

MODEL REDUCTION METHODS FOR CHAOTIC SYSTEMS

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A survey of possible model reduction methods is given for chaotic and other nonlinear systems. Problems specific to the model reduction of nonlinear systems are also presented. This area is important for providing relatively low order models of nonlinear phenomena for analysis, control design and control evaluation.

INTRODUCTION

The essential problem to be addressed in this paper is the model reduction of nonlinear systems with a particular emphasis on dissipative chaotic systems. This problem is significant in at least two areas of study. One is the extraction of the essential dynamics of a system from a given set of higher order equations based on physical principles. For example, spatially distributed systems whose dynamics are described by partial differential equations are inherently infinite dimensional in state space and it is often of interest to extract a set of low order ordinary equations that describe a particular phenomenon. Considerable effort has been applied to problems of this sort through the use of truncated modal expansions, or spectral methods ([18, 21, 5]). One notable example is the Lorenz equations which were obtained via a truncated modal expansion [16].

The second area of application for the model reduction of nonlinear systems is in the area of control systems design. Although there are many classical control design methods that use a very low order controller (one or two states) [15], many of the modern control designs based on state feedback usually require an observer which has the same number of states as the particular system to be controlled [8]. This is difficult when the system to be controlled is spatially distributed and thus requires an infinite order observer. Thus the real difficulty with the modern design approach is that the necessary compensator must have as many states as the particular system model.

With respect to the control of nonlinear systems, many techniques are available. Some techniques operate on a small perturbation model at an operating point, that is, a locally linearized system. The reason for this is that the linear control techniques are well understood, and some are sufficiently robust as to allow fairly large

perturbations [4]. Other techniques seek a low order model that maintains its nonlinear nature to test a given control design over a large range of operating points. This approach is often used for evaluating aircraft and aircraft engine controllers [19]. Thus, controller design for nonlinear systems requires both reduced order linearized systems and reduced order nonlinear systems.

The approach taken in this paper is somewhat general in nature. The viewpoint will be that of the control system designer and the focus is on what this designer's requirements are for a reduced order system. First, general considerations in the reduction process are highlighted and discussed. Many qualitatively important results are discussed with some quantitative results given for selected examples. The last section gives an overview of the currently useful methods for the model reduction of chaotic systems and gives specific examples of the application of these methods. It should be understood that much more work is required in this area.

1. GENERAL ISSUES IN THE MODEL REDUCTION OF CHAOTIC SYSTEMS

It is important to note that model reduction of chaotic systems depends on the desired characteristics the control designer is trying to preserve. In this section we discuss three different characteristics which then lead to a discussion of different model reduction methods.

A. Preservation of attractor stability

Consider a system whose asymptotic solution lies on a chaotic attractor. The control designer may only require that the reduced order model exhibit a stable solution which may or may not have the same chaotic nature as the original system. Further discussion of the subtleties of this requirement are probably best facilitated by considering possible Lyapunov spectra (LCE's).

Consider a chaotic attractor with the Lyapunov spectrum $(+, 0, -, -, \dots, -)$. Model reduction using a standard singular perturbation approach would eliminate the fast states (large negative LCE's) and keep only the more important slow states. Although 'fastness' of a given state cannot necessarily be mapped into a large negative Lyapunov exponent, it does appear to be a reasonable assumption. Unfortunately, the elimination of a very fast state can be destabilizing for a chaotic system because the fastest states significantly contribute to the attractor having a negative divergence. The arbitrary elimination may yield a positive divergence, for example a $(+, 0, -\epsilon, -)$ spectrum becomes $(+, 0, -\epsilon)$ where ϵ is small, and thus the system is unstable. As the later examples will show, although singular perturbation can be effective in reducing the order of a chaotic system, its use requires some caution.

A slightly different problem arises if we consider a system with Lyapunov spectrum $(+, 0, 0, \dots, 0, 0, -, \dots, -, -)$. The multiple zero exponents indicate multiple flow directions on a high dimensional toroidal structure. Clearly it should be possible to eliminate these flow directions from the system and maintain attractor stability as they essentially add nothing to the divergence. Elimination of the flow directions will certainly change the qualitative and quantitative nature, for example

the fractal dimension, of the model's behavior. The question that arises is how important is it to preserve the qualitative nature of the chaotic attractor for control applications.

