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FREE END-POINT LINEAR-QUADRATIC CONTROL SUBJECT TO IMPLICIT CONTINUOUS-TIME SYSTEMS: NECESSARY AND SUFFICIENT CONDITIONS FOR SOLVABILITY

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For an *implicit* continuous-time system with arbitrary constant coefficients we derive necessary and sufficient conditions for solvability of the associated free end-point linear-quadratic optimal control problem. In particular, this problem turns out to be solvable if and only if the underlying system is output stabilizable, as is the case for a *standard* system.

1. INTRODUCTION AND PRELIMINARIES

Given the implicit continuous-time system Σ:

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1.1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1.1b}$$

with $u(t) \in \mathbb{R}^m, x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^r$ for all $t \in \mathbb{R}^+ := [0, \infty)$. Let k denote the number of equations in (1.1a) and let $e = \operatorname{rank}(E)$. All matrices involved are real-valued and constant. We may, and hence will, assume that $[EAB]$ is of full row rank. If E is invertible, then the solutions of (1.1a) are

$$x(t) = \exp(E^{-1}At)x_0 + \int_0^t \exp(E^{-1}A(t-\tau))E^{-1}Bu(\tau)\,\mathrm{d}\tau \tag{1.2}$$

($x_0 \in \mathbb{R}^n$ arbitrary) and hence every x_0 is consistent, i.e., for every x_0, (1.1a) has a solution x with $x(0^+) = x_0$. If E is not invertible, however, this need not be the case and inconsistent initial conditions may give rise to impulsive solutions of (1.1a), see e.g. [12], [2]. The most natural way to deal with such phenomena is the use of distributions [11], as was done earlier in e.g. [2]. Instead of (1.1), we will consider its distributional interpretation:

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$$E\delta^{(1)} * x = Ax + Bu + Ex_0\delta, \tag{1.3a}$$

$$y = Cx + Du, \tag{1.3b}$$

where δ, $\delta^{(1)}$ denote the Dirac distribution and its distributional derivative, respectively, $*$ stands for convolution of distributions, $x_0 \in \mathbb{R}^n$, arbitrary. Moreover, $u \in \mathcal{C}_{\text{imp}}^m$, the m-vector version of $\mathcal{C}_{\text{imp}}$, the commutative algebra (over $\mathbb{R}$) of *impulsive-smooth* distributions [10, Def. 3.1], [9]. A distribution is impulsive-smooth if it can be decomposed (uniquely) in an *impulse* (any linear combination of δ and its derivatives $\delta^{(i)}, i \geq 1$) and a *smooth* distribution. A distribution is called smooth if it corresponds to a function that is smooth on $\mathbb{R}^+$ and zero elsewhere. Let $\mathcal{C}_{\text{sm}}$ denote the subalgebra of smooth distributions. The distributional derivative of $u \in \mathcal{C}_{\text{sm}}, u^{(1)} = \delta^{(1)} * u$, equals $\dot{u} + u(0^+)\delta$, where $\dot{u} \in \mathcal{C}_{\text{sm}}$ denotes the ordinary derivative of u on $\mathbb{R}^+$. Example: Let $u \in \mathcal{C}_{\text{sm}}$ correspond to $2\exp(t)$ on $\mathbb{R}^+$. Then $u^{(1)} = \dot{u} + 2\delta$. For more details on $\mathcal{C}_{\text{imp}}$, see [9]–[10], also [6]–[8]; because of its nice properties we can keep our treatment fully *algebraic*. It can be readily shown that, for every real-valued square matrix $H, (I\delta^{(1)} - H\delta)$ is invertible (w. r. t. convolution); its inverse corresponds to $\exp(Ht)$ on $\mathbb{R}^+$. Hence the solutions of (1.3a) reduce to the ordinary ones ((1.2)) if E is invertible and $u \in \mathcal{C}_{\text{sm}}^m$; for every pair (x_0, u), (1.3a) has exactly one solution. Also, note that (1.3a) reduces to (1.1a) if u and x are smooth. In general, however, the *solution set*

$$S(x_0, u) = \left\{x \in \mathcal{C}_{\text{imp}}^n | [E\delta^{(1)} - A\delta] * x = Bu + Ex_0\delta\right\}, \tag{1.4}$$

may be empty or contain infinitely many elements, see [6]. We are ready for the definition of the free end-point linear-quadratic control problem subject to (1.3).

