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A ROBUSTNESS RESULT FOR A VON KÁRMÁN PLATE[1]

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This paper considers the problem of the robustness of boundary feedback controls for a von Kármán plate with respect to a small parameter γ. This parameter enters the problem through a term representing rotational inertia for the plate and is assumed to be quite small (i.e. proportional to the plate's thickness). This paper proves that the exponential decay rates produced for the energy of the total system (with $\gamma \neq 0$) are preserved as we pass with a limit on $\gamma \to 0^+$.

1. INTRODUCTION

1.1. Statement of the Problem

Let Ω be a bounded open domain in R^2 with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_i are relatively open, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. We consider the von Kármán system in the variables $w(t,x)$ and $\chi(w(t,x))$:

$$\left.\begin{array}{ll} w_{tt} - \gamma^2 \Delta w_{tt} + \Delta^2 w + b(x) w_t = [w, \chi(w)] & \text{in } Q \\ w(0,\cdot) = w_0 \;;\; w_t(0,\cdot) = w_1 & \text{in } \Omega \\ w = \frac{\partial}{\partial \nu} w = 0 & \text{on } \Sigma_0 \\ \Delta w + (1-\mu) B_1 w = -\frac{\partial}{\partial \nu} w_t & \text{on } \Sigma_1 \\ \frac{\partial}{\partial \nu} \Delta w + (1-\mu) B_2 w - \gamma^2 \frac{\partial}{\partial \nu} w_{tt} = w_t - \frac{\partial^2}{\partial \tau^2} w_t + w & \text{on } \Sigma_1 \end{array}\right\} \qquad (1.1)$$

where $Q \equiv \Omega \times (0,T)$ and $\Sigma_i \equiv \Gamma_i \times (0,T)$, for $i = 0, 1$. Here, $b(x) \in L^\infty(\Omega)$ satisfies $b(x) > 0$ *a.e.* in Ω, $0 < \mu < \frac{1}{2}$ is Poisson's ratio and the operators B_1 and B_2 are given by

$$\left.\begin{array}{rcl} B_1 w &=& 2 n_1 n_2 w_{xy} - n_1^2 w_{yy} - n_2^2 w_{xx} \\ B_2 w &=& \frac{\partial}{\partial \tau}[(n_1^2 - n_2^2) w_{xy} + n_1 n_2 (w_{yy} - w_{xx})]. \end{array}\right\} \qquad (1.1)(b)$$

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Also, $\chi(w)$ satisfies the system of equations

$$\left.\begin{array}{l}\Delta^2\chi = -[w, w] \\ \chi = \frac{\partial}{\partial\nu}\chi = 0 \qquad \text{on } \Sigma = \Gamma \times (0, \infty)\end{array}\right\} \tag{1.2}$$

where

$$[\phi, \psi] = \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\psi}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2\psi}{\partial x\partial y}.$$

Define the bilinear form

$$a(w, v) = \int_\Omega (\Delta w\Delta v + (1-\mu)(2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx}))\,\mathrm{d}\Omega \tag{1.3}$$

and the energy functional is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega \{|w_t|^2 + \gamma^2|\nabla w_t| + |\Delta\chi|^2\}\,\mathrm{d}\Omega + \frac{1}{2}a(w, w) + \frac{1}{2}\int_{\Gamma_1} w^2\mathrm{d}\Gamma_1 \\ &\equiv E_1(t) + E_2(t), \end{aligned} \tag{1.4}$$

where $E_2(t)$ is defined by

$$E_2(t) = \frac{1}{2}\int_\Omega |\Delta\chi|^2\mathrm{d}\Omega.$$

In [2] it was proven that by implementing stabilizing controls acting through forces w_t and bending moments $\frac{\partial}{\partial\nu}w_t$ and $\frac{\partial^2}{\partial\tau^2}w_t$ along a portion of the plate's edge, we achieve an exponential decay for the energy (1.4). Our goal in this paper is to prove that this exponential decay of energy is "robust" with respect to the parameter γ (i.e. when $\gamma \to 0^+$ the energy for the resulting system also decays exponentially in an appropriate topology). This problem is of interest, since the parameter γ is assumed to be small (proportional to the thickness of the plate). Also, by taking $\gamma = 0$, the resulting limit equation (with zero right hand side) is precisely the Euler-Bernoulli plate model, which is a well-known and frequently studied plate equation from classical mechanics.

1.2. Literature and Orientation

The problem of stabilization and controllability for the von Kármán plate described above (so named for the nonlinear right hand side which was developed by von Kármán) has attracted much attention in recent years. Indeed, the results on local controllability and stabilization can be found in [4, 5, 3, 2]. As for the question of global decay rates (such as we consider here) we refer to [4]. There, it was proved that for the von Kármán model without rotational inertia or viscous damping (i.e. setting $\gamma = 0$ and $b = 0$) the energy of the resulting system is exponentially stable, provided that Ω is star-shaped. This stability was achieved by means of the boundary feedback acting on the whole boundary (i.e. $\Gamma_0 = \emptyset$), where results were achieved by means of a Lyapunov function argument. This contrasts with our approach (see [2]) which is based on proving certain functional relations directly for the energy function. This allows us to "build in" an appropriately developed nonlinear compactness-uniqueness argument to "absorb" nonlinear boundary traces and undesirable lower order terms arising from energy estimates. In order to dispense with geometric conditions, we shall use "sharp" regularity results for the traces of the linear problem, which were proved in [6] by using microlocal analysis.

Our main contribution in this paper is that we produce a cohesive theory tying together the results of [4] and those of [2] by proving that the *same* type of feedbacks will provide similar stability results for both Euler-Bernoulli model of dynamics (i. e. $\gamma = 0$ as in [4]) as well as the Kirchhoff model (i. e. $\gamma > 0$ as in [2]) in their respective appropriate topologies. In producing this result, it is critical that we carefully track the appearance of γ in the estimates which we use to bound the energy (1.4). We must then prove that by passing with a limit as $\gamma \to 0^+$ we do not destroy the exponential decay of the energy.