For spatially distributed chaotic systems represented by partial differential equations, the problem is clearly more difficult. There are theoretically an infinite number of Lyapunov exponents, although it would seem that nature would limit the number of positive ones to be a finite number. Model reduction techniques for such systems may use a combination of singular perturbation and elimination of flow directions (or center manifolds). The problem of mapping the flow directions, as denoted by the LCE's, into the system states is still present. A further problem is that the qualitative nature of the model is now dependent on a particular spatial location.

B. Preserving attractor number and structure

Many systems have multiple attractors or repellers, each having their own dimension and basin of attraction. The specific model reduction technique for such systems will depend on which attractors (repellers) as well as the size of their respective basins should be kept in the reduced order model. For the case where the controlled system may easily travel across basin boundaries, preservation of multiple attractors and basin size would be important. Clearly, the model reduction process requires some decision about what one wants the results to do.

Another problem is the preservation of the bifurcation structure of the system, that is the type of attractor that exists as a given parameter is changed. Since a good control design should be somewhat robust to small parameter variations, a reduced order model should, in some sense, reproduce some of the bifurcation structure of the true plant. The dimension of the attractor that exists for a given parameter is perhaps not as important as having a similar type of bifurcation occur at the given parameter value.

C. Preserving of time series prediction

A reduced order model which closely follows the system state(s) to be controlled would be desirable. This property is especially desirable in adaptive controllers which use an n -step ahead prediction. Long term prediction of a chaotic trajectory is inherently impossible because of sensitivity to initial conditions. Short term prediction has been somewhat successful through the use of very large order models [1].

The question that arises is which metric of a chaotic attractor could be used to develop a reduced order model with the desired characteristic trajectories. Traditionally, chaotic attractors have been characterized by their dimension or their LCE spectrum. Although preservation of fractal information or correlation dimension may result in a useful reduced order model, it is not clear that this corresponds to good time series prediction. For example, many completely different structural attractors can have the same dimension. This is also the case if we preserve the significant aspects of the LCE spectrum or alternatively the average divergence of the system. Note that neither the divergence nor the fractal dimension of the attractor

give a good idea of the size of the attractor in state space, or reduced state space. Perhaps more heuristic methods are needed. Some suggestions for a characteristic to be preserved are the average value, the root-mean-square (RMS) value, the variance, some other norm of the states, or the state vector and interstate correlation. For example, states that have very small RMS values could be replaced by their constant average value, thus reducing the order of the model. Also, states that are strongly correlated with other states could be replaced by these other states, since they are the "same" in nature.

A totally different approach is available from the classical control literature. This will be called preservation of frequency response, and will only be applicable to systems that can be separated into a dynamic linear forward path with a single static nonlinearity in the feedback path. The idea is to reduce the order of the linear forward path while maintaining its frequency response characteristic with respect to the describing function of the static nonlinearity [11, 12, 9]. The problem is that all systems cannot be easily decomposed into this form. The benefit is that many systems can be decomposed into this form, and that readily available linear model reduction methods are available for the dynamic linear forward path system. An example of this is given in the next section.

2. AVAILABLE METHODS AND EXAMPLE APPLICATIONS

This section presents four specific methods for obtaining reduced order models of chaotic systems for control system design. The strengths and weaknesses of each method with regard to preservation of system properties are discussed.

A. Singular perturbation

Singular perturbation is the most used method for the model reduction of high order PDE models via modal truncations. It can be applied directly to a given ODE with the reservation discussed above. Here we present an idealized example which shows the theory of singular perturbation as applied to the modified Lorenz system of [20]. The equations of the modified Lorenz system are as follows:

$$\begin{aligned} \dot{w} &= g(x - w) \\ \dot{x} &= 10 \left(y - \frac{x+w}{2} \right) \\ \dot{y} &= (R - z) \left(\frac{x+w}{2} \right) - y \\ \dot{z} &= y \left(\frac{x+w}{2} \right) - \frac{8z}{3} \end{aligned} \quad (1)$$

with g being a very large positive constant and $R = 28$. The modifications to this system were made so that after singular perturbation was performed, one would obtain the original set of Lorenz equations.

When singular perturbation is applied to a fast state, Ω , the differential equation for that state is multiplied by a small positive constant ε and set equal to zero.

$$\varepsilon \dot{\Omega} = 0. \quad (2)$$

Choosing w as the state to be removed in Eq. (1) yields,

$$\frac{\dot{w}}{g} = 0 \tag{3}$$

and results in

$$w = x. \tag{4}$$

Substituting Eq. (4) into Eq. (1) results in the original Lorenz equations. Letting $R = 28$ and $g = 100$ the fourth order system and the reduced third order system exhibit similar chaotic oscillations. This similarity can be quantified by the LCE spectra (0.909, 0, -14.6) versus (0.919, 0, -15.5, -90) and their Lyapunov dimension, 2.062 versus 2.059.

