2 x_2 &= A_{21} x_1 + A_{22} x_2 + B_2 v \\ w &= C_1 x_1 + C_2 x_2 + D\, v. \end{aligned} \tag{1}$$

This resembles the well-known separable Roesser model, with the difference that here the "output" consists of the whole system variable w and the "input" is an auxiliary variable v (called the driving-variable).

3. REPRESENTATION OF ℓ_2 AR SYSTEMS

In this section we investigate existence of ℓ_2 MA representations for $\ell_2\, AR$ systems. This guarantees the possibility of realizing at ℓ_2 AR systems by means a state-space model of the form (1) with ℓ_2 state and ℓ_2 driving-variable.

Definition 2. $\Sigma_2 = (\mathbb{Z}^2, \mathbb{R}^q, \mathcal{B}_2)$ is said to be an ℓ_2 AR system if $\mathcal{B}_2 = \mathcal{B} \cap \ell_2^q$, with $\mathcal{B}$ an AR behavior and $\ell_2^q := \{w : \mathbb{Z}^2 \to \mathbb{R}^q \,\|\, \Sigma_{(i,j)\in\mathbb{Z}^2} \|w(i,j)\|^2 < \infty\}$.

Thus, the behavior of an ℓ_2 AR system Σ_2 can be specified as the kernel of a polynomial operator $R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$ acting on ℓ_2^q. This operator is called an ℓ_2 AR representation of Σ_2, and we denote $\Sigma_2(R)$ (and $\mathcal{B}_2 = \mathcal{B}_2(R)$).

A first representation is given in the next proposition.

Proposition 3. If $\mathcal{B}_2$ be an ℓ_2 AR behavior, then there exists a (factor) left-prime polynomial matrix $R(s_1, s_2, s_1^{-1}, s_2^{-1})$ such that $\mathcal{B}_2 = \mathcal{B}(R)$.

Proof. Let $E(s_1, s_2, s_1^{-1}, s_2^{-1})$ be an arbitrary representation of $\mathcal{B}_2$, i.e. $\mathcal{B}_2 = \mathcal{B}_2(E)$. Then E can always be factorized as $E = F\,R$, where F has full column rank and R is a (factor) left-prime polynomial matrix of size $g \times q$. So, $\mathcal{B}_2 = \{w \in \ell_2^q \mid F(Rw) = 0\}$. This means that $w \in \mathcal{B}_2$ if and only if $Rw \in (\ker F \cap \ell_2^g)$. Using the fact that F has full column rank, it is possible to show that $\ker F \cap \ell_2^g = \{0\}$. Hence $w \in \mathcal{B}_2$ if and only if $Rw = 0$, i.e. $\mathcal{B}_2 = \mathcal{B}_2(R)$. □

Given an ℓ_2 AR system $\Sigma_2(R)$ the ℓ_2 MA representation problem can be formulated as follows. Find a polynomial matrix $M(s_1^{-1}, s_2^{-1})$ such that the system behavior $\mathcal{B}(R)$ coincides with the image of the operator $M(\sigma_1^{-1}, \sigma_2^{-1})$ acting on a space ℓ_2^p, for a suitable integer p (i.e. $\mathcal{B}(R) = \{w \mid \exists\, a \in \ell_2^p \text{ s.t. } w = M\,a\}$). This image will be denoted by $\mathrm{im}_2 M$ in order to make a distinction with the image of M viewed as on operator on $(\mathbb{R}^q)^{\mathbb{Z}^2}$ (which is simply denoted by $\mathrm{im}\, M$).

The example below shows that the foregoing problem is not always solvable.