1.3. Statement of Results

We begin by defining the space of finite energy for (1.1), $\mathcal{H}_\gamma \equiv H^2_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)$ where

$$H^2_{\Gamma_0}(\Omega) = \left\{ w \in H^2(\Omega) : w = \tfrac{\partial}{\partial \nu} w = 0 \text{ on } \Gamma_0 \right\}$$

with norm

$$\|w\|^2_{H^2_{\Gamma_0}(\Omega)} = a(w, w)$$

and

$$H^1_{\Gamma_0}(\Omega) = \{ w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_0 \}$$

with norm

$$\|w\|^2_{H^1_{\Gamma_0}(\Omega)} = \int_\Omega (w^2 + \gamma^2 |\nabla w|^2) \, d\Omega.$$

We note that well-posedness and regularity results for system (1.1) with respect to the topology $\mathcal{H}_\gamma$ was proven in [1].

We now state our main result.

Theorem 1.1. Assume that the domain $\Omega \subset R^2$ has a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ and that there exists $x_0 \in R^2$ such that

$$\mathbf{h} \cdot \nu \equiv (x - x_0) \cdot \nu \le 0 \quad \text{for } x \in \Gamma_0. \tag{1.5}$$

Then for any initial data, $(w_0, w_1) \in \mathcal{H}_\gamma$ there exist constants C and α which are independent of γ such that, for $t > T$ sufficiently large, the energy for (1.1) (corresponding to $\gamma \neq 0$) satisfies

$$E_{\gamma > 0}(t) \le C e^{-\alpha t} E_{\gamma > 0}(0). \tag{1.6}$$

Remark 1. Note that the statement of Theorem 1.1 affirms a "robust" property of the stabilizing boundary feedback. Indeed, the effectiveness of this feedback, as measured by the constants C and α in (1.6), does not deteriorate when the parameter $\gamma \to 0^+$.

Theorem 1.1 allows us to obtain decay rates for the limit problem when $\gamma \to 0^+$. We will show that the solution w_γ to (1.1) with $\gamma > 0$ converges in $L^2([0,T] \times \Omega)$ to a function w which is a solution to the following limit problem:

$$\left.\begin{array}{ll} w_{tt} + \Delta^2 w + b(x) w_t = [w, \chi(w)] & \text{in } Q \\ w(0,\cdot) = w_0 \in H^2_{\Gamma_0}(\Omega)\ ;\ w_t(0,\cdot) = w_1 \in H^1_{\Gamma_0}(\Omega) & \text{in } \Omega \\ w = \frac{\partial}{\partial\nu} w = 0 & \text{on } \Sigma_0 \\ \Delta w + (1-\mu) B_1 w = -\frac{\partial}{\partial\nu} w_t & \text{on } \Sigma_1 \\ \frac{\partial}{\partial\nu}\Delta w + (1-\mu) B_2 w = w_t - \frac{\partial^2}{\partial\tau^2} w_t + w & \text{on } \Sigma_1, \end{array}\right\} \quad (1.7)$$

with the energy $E_{\gamma=0}(t)$ given by

$$E_{\gamma=0}(t) = \frac{1}{2}\int_\Omega \left\{|w_t|^2 + |\Delta\chi|^2\right\} \mathrm{d}\Omega + \frac{1}{2} a(w,w) + \frac{1}{2}\int_{\Gamma_1} w^2 \mathrm{d}\Gamma_1 \quad (1.8)$$

Moreover, the exponential decay rates (1.6) hold for a solution w of (1.7) as well. A precise statement of this result is given in the following corollary.

Corollary 1.1. Let $w \equiv \lim_{\gamma\to 0^+} w_\gamma$. Then $w \in C([0,T]; H^2(\Omega))$ with $w_t \in C([0,T]; L^2(\Omega))$ is a solution of (1.7). Moreover, there exist constants $C > 0$ and $\alpha > 0$ such that

$$E_{\gamma=0}(t) \le C e^{-\alpha t} E_{\gamma=0}(0) \quad (1.9)$$

where the constant C depends on the size of the initial data in a bounded way.

Remark 2. Exponential decay rates for the limit problem (1.7) with boundary feedbacks acting on the *whole* boundary are proven in [4]. However, the results there are established to hold for "regular" solutions only (i.e. $w \in L^\infty([0,T]; H^4(\Omega))$ and $w_t \in L^\infty([0,T]; L^2(\Omega))$). Moreover, geometric conditions that Ω be a "star-shaped" domain were assumed in [4].

The remainder of this paper is organized as follows. We first set out the preliminary energy estimates which will be used in proving that the energy for system (1.1) ($\gamma > 0$) decays exponentially. These estimates will possess lower order terms which will then be absorbed by a compactness-uniqueness argument. We then use an argument from nonlinear semigroup theory, which provides us with an energy decay rate for system (1.1) that is independent of γ. Finally, we use a limiting argument to prove that (1.6) holds for the limiting equation (1.7).

2. PROOF OF THEOREM 1.1

Our beginning strategy in proving Theorem 1.1 is to use the method of multipliers to produce preliminary energy estimates for (1.1) (with $\gamma > 0$) which may then be used in conjunction with "sharp" trace regularity results to obtain the desired energy estimate, modulo traces of the nonlinear function $\chi(w)$ and lower order terms. These energy estimates will follow closely those that are found in [2], however, here we will carefully follow all dependence of constants on the parameter γ. We summarize the results of these energy estimates below.

Note. In this section, w and $E(T)$ denote, respectively, the solution and energy to (1.1) with $\gamma > 0$. We will only distinguish w_γ and E_γ from w and $E_{\gamma=0}$ in the limiting argument, where such a distinction becomes necessary.