B. Linear-nonlinear decoupling

This approach essentially requires the decoupling of a dynamic linear part in a forward path and a static nonlinear part in the feedback. When this is possible, a good reduced order model can be obtained by reducing the linear part using standard linear model reduction techniques and closely preserving its frequency response. This approach is applied below to the infinite dimensional time delay system of [11].

This system qualitatively represents the dynamics of a supersonic inlet entering buzz mode. The equations are

$$\varepsilon dx(t)/dt = -x(t) + ax(t - T)[1 - x(t - T)] \tag{5}$$

with $\varepsilon = 0.2$, $a = 4.1$, and $T = 1$. A block diagram of the system is presented in Figure 1. Although the state space representation of this system will be infinite dimensional because of the time delay, T ; the asymptotic attractor has a dimension of only 4.6. A fifth order reduced model of the linear plant is obtained in [11] and is given below.

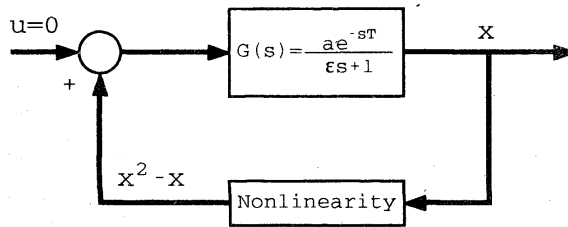


Fig. 1. Block diagram for infinite dimensional system of part B.

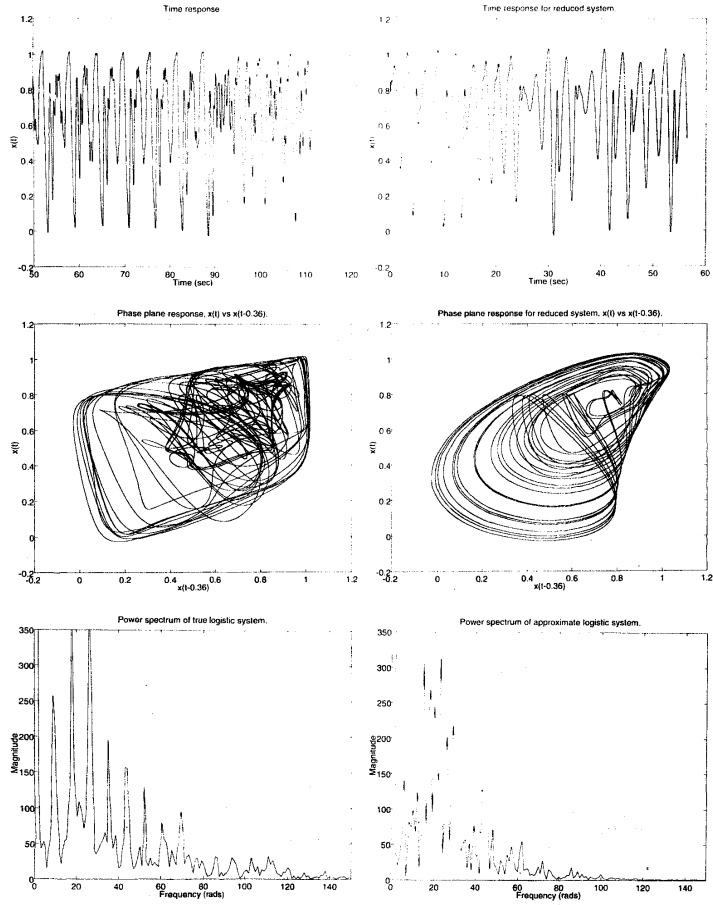


Fig. 2. Plots for the time delay system of Figure 1. (a) Time domain for the full system; (b) Time domain for the reduced order system; (c) Phase plane for the full system; (d) Phase Plane for the reduced order system; (e) Absolute value of the fft for the full system; (f) Absolute value of the fft for the reduced order system.

$$\hat{G}(s) = \frac{-20a(s^3 - 15s^2 + 90s - 210)}{(s^4 + 15.56s^3 + 120s^2 + 480s + 840)(s + 5)}. \quad (6)$$

Figure 2 compares this time response and a phase plot for the original system of Figure 1 and the reduced order model of Equation 6. Both the original infinite dimensional system and the fifth order approximation display qualitatively similar time series behavior.