Example 4. Let $\Sigma_2 = (\mathbb{Z}^2, \mathbb{R}^2, \mathcal{B}_2)$ be an ℓ_2 system in two variables such that $\mathcal{B}_2 := \mathcal{B}_2(R)$ and $R(s_1, s_2, s_1^{-1}, s_2^{-1}) := [s_2 - 1 \;\; -(s_1 - 1)]$. So, $\mathcal{B}_2 = \mathcal{B} \cap \ell_2^2$, with $\mathcal{B} := \{w : \mathbb{Z}^2 \to \mathbb{R}^2 \mid w = \mathrm{col}(w_1, w_2)\}$ and $(\sigma_2 - 1)\, w_1 = (\sigma_1 - 1)\, w_2\}$. Since the polynomial matrix R is left-prime, $\mathcal{B}$ has an image representation, namely $\mathcal{B} = \mathrm{im}\, M(\sigma_1^{-1}, \sigma_2^{-1})$, with $M(s_1^{-1}, s_2^{-1}) := \mathrm{col}(s_2^{-1}(1 - s_1^{-1}),\; s_1^{-1}(1 - s_2^{-1}))$. Thus $B_2 = \mathrm{im}\, M \cap \ell_2^2$. However it can be shown that $\mathcal{B}_2 \neq \mathrm{im}_2 M$, and that moreover there does not exists another operator $\overline{M}$ such that $\mathcal{B}_2 = \mathrm{im}_2 \overline{M}$.

A sufficient condition for the existence of an ℓ_2 MA representation is as follows.

Proposition 5. Let $\mathcal{B}_2$ be an ℓ_2 AR behavior, and let $R(s_1, s_2, s_1^{-1}, s_2^{-1})$ be a $g \times q$ (factor) left-prime 2D polynomial matrix such that $\mathcal{B}_2 = \mathcal{B}_2(R)$. Then $\mathcal{B}_2$ allows for an ℓ_2 MA if the following condition is satisfied.

$$\mathrm{rank} R(\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1}) = g \;\; \forall\, (\lambda_1, \lambda_2) \in \mathcal{P} := \{(\lambda_1, \lambda_2) \in \mathbb{C} \times \mathbb{C} \,|\, |\lambda_1| = |\lambda_2| = 1\}. \tag{C}$$

Proof. Since R is factor left-prime, R^T is an irreducible basis (cf. [3]). Let M^T be an irreducible dual basis of R^T. Then, by (C), M must have full column rank over $\mathcal{P}$ (cf. [3], Lemma 2.5). This implies that there exists a $2D$ polynomial matrix L such that $L\,M = N$, with N square, $\det N \not\equiv 0$, and $\det N(\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1}) \neq$

$0 \; \forall \, (\lambda_1, \lambda_2) \in \mathcal{P}$. Given $w \in \mathcal{B}_2$ define a as the ℓ_2 solution of the equation $N\, a = L\, w$. Such a solution always exists since $L\, w$ is ℓ_2 and N is a full row rank polynomial matrix without zeros in $\mathcal{P}$. We now claim that a is such that $w = M\, a$. Clearly, $L(w - M\, a) = 0$; moreover, since M^T is a dual basis of R^T, $RM = 0$ and hence $R(w - M\, a) = 0$. Combining the two equations in $w - M\, a$ yields $S(w - M\, a) = 0$, with $S := \mathrm{col}(R, L)$. Finally, it can be shown that S has full column rank, so that $\ker S \cap \ell_2^q = \{0\}$. This implies that $w = M\, a$, and therefore $\mathcal{B}_2 \subseteq \mathrm{im}_2 M$. The reciprocal inclusion is obvious. □

Corollary 6. Every ℓ_2 2D system $\Sigma_2 = (\mathbb{Z}^2, \mathbb{R}^2, \mathcal{B})$ satisfying the conditions of Proposition 5 can be realized by means of a state model of the form (1) with ℓ_2 driving-variables v and ℓ_2 state trajectories $x := \mathrm{col}(x_1, x_2)$.