2.1. Energy Estimates

The first preliminary energy estimate we will prove is

Proposition 2.1. Let $(w_0, w_1) \in \mathcal{H}_\gamma \equiv H^2_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)$, then the energy of system (1.1) as given by (1.4) satisfies the following estimate

$$\begin{aligned}
&\int_0^T E(t)\mathrm{d}t - C(1+\gamma^2)E(T) \\
&\leq \; C_T \left\{ (1+\gamma^2)\int_{\Sigma_1} (w_t^2 + |\nabla w_t|^2)\mathrm{d}\Sigma_1 + l.o.(w) \right. \\
&\qquad + (1+\gamma^2)\int_Q b(x)w_t^2 \,\mathrm{d}Q + \int_\Sigma |\Delta\chi|\mathrm{d}\Sigma \\
&\qquad \left. + \int_{\Sigma_1} \left(\left|\frac{\partial^2 w}{\partial\tau^2}\right|^2 + \left|\frac{\partial^2 w}{\partial\nu^2}\right|^2 + \left|\frac{\partial^2 w}{\partial\nu\partial\tau}\right|^2 \right) \mathrm{d}\Sigma_1 \right\}
\end{aligned} \tag{2.1}$$

for all T sufficiently large where C_T depends on $E(0)$ in an increasing manner but does not depend on γ. Here, $l.o.(w)$ represents terms in w having order lower than the energy:

$$l.o.(w) = \|w\|^2_{L^2([0,T];H^{3/2+\varepsilon}(\Omega))} + \|w_t\|^2_{L^2(Q)} \tag{2.2}$$

where $0 < \varepsilon < \frac{1}{2}$.

In the following computations, we will assume the necessary regularity of solutions for (1.1) as guaranteed by [1] for sufficiently smooth initial data. Then the final result will follow by a standard density argument.

Proof. Recalling (1.4) and by using the multiplier w_t, it is straightforward to calculate the identity

$$E(t) + 2\left(\int_0^t \int_{\Gamma_1} \{w_t^2 + |\nabla w_t|^2\}\,\mathrm{d}\Gamma_1\,\mathrm{d}t + \int_0^t \int_\Omega b(x)w_t^2 \mathrm{d}\Omega\,\mathrm{d}t \right) = E(0). \tag{2.3}$$

This proves that energy is nonincreasing for the controlled system.

If we now multiply the state equation in (1.1) by w and integrate by parts, we see that

$$\int_Q \{w_t^2 + \gamma^2 |\nabla w_t|^2\} \,\mathrm{d}Q - \int_0^T a(w,w)\mathrm{d}t - \int_Q (\Delta\chi)^2 \mathrm{d}Q$$
$$= (w_t, w)_\Omega|_0^T + \gamma^2(\nabla w_t, \nabla w)_\Omega|_0^T + \tfrac{1}{2}(b(x)w, w)_\Omega|_0^T$$
$$+ \int_{\Sigma_1} w^2 \mathrm{d}\Sigma_1 + (w,w)_{\Gamma_1}|_0^T + (\nabla w, \nabla w)_{\Gamma_1}|_0^T. \tag{2.4}$$

Subsequently, we multiply (1.1) by $\mathbf{h}\cdot\nabla w$ and again integrate by parts, we obtain

$$\int_Q [w,\chi](\mathbf{h}\cdot\nabla w)\mathrm{d}Q$$
$$= (w_t, \mathbf{h}\cdot\nabla w)_\Omega|_0^T + \gamma^2(\nabla w_t, \mathbf{h}\cdot\nabla w)_\Omega|_0^T + \int_Q w_t^2 \mathrm{d}Q$$
$$+ \int_0^T a(w,w)\,\mathrm{d}t + \int_Q b(x) w_t (\mathbf{h}\cdot\nabla w)\,\mathrm{d}Q$$
$$- \frac{1}{2}\int_{\Sigma_1} (w_t^2 + \gamma^2|\nabla w_t|^2)(\mathbf{h}\cdot\nu)\mathrm{d}\Sigma_1 - \frac{1}{2}\int_{\Sigma_0} (\mathbf{h}\cdot\nu)(\Delta w)^2 \mathrm{d}\Sigma_0$$
$$+ \frac{1}{2}\int_{\Sigma_1} (\mathbf{h}\cdot\nu)\left\{(\Delta w)^2 + 2(1-\mu)(w_{xy}^2 - w_{xx}w_{yy})\right\} \mathrm{d}\Sigma_1$$
$$+ \int_{\Sigma_1} \left\{ \left(w_t - \frac{\partial^2 w_t}{\partial \tau^2} + w \right)(h\cdot\nabla w) + \tfrac{\partial}{\partial\nu}(\mathbf{h}\cdot\nabla w)\tfrac{\partial}{\partial\nu} w_t \right\} \mathrm{d}\Sigma_1. \tag{2.5}$$

From [4] (see page 115) we have

$$\int_Q [w,\chi](\mathbf{h}\cdot\nabla w)\,\mathrm{d}Q = -\frac{1}{2}\int_Q (\Delta\chi)^2 \mathrm{d}Q - \frac{1}{2}\int_\Sigma \mathbf{h}\cdot\nu(\Delta\chi)^2 \mathrm{d}\Sigma.$$