(LQCP)$^-$: For all x_0, determine

$$J^-(x_0) := \inf\left\{\int_0^\infty y'y \, \mathrm{d}t | u \in \mathcal{C}_{\text{sm}}^m, x \in S(x_0, u) \cap \mathcal{C}_{\text{sm}}^n\right\}, \tag{1.5}$$

and if, for every $x_0, J^-(x_0) < \infty$, then compute (if possible) optimal controls $\bar{u} \in \mathcal{C}_{\text{sm}}^m$ and associated optimal state trajectories $\bar{x} \in S(x_0, \bar{u})$. The problem (LQCP)$^-$ is *solvable* if both requirements are met.

In the sequel we will need several subspaces of interest. Let

$$\mathcal{S}(\Sigma) := \left\{x_0 \in \mathbb{R}^n | \exists\, u \in \mathcal{C}_{\text{sm}}^m \exists\, x \in S(x_0, u) \cap \mathcal{C}_{\text{sm}}^n : \lim_{t\to\infty} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} = 0\right\},$$
$$\mathcal{V}_C(\Sigma) := \left\{x_0 \in \mathbb{R}^n | \exists\, u \in \mathcal{C}_{\text{sm}}^m \exists\, x \in S(x_0, u) \cap \mathcal{C}_{\text{sm}}^n : y = 0, x(0^+) = x_0\right\},$$
$$\mathcal{O}(\Sigma) := \left\{x_0 \in \mathbb{R}^n | \exists\, u \in \mathcal{C}_{\text{sm}}^m \exists\, x \in S(x_0, u) \cap \mathcal{C}_{\text{sm}}^n : \lim_{t\to\infty} y(t) = 0\right\} \tag{1.6}$$

and let $\mathcal{S}_B(\Sigma)$, $\mathcal{O}_B(\Sigma)$ denote those subspaces of $\mathcal{S}(\Sigma)$ and $\mathcal{O}(\Sigma)$, for which u and x in the respective definitions are of the Bohl type (a Bohl function is any linear combination of functions $t^k \exp(\lambda t)$, $k \geq 0$). For $\mathcal{V}_C(\Sigma)$ we have the following result.

Proposition 1.1. [7, Prop. 3.5, Theorem 3.6]. $\mathcal{V}_C(\Sigma)$ is the largest subspace $\mathcal{L} \subset \mathbb{R}^n$ for which there exists a matrix $F \in \mathbb{R}^{m \times n}$ such that $(A + BF)\mathcal{L} \subset E\mathcal{L}$, $(C + DF)\mathcal{L} = 0$.

If, moreover,

$$\mathcal{V}(\Sigma) := \{x_0 \in \mathbb{R}^n | \exists\, u \in \mathcal{C}^m_{\text{sm}} \exists\, x \in S(x_0, u) \cap \mathcal{C}^n_{\text{sm}} : y = 0\}, \tag{1.7}$$

then [7, Prop. 3.4] tells us that

$$\mathcal{V}(\Sigma) = \mathcal{V}_C(\Sigma) + \ker(E). \tag{1.8}$$

In [10], [7] a point $x_0 \in \mathcal{V}(\Sigma)$ is called *weakly unobservable*; we establish that all points in $\mathcal{V}_C(\Sigma)$ are also *consistent*. Let, for any subspace T and η any complex row vector of compatible size, ηT stand for $\{\eta t | t \in T\}$. The next result is stated in [3].

Proposition 1.2. Let E be invertible. Then $\mathcal{S}(\Sigma) + \mathcal{V}(\Sigma) = \mathcal{O}(\Sigma) = \{x_0 \in \mathbb{R}^n | J^-(x_0) < \infty\}$, $\mathcal{O}_B(\Sigma) = \mathcal{O}(\Sigma), \mathcal{S}_B(\Sigma) = \mathcal{S}(\Sigma)$ and $\mathcal{O}(\Sigma) = \mathbb{R}^n$ if and only if, for all $\lambda \in \mathbb{C}$ with $Re(\lambda) \geq 0$,

$$\eta[\lambda E - A, -B] = 0 \text{ and } \eta E\mathcal{V}(\Sigma) = 0 \text{ only if } \eta = 0. \tag{1.9}$$

If in Proposition 1.2, $C = I$ and $D = 0$, then $\mathcal{V}(\Sigma) = 0$ and we reobtain the well-known statement that $\mathcal{S}(\Sigma) = \mathbb{R}^n$ if and only if Σ is *(state) stabilizable*. We will say that Σ is *output stabilizable* if $\mathcal{O}(\Sigma) = \mathbb{R}^n$.