Proof. By Proposition 5 $\mathcal{B} = \{w \,|\, \exists\, v \in \ell_2 \text{ s.t. } w = M(\sigma_1^{-1}, \sigma_2^{-1})\, v\}$. Factorizing $M(s_1^{-1}, s_2^{-1})$ as $M(s_1^{-1}, s_2^{-1}) = M_2(s_2^{-1})\, M_1(s_1^{-1})$ shows that $\mathcal{B}$ can be viewed as the output behavior of two 1D FIR filters acting in series and driven by an ℓ_2 input v. The desired 2D realization can be obtained based on 1D realization with ℓ_2 state for M_1 and M_2. For more detail we refer to [6]. □

An ℓ_2 AR behavior $\mathcal{B}_2 = \mathcal{B}(R) \cap \ell_2^q$ is said to have a maximal degree of freedom if the number of ℓ_2 free variables in $\mathcal{B}_2$ equals the number of free variables in $\mathcal{B}(R)$. (This does not happen, for instance, for the behavior $\mathcal{B}_2$ of Example 4.)

It turns out that for ℓ_2 behaviors with a maximal degree of freedom the sufficient condition of Proposition 5 is also necessary.

Theorem 7. Let $\mathcal{B}_2$ be an ℓ_2 AR behavior given by $\mathcal{B}_2 = \mathcal{B}_2(R)$, with R a $g \times q$ left-prime 2D polynomial matrix. Further, assume that $\mathcal{B}_2$ has a maximal degree of freedom. Then $\mathcal{B}_2$ allows for an ℓ_2 MA representation if and only if the condition (C) of Proposition 5 is satisfied.

Proof. Suppose that $\mathcal{B}_2$ has an ℓ_2 MA representation $w = M\, a$. Then M must be a dual basis of R, and its column rank drops wherever the row rank of R does. So, if (C) is not satisfied there exists $(\lambda_1^*, \lambda_2^*) \in \mathcal{P}$ such that every $(q-g) \times (q-g)$ minor of M vanishes at $(\lambda_1^*, \lambda_2^*)$. Assume now, w. l. g., that the first $q-g$ components $\tilde{w}$ of w are free in ℓ_2, and denote by P the $q-g$ first rows of M. Then for every $\tilde{w} \in \ell_2^{(q-g)}$ there must exist $a \in \ell_2^{(q-g)}$ such that $P\, a = \tilde{w}$. In particular P^{-1} should have an ℓ_2 impulse response, which is absurd since $\det P(\lambda_1^*, \lambda_2^*) = 0$. □

Example 8. Let $\mathcal{B} = \mathcal{B}_2(R)$ with $R(s_1, s_2, s^{-1}, s_2^{-1}) := [(1 - s_1)(s_2 - 1) \;\; 2 s_2 s_1 - s_1 - s_2]$. Clearly $\mathcal{B}(R)$ has one free variable. Moreover, it is shown in [1] that the 2D transfer function $t(z_1, z_2) = (z_1 - 1)(z_2 - 1) \,/\, (2 z_2 z_1 - z_1 - z_2)$ has an ℓ_2 impulse response. This implies that the second variable in $\mathcal{B}_2$ is free in ℓ_2, and so $\mathcal{B}_2$ has a maximal degree of freedom. Now, if $\mathcal{B}_2$ has an ℓ_2 MA representation, this must be of the following form:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} (\sigma_1 - 1)(\sigma_2 - 1) \\ 2\sigma_1\sigma_2 - \sigma_1 - \sigma_2 \end{pmatrix} a.$$

However, if w_2 is the 2D impulse there is no ℓ_2 variable a satisfying $(2\sigma_1\sigma_2 - \sigma_1 - \sigma_2)\, a = w_2$ (since the impulse response of $(2z_1z_2 - z_1 - z_2)^{-1}$ is not in ℓ_2). This shows that $\mathcal{B}_2$ does not allow an ℓ_2 MA representation. □

4. CONCLUSIONS

In this paper we present preliminary results on the solvability of the $\ell_2\, MA$ representation problem for the class of ℓ_2 AR systems. This problem is of particular interest due to its connection with the construction of state space realizations for that class of systems. The necessity of the condition (C) in Proposition 5 for ℓ_2 behaviors without a maximal degree of freedom is still under investigation.

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