Putting these together, keeping track of the dependence on γ, and bounding the traces in the last term of (2.5) and using (1.5) to eliminate the boundary integral on Σ_0 we obtain

$$\int_Q w_t^2 dQ + \int_0^T a(w,w)\,\mathrm{d}t + \frac{1}{2}\int_Q (\Delta\chi)^2 \mathrm{d}Q$$
$$\leq C_1 \int_Q b w_t^2 \mathrm{d}Q + C_2 \int_\Sigma (\Delta\chi)^2 \mathrm{d}\Sigma$$
$$+ \left|(w_t, \mathbf{h}\cdot\nabla w)_\Omega|_0^T\right| + \gamma^2 \left|(\nabla w_t, \mathbf{h}\cdot\nabla w)_\Omega|_0^T\right|$$
$$+ C_3(1+\gamma^2)\int_{\Sigma_1} \{w_t^2 + |\nabla w_t|^2\}\,\mathrm{d}\Sigma_1 + l.o.(w)$$
$$+ C_4 \int_{\Sigma_1} \left(\left|\frac{\partial^2 w}{\partial\tau^2}\right|^2 + \left|\frac{\partial^2 w}{\partial\nu^2}\right|^2 + \left|\frac{\partial^2 w}{\partial\nu\partial\tau}\right|^2 \right) \mathrm{d}\Sigma_1. \tag{2.6}$$

Here, we have also used the estimate

$$\begin{aligned} -\int_Q bw_t(\mathbf{h}\cdot\nabla w)\,\mathrm{d}Q &\le C_1\int_Q bw_t^2\mathrm{d}Q + C_2\int_Q |\nabla w|^2\mathrm{d}Q \\ &\le C_1\int_Q bw_t^2\mathrm{d}Q + l.o.(w). \end{aligned} \tag{2.7}$$

From (2.4) and (2.6), we obtain

$$\begin{aligned} \int_0^T E(t)\mathrm{d}t \le\ & C\Big(\left|(w_t,w)_\Omega|_0^T]\right| + \gamma^2\left|(\nabla w_t,\nabla w)_\Omega|_0^T\right| + \left|(bw,w)_\Omega|_0^T\right| \\ &+ \left|(w_t,\mathbf{h}\cdot\nabla w)_\Omega|_0^T\right| + \gamma^2\left|(\nabla w_t,\mathbf{h}\cdot\nabla w)_\Omega|_0^T\right| + \left|(w,w)_{\Gamma_1}|_0^T\right| \\ &+ \left|(\nabla w,\nabla w)_{\Gamma_1}|_0^T\right|\Big) + C_1\int_\Sigma (\Delta\chi)^2\mathrm{d}\Sigma + l.o.(w) \\ &+ C_2(1+\gamma^2)\int_{\Sigma_1}(w_t^2+|\nabla w_t|^2)\,\mathrm{d}\Sigma_1 + C_3\int_Q bw_t^2\mathrm{d}Q \\ &+ C_4\int_{\Sigma_1}\left(\left|\frac{\partial^2 w}{\partial\tau^2}\right|^2 + \left|\frac{\partial^2 w}{\partial\nu^2}\right|^2 + \left|\frac{\partial^2 w}{\partial\nu\partial\tau}\right|^2\right)\mathrm{d}\Sigma_1. \end{aligned} \tag{2.8}$$

By the Sobolev imbeddings and trace theory, we see that

$$\begin{aligned} & C_1(1+\gamma^2)E(T) + C_2(1+\gamma^2)E(0) \\ \ge\ & \left|(w_t,w)_\Omega|_0^T\right| + \gamma^2\left|(\nabla w_t,\nabla w)_\Omega|_0^T\right| + \left|(w_t,\mathbf{h}\cdot\nabla w)_\Omega|_0^T\right| \\ & + \gamma^2\left|(\nabla w_t,\mathbf{h}\cdot\nabla w)_\Omega|_0^T\right| + \left|(w,w)_{\Gamma_1}|_0^T\right| + \left|(\nabla w,\nabla w)_{\Gamma_1}|_0^T\right|. \end{aligned} \tag{2.9}$$

We now estimate

$$\begin{aligned} \left|(bw,w)_\Omega|_0^T\right| &\le \overline{C}\int_\Omega (w^2(T)+w^2(0))\,\mathrm{d}\Omega \\ &\le C\{E(T)+E(0)\}, \end{aligned} \tag{2.10}$$

since $b\in L^\infty(\Omega)$.

Using the results in [2], we obtain the bound

$$\int_\Sigma |\Delta\chi(w)|^2\mathrm{d}\Sigma \le \varepsilon C\cdot E^2(0)\int_0^T E(t)\,\mathrm{d}t + \frac{1}{4\varepsilon}\int_\Sigma |\Delta\chi(w)|\mathrm{d}\Sigma. \tag{2.11}$$

By appropriately selecting $\varepsilon = \frac{\tilde{C}}{E^2(0)}$ and using (2.9)–(2.11) in (2.8) along with the estimate (2.3), we obtain the estimate

$$\begin{aligned} & \int_0^T E(t)\,\mathrm{d}t - C(1+\gamma^2)E(T) \\ \le\ & C_1\left\{(1+\gamma^2)\int_{\Sigma_1}(w_t^2+|\nabla w_t|^2)\,\mathrm{d}\Sigma_1\right. \end{aligned}$$

$$+(1+\gamma^2)\int_Q bw_t^2 \mathrm{d}Q + l.o.(w) + \widehat{C}E^2(0)\int_\Sigma |\Delta\chi|\,\mathrm{d}\Sigma$$
$$+\int_{\Sigma_1}\left(\left|\frac{\partial^2 w}{\partial\tau^2}\right|^2 + \left|\frac{\partial^2 w}{\partial\nu^2}\right|^2 + \left|\frac{\partial^2 w}{\partial\nu\partial\tau}\right|^2\right)\mathrm{d}\Sigma_1\Bigg\}, \tag{2.12}$$

where the constants C, C_1, and $\widehat{C}$ do not depend on γ. This proves Proposition 2.1. □

Our next step is to develop appropriate estimates for the traces of the solution, w, on the portion of the boundary, Γ_1. To accomplish this, we shall use the following result proved in [2].