Now, we consider Σ with arbitrary E. From [6, Theorem 4.5] we borrow

Proposition 1.3.

$$\begin{aligned} &\forall x_0 \in \mathbb{R}^n \exists\, u \in \mathcal{C}^m_{\text{sm}}\, \exists\, x \in S(x_0, u) \cap \mathcal{C}^n_{\text{sm}} \iff \\ &\text{im}(E) + \text{im}(B) + A(\ker(E)) = \mathbb{R}^k. \end{aligned} \tag{1.10}$$

2. MAIN RESULTS

Without loss of generality, we may rewrite Σ in the form

$$\begin{aligned} &\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \delta^{(1)} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \delta, \\ &y = [C_1\; C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du \end{aligned} \tag{2.1}$$

Assume that (1.10) is satisfied, i.e., that $[A_{22}\ B_2]$ is of full row rank. Let $T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \in \mathbb{R}^{(n+m-e)\times(n+m-k)}$, of full column rank, be such that $[A_{22}\ B_2]T = 0$. Set $N := A_{22}A'_{22} + B_2B'_2 > 0,\ L := T'T > 0$. Then

$$Q := \begin{bmatrix} A'_{22} & T_1 \\ B'_2 & T_2 \end{bmatrix} \text{ is invertible, } Q^{-1} = \begin{bmatrix} N^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} Q'. \tag{2.2}$$

If $\bar{\Sigma}$ denotes the standard system

$$\delta^{(1)} * z = \bar{A}z + \bar{B}v + z_0\delta, \tag{2.3a}$$

$$w = \bar{C}z + \bar{D}v, \tag{2.3b}$$

with

$$\begin{aligned} \bar{A} &:= A_{11} - [A_{12}\ B_1] \begin{bmatrix} A'_{22} \\ B'_2 \end{bmatrix} N^{-1}A_{21}, \bar{B} := [A_{12}\ B_1]T, \\ \bar{C} &:= C_1 - [C_2\ D] \begin{bmatrix} A'_{22} \\ B'_2 \end{bmatrix} N^{-1}A_{21}, \bar{D} := [C_2\ D]T, \end{aligned} \tag{2.3c}$$

then it turns out that all solutions for (1.3) can be expressed in solutions for (2.3) and vice versa.

Theorem 2.1. Let $\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \in \mathbb{R}^n, u \in \mathcal{C}^m_{\text{imp}}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S\left(\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}, u\right)$. Then $x_1 = z(x_{01}, v), \begin{bmatrix} x_2 \\ u \end{bmatrix} = \begin{bmatrix} A'_{22} \\ B'_2 \end{bmatrix} N^{-1}(-A_{21})(z(x_{01}, v)) + Tv$ with $v = L^{-1}[T'_1x_2 + T'_2u] \in \mathcal{C}^{n+m-k}_{\text{imp}}$. Moreover, $y = w(x_{01}, v)$. Conversely, let $z_0 \in \mathbb{R}^e$, $v \in \mathcal{C}^{n+m-k}_{\text{imp}}$, and $z = z(z_0, v)$. Then $u = -B'_2N^{-1}A_{21}z + T_2v \in \mathcal{C}^m_{\text{imp}}$ and, for all x_{02}, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S\left(\begin{bmatrix} z_0 \\ x_{02} \end{bmatrix}, u\right)$ with $x_1 = z$ and $x_2 = -A'_{22}N^{-1}A_{21}z + T_1v$. In addition, $y = w(z_0, v)$.

Proof. First half. If in (2.3a) with $z_0 = x_{01}$ we insert v as prescribed, then $\delta^{(1)} * z = \bar{A}z + [A_{12}\ B_1]Q \begin{bmatrix} N^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \left\{Q' \begin{bmatrix} x_2 \\ u \end{bmatrix} + \begin{bmatrix} A_{21}x_1 \\ 0 \end{bmatrix}\right\} + x_{01}\delta = \bar{A}z + [A_{12}\ B_1] \begin{bmatrix} x_2 \\ u \end{bmatrix} + (A_{11} - \bar{A})x_1 + x_{01}\delta = \bar{A}z + (\delta^{(1)} * x_1 - A_{11}x_1 - x_{01}\delta) + (A_{11} - \bar{A})x_1 + x_{01}\delta = \delta^{(1)} * x_1 + \bar{A}(z - x_1)$, by (2.1)–(2.2). Hence $[I_e\delta^{(1)} - \bar{A}\delta] * (z - x_1) = 0$ and $z - x_1 = 0$. Since $\begin{bmatrix} x_2 \\ u \end{bmatrix} = QQ^{-1} \begin{bmatrix} x_2 \\ u \end{bmatrix} = \begin{bmatrix} A'_{22} \\ B'_2 \end{bmatrix} N^{-1}(-A_{21}x_1) + Tv$, the rest is clear. The second half is now trivial. □

Observe that if in (2.1), $e = k$ (i.e., E is of full row rank), then T is invertible and $\bar{A} = A_{11}$, $\bar{C} = C_1$ in (2.3). Here is our first main result.