Interesting, the reduced order model does preserve some simple statistical metrics. For the original system, the mean value of the output is 0.667 with standard deviation 0.243; and for the approximate system the mean value of the output is 0.666 with standard deviation 0.244. Although qualitatively and statistically similar, the phase plane plots indicate that the reduced order system may have a significantly lower dimensional attractor than the original system. The absolute values of the fact Fourier transform are given in Figure 2(e) and (f) and show no particularly interesting differences.

C. Linearization at an operating point

This approach is most applicable to stabilizing a chaotic system to some predefined unstable operating point and thus allowing standard linear small perturbation control design [12]. Although it can be applied to any system, many standard linear model reduction methods are known to have difficulty in reducing unstable systems. In addition, chaotic systems linearized at most operating points in state space will be unstable in the small perturbation sense. This approach is applied here to the infinite dimensional system of Eq. (5).

The first step in solving this problem is to determine the system fixed points. For the system used in example B, the fixed points are determined to be 0 and $(a - 1)/a$. Thus the system linearized about the $(a - 1)/a$ fixed point is

$$\varepsilon \delta \dot{x} = -\delta x + \left(\frac{2 - a}{a}\right) e^{-sT} \delta x + \delta u. \quad (7)$$

This results in a transfer function of the form

$$G(s) = \frac{\frac{1}{\varepsilon}}{s + \frac{1}{\varepsilon} + \frac{a-1}{\varepsilon a} e^{-sT}}. \quad (8)$$

Using a Pade (3,4) approximation for the e^{-sT} term, the reduced order system is determined to be

$$\hat{G}(s) = \frac{5s^4 + 80s^3 + 600s^2 + 2400s + 4200}{s^5 + 21s^4 + 189.67s^3 + 1541.15s^2 + 2317.9s + 6351.24}. \quad (9)$$

Note that this final reduced order linear model does not preserve attractor stability.

Most control system designs for nonlinear systems are based on this method of linearizing the particular system at the desired operating point and then designing

a controller to obtain the desired response. This approach assumes that any deviations from the particular operating point are small with respect to the degree of nonlinearity. One particular problem with using this approach for chaotic systems is choosing when to turn the controller on because it is necessary to have all the states inside the basin of attraction for the desired operating point. Since the underlying system is chaotic, this basin is likely to be fractal thus making it difficult to ensure that all the initial conditions and any deviations from the operating point are in the basin of attraction. This problem of being outside the basin of attraction has been considered in controller design in the aerospace industry via gain scheduling. Here a series of operating points with their controller gains are stored in a computer and are recalled whenever there is a change in operating region (i. e., change to another basin).

D. Removal of insignificant manifolds

This is a somewhat novel approach that attempts to eliminate the states going in the flow direction for a system with several near zero Lyapunov exponents. The method of model reduction in this case relies heavily upon the center manifold theorem. In order to determine which components of the state vector propagate in the flow direction, a set of orthogonal vectors tangent to the chaotic attractor are integrated with time. Periodically, the angles between all of the vectors are examined using a dot product rule. Vectors that propagate in the flow direction should remain close to a 90 degree angle from vectors that are in a diverging direction. The problem of determining which states corresponds on the average most closely to the diverging, flow, and converging directions of the attractor has not been resolved in a general sense.

The seventh order system due to [7] will be used to demonstrate this method.

$$\begin{aligned}
 \dot{x}_1 &= -2x_1 + 4\sqrt{5}x_2x_3 + 4\sqrt{5}x_4x_5 \\
 \dot{x}_2 &= -9x_2 + 3\sqrt{5}x_1x_3 + 3\sqrt{5}x_6x_7 \\
 \dot{x}_3 &= -5x_3 + 9x_1x_7 - 7\sqrt{5}x_1x_2 + R \\
 \dot{x}_4 &= -5x_4 - \sqrt{5}x_1x_5 \\
 \dot{x}_5 &= -x_5 - 3\sqrt{5}x_1x_4 \\
 \dot{x}_6 &= -8x_6 - 4\sqrt{5}x_2x_7 \\
 \dot{x}_7 &= -5x_7 + \sqrt{5}x_2x_6 - 9x_1x_3
 \end{aligned} \tag{10}$$

For $R = 300$, the system exhibits chaos with a Lyapunov dimension of 3.19. Table 1 contains the average cosine of the angles between the tangent vectors after many orbits around the attractors.