Proposition 2.2. Let w satisfy (1.1). Then for any $\alpha > 0$ and $\varepsilon > 0$ we have

$$\begin{aligned}\int_\alpha^{T-\alpha}\int_{\Gamma_1}&\left(\left|\frac{\partial^2 w}{\partial\tau^2}\right|^2 + \left|\frac{\partial^2 w}{\partial\nu^2}\right|^2 + \left|\frac{\partial^2 w}{\partial\nu\partial\tau}\right|^2\right)\mathrm{d}\Gamma_1\,\mathrm{d}t \\ \le\ & C_{T,\alpha,\varepsilon}\left\{\|w_t\|^2_{L^2(\Sigma_1)} + \|\nabla w_t\|^2_{L^2(\Sigma_1)}\right. \\ &\left. + E^2(0)\,\|\chi(w)\|_{L^1([0,T];H^{3-\varepsilon}(\Omega))} + l.o.(w)\right\}.\end{aligned} \tag{2.13}$$

Here, $C_{T,\alpha,\varepsilon}$ does not depend on γ for $0 \le \gamma \le M < \infty$.

We are now in a position to prove our main energy estimate.

Lemma 2.1. Let w satisfy the system (1.1) and let $0 < \alpha < T$ and let $\varepsilon > 0$ be arbitrary. Then

$$\begin{aligned}E(T)\ \le\ & C_{T,\alpha,\varepsilon}(E(0))\Bigg\{(1+\gamma^2)\int_{\Sigma_1}(w_t^2 + |\nabla w_t|^2)\,\mathrm{d}\Sigma_1 + l.o.(w) \\ & +(1+\gamma^2)\int_Q bw_t^2\mathrm{d}Q + \int_\Sigma |\Delta\chi(w)|\mathrm{d}\Sigma + \int_0^T \|\chi(w)\|_{H^{3-\varepsilon}(\Omega)}\mathrm{d}t\Bigg\}\end{aligned}$$

where $C_{T,\alpha,\varepsilon}(E(0))$ is an increasing function of $E(0)$ but does not depend on γ.

Proof. Applying the result of Proposition 2.1 on the interval $[\alpha, T-\alpha]$ yields

$$\begin{aligned}&\int_\alpha^{T-\alpha} E(t)\,\mathrm{d}t - (1+\gamma^2)E(T-\alpha) \\ &\le C_T(E(\alpha))\Bigg\{(1+\gamma^2)\int_{\Sigma_{1\alpha}}(w_t^2 + |\nabla w_t|^2)\,\mathrm{d}\Sigma_{1\alpha} + \int_\Sigma|\Delta\chi(w)|\mathrm{d}\Sigma \\ &\quad +(1+\gamma^2)\int_Q bw_t^2\mathrm{d}Q + l.o.(w) \\ &\quad +\int_{\Sigma_{1\alpha}}\left(\left|\frac{\partial^2 w}{\partial\tau^2}\right|^2 + \left|\frac{\partial^2 w}{\partial\nu^2}\right|^2 + \left|\frac{\partial^2 w}{\partial\tau\partial\nu}\right|^2\right)\mathrm{d}\Sigma_{1\alpha}\Bigg\}\end{aligned} \tag{2.14}$$

where $\Sigma_{1\alpha} \equiv \Gamma_1 \times [\alpha, T-\alpha]$. By the dissipation of energy given in (2.3), we have $E(T) \leq E(T-\alpha)$. Also, since $C_T(E(t))$ is an increasing function of $E(t)$, we have

$$C_T(E(\alpha)) \leq C_T(E(0)). \tag{2.15}$$

Applying Proposition 2.2 to the last term on the right hand side of (2.14) and recalling (2.3) and (2.15) leads to the desired right hand side of the inequality in Lemma 2.1. Again using (2.3), we see that

$$TE(T) \leq TE(T-\alpha) \leq \int_{\alpha}^{t-\alpha} E(t)\,\mathrm{d}t.$$

Then since, $\gamma < 1$, selecting $T > T_0 = 2C$, we have the lemma. □

2.2. Compactness–Uniqueness Argument

Making the observation that

$$\begin{aligned} \int_{\Sigma} \|\Delta\chi\|\,\mathrm{d}\Sigma &\leq mes(\Gamma)^{1/2} \int_0^T \|\Delta\chi\|_{L^2(\Gamma)}\mathrm{d}t \\ &\leq C\int_0^T \|\Delta\chi\|_{H^{1/2+\epsilon}(\Omega)}\mathrm{d}t \leq C\int_0^T \|\chi\|_{H^{3-\epsilon}(\Omega)}\mathrm{d}t \end{aligned} \tag{2.16}$$

and using the results of Lemma 2.1, we now have a bound on our energy in terms of the controls and some undesirable terms (i.e. lower order and nonlinear terms) and in which the dependence on γ is explicit. We now use the following result proven in [2] which proves that the energy of (1.1) is bounded in terms of the controls alone.

Lemma 2.2. Let (w, w_t) be a solution pair for (1.1). Then for any $\varepsilon > 0$,

$$\begin{aligned} &\int_0^T \|\chi(w)\|_{H^{3-\epsilon}(\Omega)}\mathrm{d}t + l.o.(w) \\ &\leq C_\gamma(E(0))\left\{\int_{\Sigma_1}\{w_t^2 + |\nabla w_t|^2\}\mathrm{d}\Sigma_1 + \int_Q b(x)w_t^2\mathrm{d}Q\right\} \end{aligned} \tag{2.17}$$

where $C_\gamma(E(0))$ is an increasing function of the initial energy, $E(0)$, and may depend on γ. We note that $l.o.(w)$ are as in (2.2).

We now must prove that the estimate (2.17) holds where $C(E(0))$ is increasing in $E(0)$ *but does not depend on* γ.