Theorem 2.2. If the system (1.3) satisfies (1.10), then $\mathcal{S}(\Sigma) + \mathcal{V}(\Sigma) = \mathcal{O}(\Sigma) = \{x_0 \in \mathbb{R}^n | J^-(x_0) < \infty\}, \mathcal{S}_B(\Sigma) = \mathcal{S}(\Sigma)$ and $\mathcal{O}_B(\Sigma) = \mathcal{O}(\Sigma)$. Moreover, (1.3) is output stabilizable if and only if (1.9)–(1.10) are satisfied.

Proof. Consider (2.1)– (2.3). Then $[\eta_1\ \eta_2]\begin{bmatrix} \lambda I & - & A_{11} & -A_{12} & -B_1 \\ & - & A_{21} & -A_{22} & -B_2 \end{bmatrix} = 0$ if and only if $\eta_1[\lambda I_e - \bar{A}, -\bar{B}] = 0$ and η_2 equals $-\eta_1[A_{12}\ B_1]\begin{bmatrix} A'_{22} \\ B'_2 \end{bmatrix} N^{-1}$, for every $\lambda \in \mathbb{C}$. Since $\ker(E)$ is contained in all subspaces involved, both claims follows immediately from Propositions 1.2, 1.3 and Theorem 2.1. □

Now, let us consider (LQCP)$^-$. By Theorem 2.2, it is obvious that output stabilizability is *necessary* for solvability. Output stabilizability turns out to be *sufficient* for solvability as well.

Theorem 2.3. For every $x_0 \in \mathbb{R}^n, J^-(x_0) < \infty$ if and only if the system (1.3) is output stabilizable. Assume this to be the case. Then there exists a unique real symmetric matrix $P^- \geq 0$, with $\ker(E) \subset \ker(P^-)$, such that, for all x_0, $J^-(x_0) = x_0'P^-x_0$. If

$$\ker\left(\begin{bmatrix} E & 0 \\ C & D \end{bmatrix}\right) \cap [A\ B]^{-1}\mathrm{im}(E) = 0, \tag{2.4}$$

then for every x_0 there exists a unique optimal control $\bar{u}$ and a unique optimal state trajectory $\bar{x} \in S(x_0, \bar{u})$, both of the Bohl type. If (2.4) is not satisfied, then for every x_0 there exist $u \in \mathcal{C}^m_{\text{imp}}$ and $x \in S(x_0, u)$ such that $y \in \mathcal{C}^r_{\text{sm}}$ and $J^-(x_0) = \int_0^\infty y'y\,dt$.

Proof. Assume that Σ is output stabilizable. Consider the subsystem $\bar{\Sigma}$ (2.3), and let $\bar{J}^-(z_0) := \inf\{\int_0^\infty w'w\,dt | v \in \mathcal{C}^{n+m-k}_{\text{sm}}\}$. It follows from Theorem 2.1 that, for every $z_0 \in \mathbb{R}^e, \bar{J}^-(z_0) < \infty$ if and only if, for every $x_0 \in \mathbb{R}^n, J^-(x_0) < \infty$. Hence, by Theorem 2.2, $\bar{\Sigma}$ is output stabilizable. Then there exists a unique $\bar{P}^- \geq 0$ such that, for all $z_0 \in \mathbb{R}^e, \bar{J}^-(z_0) = z_0'\bar{P}^-z_0$ [3]–[4]. Hence there exists a unique $P^- \geq 0$, with $\ker(E) \subset \ker(P^-)$, such that, for every $x_0 \in \mathbb{R}^n, J^-(x_0) = x_0'P^-x_0$. Next, for every z_0 there *exist* a unique input v and (thus) a unique resulting state trajectory z, both of the Bohl type, such that $z_0'\bar{P}^-z_0 = \int_0^\infty w'w\,dt$, if $\ker(\bar{D}) = 0$, i.e., if the LQCP without stability subject to $\bar{\Sigma}$ is *regular* [4]. If $\ker(\bar{D}) \neq 0$, i.e., if this LQCP is *singular*, then for every z_0 there *exist* $v \in \mathcal{C}^{n+m-k}_{\text{imp}}$ and $z \in \mathcal{C}^e_{\text{imp}}$