Table 1. Cosine of angles between state vectors

	V ₂	V ₃	V ₄	V ₅	V ₆	V ₇
V ₁	.2431	-.193875	.311181	.002186	-.039807	-.092461
V ₂		-0.2355	.062205	-.01384	.03834	.005686
V ₃			-.065598	.003047	.029489	-.025053
V ₄				.010508	-.015768	-.020141
V ₅					.002091	.005847
V ₆						.084183

The largest value of the cosine (which corresponds to the angles furthest from 90°) appears between vectors V₁, V₂, V₃, and V₄. For the system it happens that vectors 5–7 correspond mostly closely to states 5–7 respectively. It is then chosen that states x₅, x₆ and x₇ will be eliminated from the system by replacing them with their average values. The reduced order nonlinear system is thus:

$$\begin{aligned}
 \dot{x}_1 &= -2x_1 + 4\sqrt{5}x_2x_3 + 4\sqrt{5}x_4(7.754) \\
 \dot{x}_2 &= -9x_2 + 3\sqrt{5}x_1x_3 + 3\sqrt{5}(0.966)(-4.72) \\
 \dot{x}_3 &= -5x_3 + 9x_1(-4.72) - 7\sqrt{5}x_1x_2 + R \\
 \dot{x}_4 &= -5x_4 - \sqrt{5}x_1(7.754).
 \end{aligned}
 \tag{11}$$

The reduced order model of Eq. (12) is chaotic (Lyapunov dimension = 2.83). As can be seen from the time domain series of Figure 3, the reduced order model maintains some qualitative relationship with the seventh order system. By observation of the absolute value of the fast Fourier transform in Figure 4, it can be seen that the fourth order model preserves most of the primary harmonics of the seventh order model.

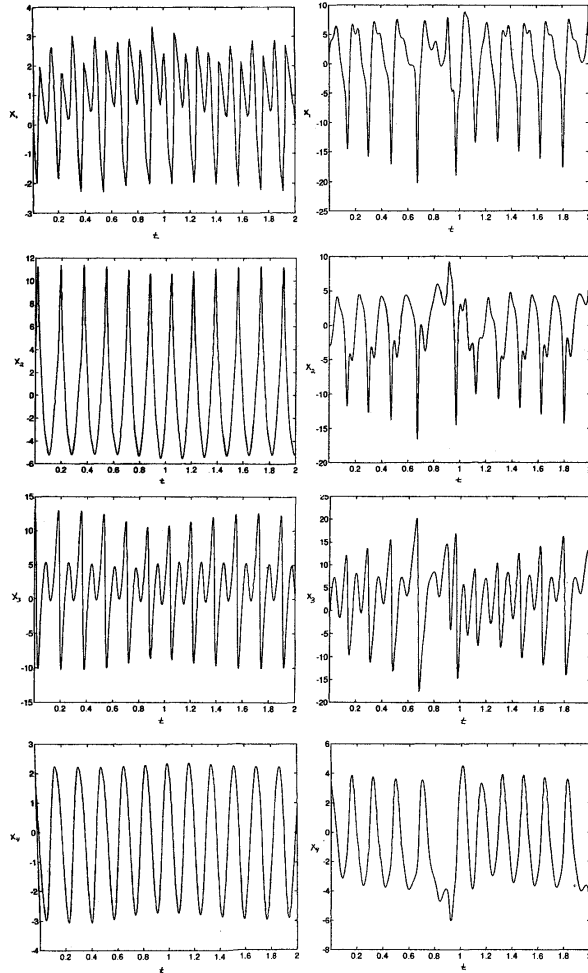


Fig. 3. Time series plots for the model due to Franceschini. (a), (b), (c), and (d) are due to the seventh order model; (e), (f), (g), and (h) are due to the fourth order approximation.