Lemma 2.3. Let (w, w_t) be a solution pair for (1.1). Then for any $\varepsilon > 0$,

$$\begin{aligned} &\int_0^T \|\chi(w)\|_{H^{3-\epsilon}(\Omega)}\mathrm{d}t + l.o.(w) \\ &\leq C(E(0))\left\{\int_{\Sigma_1}\{w_t^2 + |\nabla w_t|^2\}\mathrm{d}\Sigma_1 + \int_Q b(x)w_t^2\mathrm{d}Q\right\} \end{aligned} \tag{2.18}$$

where $C(E(0))$ is an increasing function of the initial energy, $E(0)$, and does not depend on γ.

Proof. The proof is by contradiction. Suppose (2.18) does not hold for $C(E(0))$ independent of γ. Then there exists a sequence of functions $\{w_\gamma(t)\}$ in $\mathcal{H}_\gamma$ which satisfies the system

$$\left.\begin{aligned}
&w_\gamma'' - \gamma^2\Delta w_\gamma'' + \Delta^2 w_\gamma + bw_\gamma' = [w_\gamma, \chi(w_\gamma)] && \text{in } Q\\
&w_\gamma(0,\cdot) = w_{\gamma 0}\ ;\ w_\gamma'(0,\cdot) = w_{\gamma 1} && \text{in } \Omega\\
&w_\gamma = \tfrac{\partial}{\partial\nu} w_\gamma = 0 && \text{on } \Sigma_0\\
&\Delta w_\gamma + (1-\mu)B_1 w_\gamma = -\tfrac{\partial}{\partial\nu} w_\gamma' && \text{on } \Sigma_1\\
&\tfrac{\partial}{\partial\nu}\Delta w_\gamma + (1-\mu)B_2 w_\gamma - \gamma^2\tfrac{\partial}{\partial\nu} w_\gamma'' = w_\gamma' - \tfrac{\partial^2}{\partial\tau^2} w_\gamma' + w_\gamma && \text{on } \Sigma_1
\end{aligned}\right\}\tag{2.19}$$

and such that

$$\lim_{\gamma\to 0+}\frac{l.o.(w_\gamma) + \int_0^T \|\chi(w_\gamma)\|_{H^{3-\epsilon}(\Omega)}\mathrm{d}t}{\int_{\Sigma_1}\left((w_\gamma')^2 + |\nabla w_\gamma'|^2\right)\mathrm{d}\Sigma_1 + \int_Q b(w_\gamma')^2\mathrm{d}Q} = \infty \tag{2.20}$$

where the initial energy (as prescribed by initial data $(w_{\gamma 0}, w_{\gamma 1})$) are uniformly bounded in γ. (Note: for convenience, we denote the time derivatives by $'$.)

Denoting the sequence

$$c_\gamma \equiv \left\{l.o.(w_\gamma) + \int_0^T \|\chi(w_\gamma)\|_{H^{3-\epsilon}(\Omega)}\mathrm{d}t\right\}^{1/2} \tag{2.21}$$

we introduce the new variable

$$v_\gamma \equiv \frac{w_\gamma}{c_\gamma}. \tag{2.22}$$

We observe that v_γ satisfies the system

$$\left.\begin{aligned}
&v_\gamma'' - \gamma^2\Delta v_\gamma'' + \Delta^2 v_\gamma + bv_\gamma' = [v_\gamma, \chi(w_\gamma)] && \text{in } Q\\
&v_\gamma(0,\cdot) = v_{\gamma 0}\ ;\ v_\gamma'(0,\cdot) = v_{\gamma 1} && \text{in } \Omega\\
&v_\gamma = \tfrac{\partial}{\partial\nu} v_\gamma = 0 && \text{on } \Sigma_0\\
&\Delta v_\gamma + (1-\mu)B_1 v_\gamma = -\tfrac{\partial}{\partial\nu} v_\gamma' && \text{on } \Sigma_1\\
&\tfrac{\partial}{\partial\nu}\Delta v_\gamma + (1-\mu)B_2 v_\gamma - \gamma^2\tfrac{\partial}{\partial\nu} v_\gamma'' = v_\gamma' - \tfrac{\partial^2}{\partial\tau^2} v_\gamma' + v_\gamma && \text{on } \Sigma_1.
\end{aligned}\right\}\tag{2.23}$$

By using (2.20), we see that v_γ satisfies (by the quadratic dependence of χ on w_γ)

$$l.o.(v_\gamma) + \int_0^T \|\chi(v_\gamma)\|_{H^{3-\epsilon}(\Omega)}\mathrm{d}t$$

$$= \frac{l.o.(w_\gamma) + \int_0^T \|\chi(w_\gamma)\|_{H^{3-\epsilon}(\Omega)} dt}{l.o.(w_\gamma) + \int_0^T \|\chi(w_\gamma)\|_{H^{3-\epsilon}(\Omega)} dt} \equiv 1 \tag{2.24}$$

so that

$$\lim_{\gamma \to 0^+} \int_{\Sigma_1} ((v'_\gamma)^2 + |\nabla v'_\gamma|^2) d\Sigma_1 + \int_Q b(v'_\gamma)^2 dQ = 0 \tag{2.25}$$

By (2.25), we have the following convergence properties:

$$\begin{aligned} &\text{(i)} \quad v'_\gamma \to 0 \quad \text{in } L^2(Q) \\ &\text{(ii)} \quad v'_\gamma \to 0 \quad \text{in } H^1(\Sigma_1). \end{aligned} \tag{2.26}$$

In order to pass with a limit on (2.23), we need first to determine the convergence properties for v_γ and for our nonlinear terms.