such that $z_0'\bar{P}^- z_0 = \int_0^\infty w'w \, dt$ [13], [5]; however, in general these optimal controls and optimal state trajectories have nonzero impulsive components. Observe that, in terms of (2.1) – (2.3), $\ker(\bar{D}) = 0$ if and only if $\ker\left(\begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix}\right) = 0$, and it is clear that the latter condition is equivalent to (2.4). The proof is now completed by application of Theorem 2.1.

The condition (2.4) can be interpreted as a system property for Σ. In [8, Theorem 3.2] it is proven that (2.4) holds if and only if

$$y \in \mathcal{C}^r_{\rm sm} \Longleftrightarrow u \in \mathcal{C}^m_{\rm sm}, x \in S(x_0, u) \cap \mathcal{C}^n_{\rm sm}. \tag{2.5}$$

In other words, (2.4) stands for the property that outputs for Σ are *functions* only if the output generating controls and state trajectories are *functions* as well. Therefore (LQCP)$^-$ is called *regular* in [8] if (2.5) is satisfied; note that (2.4) reduces to $\ker(D) = 0$ if E is invertible. The linear-quadratic control problems considered in [1] - [2] are regular in the sense of (2.4), since it is assumed there that $\ker\left(\begin{bmatrix} E & 0 \\ C & D \end{bmatrix}\right) = 0$. An example of a regular linear-quadratic problem for which $\ker(\begin{bmatrix} E & 0 \\ C & D \end{bmatrix}) \neq 0$ is given in [8].

Observe that Theorem 2.3 states the *existence* of the matrix P^-; an explicit *characterization* of P^-, generalizing results in [4] – [5], will be given elsewhere. To the best of our knowledge, Theorem 2.3 contains the first general statements on (possibly) singular linear-quadratic control subject to implicit systems. Also, unlike in [1] – [2], we allow the state trajectories to diverge.

We will conclude this short paper with a by-result on uniqueness of optimal controls and optimal state trajectories for (LQCP)$^-$.

If Σ is output stabilizable and (2.4) is not satisfied, then we may still assume $\begin{bmatrix} E & 0 \\ A & B \\ C & D \end{bmatrix}$ to be of full column rank. Let this be the case. Now the distributional optimal controls and state trajectories for (LQCP)$^-$ (see Theorem 2.3) are in general *not* unique. This follows from Theorem 2.1, since it is proven in [5] that optimal controls and state trajectories for (LQCP)$^-$ subject to a standard system Σ are unique if and only if Σ is left invertible [10, Theorem 3.26], i.e., if in (1.3) with E invertible, $y = 0$ and $x_0 = 0$ imply that $u = 0$ (and hence also $x = 0$). Moreover, the smooth parts of these unique optimal controls and state trajectories are of the Bohl type.

Two different concepts for left-invertibility for *implicit* systems are given in [7]. There, a system (1.3) is defined left invertible in the *strong* sense if $x_0 = 0$ and $y = 0$ imply that $u = 0$ and $Ex = 0$ (and left invertible in the *weak* sense if merely $u = 0$), see [7, Defs. 4.1, 4.10]. Under the above-mentioned rank condition, it is proven in [7, Corollary 4.15] that Σ is left invertible in the strong sense if and only if $x_0 = 0, y = 0$ imply that $u = 0, x = 0$. Hence, again by Theorem 2.1, Σ is left invertible in the strong sense if and only if (2.3) is left invertible in the sense of [10] and thus

Corollary 2.4. Let Σ be output stabilizable and $\ker \left(\begin{bmatrix} E & 0 \\ A & B \\ C & D \end{bmatrix} \right) = 0$. Then for every x_0 there exists exactly one (possibly distributional) $\bar{u}$ and exactly one (possibly distributional) $\bar{x}$ such that $\bar{y} \in \mathcal{C}^r_{\rm sm}$ and $\int_0^{\infty-} \bar{y}'\bar{y}\,dt = J^-(x_0)$ if and only if Σ is left invertible in the strong sense. Moreover, if $\bar{u}_2, \bar{x}_2$ denote the smooth parts of $\bar{u}$ and $\bar{x}$, then $\bar{u}_2$ and $\bar{x}_2$ are of the Bohl type.

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