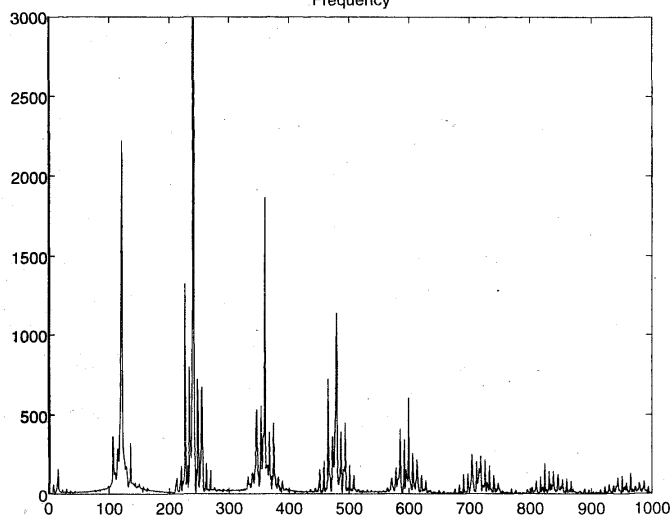
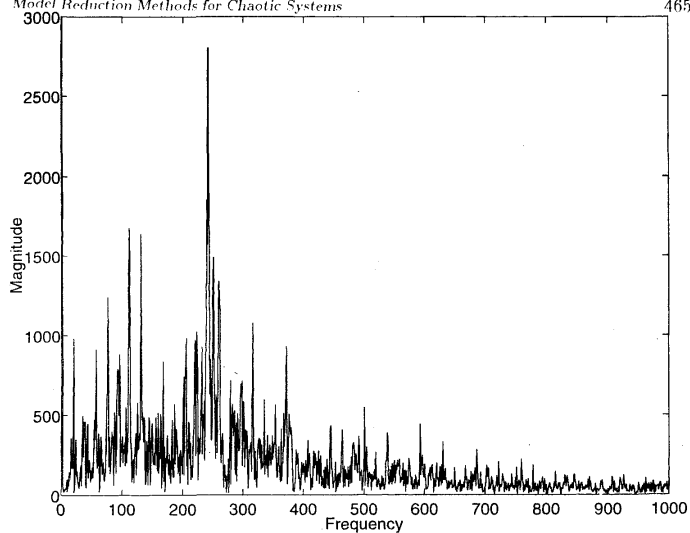


Fig. 4. (a) Absolute value of the fft for the seventh order Franceschini model; (b) Absolute value of the fft for the fourth order approximation of the Franceschini model.

3. CONCLUSIONS

In this paper we have presented four methods for obtaining reduced order models of chaotic systems where the resulting models have preserved some system properties. There are of course, additional considerations in applying these approximate models in control system design. These issues include the effect of nonlinearities, model uncertainty, stabilization of high dimensional attractors, and the robustness of the controlled system. The question of using the reduced order model for control system design will now be briefly considered. For linear dynamic systems, the robust control area has adequately addressed the problem of control with uncertainty in the model [17]. The robust control area also addresses the problem of stabilizing a nonlinear system to a point attractor [17,6]. Here, both model errors and some bounds on the nonlinearity are combined into the uncertainty in order to design a controller that is robust for perturbations of a predetermined size. Unfortunately, very little work has been done in the area of stabilizing an attractor of dimension greater than one, particularly with respect to robustness to model uncertainties. As robust control fundamentally uses an input output representation for control design, perhaps the input output approach (as opposed to state space approach) will be the most successful approach for stabilizing higher dimensional attractors. With this in mind it is suggested that the linear nonlinear decoupling approach discussed above will probably be the most fruitful area for stabilizing higher dimensional attractors. In particular, the methods used in [9] and [12], which are based on harmonic balance, are probably a good place to begin considering robustness of chaotic attractors to modelling errors in the control design.

Table 2. Comparison of discussed methods (preservation of characteristics)

Characteristic to be preserved	Singular perturbation	Decoupling	Linearization at Operation Point	Removal of Manifold
Attractor stability	Maybe	Probably	No	Probably
Attractor number	Maybe	Probably	No	Maybe
Basins	Maybe	Probably	No	Maybe
Bifurcation structure	Probably	Probably	No	Maybe
Attractor dimension	Maybe	Probably	Only at Fixed point	No
Attractor volume	Probably	Probably	Only at fixed point	Maybe
Some frequency characteristics	Probably	Yes	Probably	Maybe
Applicable to most systems	Yes	No	Yes	No
Control chaos to a point	Probably	Yes	Yes	Maybe
Stabilize attractor of dimension > 1	Yes	Yes	Yes	Maybe

The purpose of this paper has been to lay some framework for the development of model reduction methods for chaotic systems. Some ideas on the required characteristics of the reduced order model as applied to controller design have been presented as well as some suggested model reduction techniques. These are cross references in Table 2. It is hoped that this paper will stimulate further work in this area.

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