To determine the convergence properties of v_γ, we will use the energy estimates which were derived in the previous section. Using an argument similar to that found in [2], we apply the results of our well-posedness theorem (see [1]) to (2.23) to obtain that

$$\|v_\gamma\|_{C([0,T];H^2(\Omega))} + \|v'_\gamma\|_{C([0,T];L^2(\Omega))} + \gamma^2 \|\nabla v'_\gamma\|_{C([0,T];L^2(\Omega))} \le C \tag{2.27}$$

which implies, in particular, that $\{v_\gamma\}$ are uniformly bounded in $H^1([0,T] \times \Omega)$. Hence, by the compact Sobolev imbeddings and trace theory, we see that

$$\begin{aligned} & v_\gamma \xrightarrow{w} v \text{ in } L^2([0,T]; H^2(\Omega)) \text{ and } v'_\gamma \xrightarrow{w} v' \text{ in } L^2([0,T]; L^2(\Omega)) \\ \Longrightarrow \quad & v_\gamma \xrightarrow{w} v \text{ in } H^1([0,T] \times \Omega) \\ \Longrightarrow \quad & v_\gamma \to v \text{ in } L^2(\Sigma). \end{aligned} \tag{2.28}$$

Also, since $E_\gamma(0) \le M$, the well-posedness for (1.1) yields similar convergence properties for w_γ. □

We now state some convergence properties of the von Kármán nonlinearity, $[v_\gamma, \chi(w_\gamma)]$, which were proven (or are similar to results proven) in [2].

Proposition 2.3. Let $w_\gamma \xrightarrow{w} w$ in $H^2(\Omega)$. Then $\chi(w_\gamma) \xrightarrow{w} \chi(w)$ in $H_0^2(\Omega)$.

We now seek to obtain the convergence of $\chi(w_\gamma)$ in the space-time cylinder, Q.

Proposition 2.4. Assume that

$$\|w_\gamma\|_{C([0,T];H^2(\Omega))} + \|w'_\gamma\|_{C([0,T];L^2(\Omega))} + \gamma^2 \|\nabla w'_\gamma\|_{C([0,T];L^2(\Omega))} \le C$$

and

$$\begin{aligned} w_\gamma &\xrightarrow{w} w \quad \text{in } L^2([0,T]; H^2(\Omega)) \\ w'_\gamma &\xrightarrow{w} w' \quad \text{in } L^2([0,T]; L^2(\Omega)). \end{aligned}$$

Then for every $0 < \varepsilon < \frac{1}{2}$,

$$\chi(w_\gamma) \to \chi(w) \text{ in } C([0,T]; H^{3-\varepsilon}(\Omega)).$$

Proposition 2.5. Suppose that $v_\gamma \xrightarrow{w} v$ in $H^2(\Omega)$ and w_γ satisfies the assumptions of Proposition 2.4. Then $[v_\gamma, \chi(w_\gamma)] \to [v, \chi(w)]$ in the sense of distributions.

In passing with a limit on (2.23), we will consider two cases.

Case 1. $c_0 = \{l.o.(w) + \int_0^T \|\chi(w)\|_{H^{3-\varepsilon}(\Omega)} dt\}^{1/2} \neq 0$. By the result of Proposition 2.4, (2.21), the convergence properties of w_γ and by the compactness properties of $l.o.(w)$, we have $c_\gamma \to c_0$, hence $v = w/c_0$. Using (2.26), Proposition 2.5 and passing with a limit on (2.23), we obtain the limit system

$$\left.\begin{array}{ll} \Delta^2 v = [v, \chi(w)] = \frac{1}{c_0}[w, \chi(w)] & \text{in } Q \\ v = \frac{\partial}{\partial\nu} v = 0 & \text{on } \Sigma_0 \\ \Delta v + (1-\mu) B_1 v = 0 & \text{on } \Sigma_1 \\ \frac{\partial}{\partial\nu}\Delta v + (1-\mu) B_2 v = 0 & \text{on } \Sigma_1. \end{array}\right\} \tag{2.29}$$

Multiplying (2.29) by v and integrating by parts, it is easy to show that

$$0 = a(v,v) + \frac{1}{c_0^2}\int_\Omega (\Delta\chi(w))^2 d\Omega,$$

and conclude, by the positivity of $a(v,v)$ that

$$v \equiv 0 \qquad \text{in } Q. \tag{2.30}$$

By Proposition 2.4, (2.27), (2.28) and (2.30) we obtain

$$\left(l.o.(v_\gamma) + \int_0^T \|\chi(v_\gamma)\|_{H^{3-\varepsilon}(\Omega)} dt\right) \longrightarrow 0.$$

and hence the proof of Lemma 2.3 for Case 1.

Case 2. $c_0 = 0$, (i.e. $\chi(w) \equiv 0$ and $l.o.(w) = 0$.) In this case, we will again use the result of Proposition 2.4. Here we use the fact that $\chi(w_\gamma) \to 0$ in $C([0,T]; H^{3-\varepsilon}(\Omega))$ in combination with (2.27) and Proposition 2.5 in order to obtain that $[v_\gamma(t), \chi(w_\gamma)(t)] \to 0$ in the sense of distributions. By using (2.26) and passing with a limit on system (2.29), we obtain

$$\left.\begin{array}{ll} \Delta^2 v = 0 & \text{in } Q \\ v = \frac{\partial}{\partial\nu} v = 0 & \text{on } \Sigma_0 \\ \Delta v + (1-\mu) B_1 v = 0 & \text{on } \Sigma_1 \\ \frac{\partial}{\partial\nu}\Delta v + (1-\mu) B_2 v = 0 & \text{on } \Sigma_1. \end{array}\right\} \tag{2.31}$$

The same argument as in Case 1 yields a contradiction and the proof. □

2.3. Completion of Proof of Theorem 1.1

By using (2.3), (2.16) and Lemmas 2.1 and 2.2, we have shown that, for T sufficiently large, the energy for system (1.1) satisfies

$$E(T) \leq C(E(0))(1+\gamma^2)\left(\int_{\Sigma_1}(w_t^2+|\nabla w_t|^2)\,\mathrm{d}\Sigma_1+\int_Q bw_t^2\mathrm{d}Q\right), \tag{2.32}$$

where $C(E(0))$ is increasing in $E(0)$ but does not depend on γ. Now using an argument from nonlinear semigroup theory (as in [2]), we see that the energy of (1.1) satisfies

$$E_{\gamma>0}(T) \leq C\mathrm{e}^{-\alpha t}E_{\gamma>0}(0) \tag{2.33}$$

where the constants C and α depend on the initial energy, $E_{\gamma>0}(0)$, but not on γ. This proves Theorem 1.1. □

3. PROOF OF COROLLARY 1.1

We now show that the estimate (1.6) also holds for the energy of the limiting system (1.7). We first note that by the well-posedness and regularity of solutions to system (1.1), we have

$$||w_\gamma||_{C([0,T];H^2_{\Gamma_0}(\Omega))}+||w'_\gamma||_{C([0,T];L^2(\Omega))}+\gamma^2||\nabla w'_\gamma||_{C([0,T];L^2(\Omega))} \leq C \tag{3.1}$$

so that w_γ enjoys the following convergence properties:

$$\begin{aligned} w_\gamma &\overset{w^*}{\to} w \quad \text{in} \quad L^\infty([0,T];H^2_{\Gamma_0}(\Omega))\\ w'_\gamma &\overset{w^*}{\to} w' \quad \text{in} \quad L^\infty([0,T];L^2(\Omega))\\ w_\gamma &\overset{w}{\to} w \quad \text{in} \quad H^1([0,T]\times\Omega)\\ w_\gamma &\to w \quad \text{in} \quad L^2(\Sigma_1). \end{aligned} \tag{3.2}$$

In particular, this implies by the compact Sobolev imbeddings that $w_\gamma \to w$ strongly in $L^2([0,T]\times\Omega)$.

We also know that $[w_\gamma,\chi(w_\gamma)] \to [w,\chi(w)]$ in the sense of distributions (see Propositions 2.3–2.5). However, this is not sufficient to pass with a limit on system (1.1) as $\gamma \to 0^+$. We must also have the following result, which was proven in [3].

Lemma 3.1. Let (w_γ, w'_γ) be a solution pair to system (1.1). Then

$$\begin{aligned} &||w'_\gamma||^2_{L^2(\Sigma_1)}+||\nabla w'_\gamma||^2_{L^2(\Sigma_1)}\\ &\quad\leq C(||w_1||^2_{L^2(\Omega)}+\gamma^2||\nabla w_1||^2_{L^2(\Omega)}+||w_0||_{H^2_{\Gamma_0}(\Omega)}) \end{aligned}$$

where $(w_0,w_1)\in\mathcal{H}_\gamma$ are the initial data for system (1.1) and the constant C does not depend on γ.

A consequence of Lemma 2.4 is that the traces $w'_\gamma|_{(\Sigma_1)}$ and $\nabla w'_\gamma|_{(\Sigma_1)}$ are uniformly bounded in γ in $L^2(\Sigma_1)$. (This follows because the initial data are not changing with γ and the fact that γ is bounded above.)

By the result of Lemma 2.4 and a standard limiting argument, we obtain that

$$\begin{aligned} \nabla w'_\gamma|_{\Sigma_1} &\xrightarrow{w} \nabla w'|_{\Sigma_1} \quad \text{in} \quad L^2(\Sigma_1) \\ w'_\gamma|_{\Sigma_1} &\xrightarrow{w} w'|_{\Sigma_1} \quad \text{in} \quad L^2(\Sigma_1). \end{aligned} \tag{3.3}$$

We note that this result does *not* follow from general trace theory and interior regularity of solutions. In fact, $w'_\gamma \in H^1_{\Gamma_0}(\Omega)$ alone is not enough to imply that the trace $\nabla w'_\gamma|_{\Sigma_1}$ is even well-defined!

Now combining (3.1)–(3.3), we may pass with a limit on (1.1) as $\gamma \to 0^+$ to obtain that w satisfies (1.7) (in the sense of distributions). We also observe that for $t > T$ sufficiently large we have

$$\begin{aligned} &E_{\gamma=0}(t) \\ &= \tfrac{1}{2}\Big(\|w'(t)\|_{L^2(\Omega)} + a(w(t),w(t)) + \textstyle\int_\Omega |\Delta\chi(w(t))|^2 \mathrm{d}\Omega + \int_{\Gamma_1} w^2(t)\,\mathrm{d}\Gamma_1\Big) \\ &\quad \le \liminf_\gamma \tfrac{1}{2}\Big(\|w'_\gamma(t)\|^2_{L^2(\Omega)} + \gamma^2\|\nabla w'_\gamma(t)\|^2_{L^2(\Omega)} + a(w_\gamma(t),w_\gamma(t)) \\ &\qquad\qquad + \textstyle\int_\Omega |\Delta\chi(w_\gamma(t))|^2\mathrm{d}\Omega + \int_{\Gamma_1} w^2_\gamma(t) d\Gamma_1\Big) \\ &\quad \le \liminf_\gamma C\mathrm{e}^{-\alpha t}\Big\{\|w_1\|^2_{L^2(\Omega)} + \gamma^2\|\nabla w_1\|^2_{L^2(\Omega)} + a(w_0,w_0) \\ &\qquad\qquad + \textstyle\int_\Omega |\Delta\chi(w_0)|^2\mathrm{d}\Omega + \int_{\Gamma_1} (w_0|_\Gamma)^2\,\mathrm{d}\Gamma_1\Big\} \\ &\qquad\qquad = C\mathrm{e}^{-\alpha t}E_{\gamma=0}(0), \end{aligned}$$

where we have used (2.33). This gives us the proof of Corollary 1.1. □

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