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A NOTE ON T–NORM–BASED OPERATIONS ON LR FUZZY INTERVALS[1]

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The goal of this paper is to give a functional relationship between the membership functions of fuzzy intervals $M_1 \oplus \ldots \oplus M_n$ and $M_1 \odot \ldots \odot M_n$, where M_i are positive LR fuzzy intervals of the same form $M_i = M = (a, b, \alpha, \beta)_{LR}$ and the extended addition $\oplus$ and multiplication $\odot$ are defined in the sense of a triangular norm (i.e. via sup–t–norm convolution).

1. DEFINITIONS

A fuzzy interval M is a fuzzy set of the real line $\mathbb{R}$ with a continuous, compactly supported, unimodal and normalized membership function $\mu_M : \mathbb{R} \to I = [0,1]$. A fuzzy set M of $\mathbb{R}$ is said to be positive if $\mu_M(x) = 0$ for all $x < 0$. We shall use the notation $M(x)$ to abbreviate $\mu_M(x)$.

It is known [3] that any fuzzy interval M can be described as

$$M(t) = \begin{cases} 1 & \text{if } t \in [a,b] \\ L\left(\frac{a-t}{\alpha}\right) & \text{if } t \in [a-\alpha, a] \\ R\left(\frac{t-b}{\beta}\right) & \text{if } t \in [b, b+\beta] \\ 0 & \text{otherwise} \end{cases}$$

where $[a,b]$ is the peak of M; L and R are continuous and non–increasing shape functions $I \to I$ with $L(0) = R(0) = 1$ and $R(1) = L(1) = 0$. We call this fuzzy interval of LR type and refer to it by $M = (a, b, \alpha, \beta)_{LR}$. The support of M (denoted by $Supp\, M$) is $[a-\alpha, b+\beta]$.

A function $T : I^2 \to I$ is said to be triangular norm (t–norm for short) iff T is symmetric, associative, non–decreasing in each argument, and $T(x, 1) = x$ for all $x \in I$. Recall that a t–norm T is Archimedean iff T is continuous and $T(x,x) < x$ for all $x \in (0,1)$.

Every Archimedean t–norm T is representable by a continuous and decreasing function $f : I \to [0, \infty]$ with $f(1) = 0$ and

$$T(x,y) = f^{[-1]}(f(x) + f(y))$$

where $f^{[-1]}$ is the pseudo–inverse of f, defined as

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{if } y \in [0, f(0)] \\ 0 & \text{otherwise} \end{cases}$$

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The function f is called the additive generator of T.

Let T be a t–norm and let $*$ be an operation on $\mathbb{R}$. Then $*$ can be extended to fuzzy intervals in the sense of the following extension principle

$$(M_1 * M_2)(z) = \sup_{x_1 * x_2 = z} T(M_1(x_1), M_2(x_2)) \qquad z \in \mathbb{R}$$

which can be written as

$$(M_1 * M_2)(z) = \sup_{x_1 * x_2 = z} f^{[-1]}\Big(f(M_1(x_1)) + f(M_2(x_2))\Big) \qquad z \in \mathbb{R}$$

2. THE RESULT

The following theorem gives a functional relationship between the membership functions of fuzzy intervals $M_1 \oplus ... \oplus M_n$ and $M_1 \odot ... \odot M_n$, where M_i are positive LR fuzzy intervals of the same form $M_i = M = (a, b, \alpha, \beta)_{LR}$.

Theorem 1. Let T be an Archimedean t–norm with an additive generator f and let $M_i = M = (a, b, \alpha, \beta)_{LR}$ be positive fuzzy intervals of LR type. If L and R are twice differentiable, concave functions, and f is twice differentiable, strictly convex function, then

$$(M_1 \oplus ... \oplus M_n)(n \cdot z) = (M_1 \odot ... \odot M_n)(z^n) = f^{[-1]}\Big(n \cdot f(M(z))\Big) \qquad (1)$$

P r o o f. Let $z \geq 0$ be arbitrarily fixed. According to the decomposition rule of fuzzy intervals into two separate parts [5], we can assume without loss of generality that $z < a$. From Theorem 1 of [6] it follows that

$$\begin{aligned}(M_1 \oplus ... \oplus M_n)(n \cdot z) &= f^{[-1]}\left(n \cdot f\left(L\left(\frac{na - nz}{n\alpha}\right)\right)\right) = \\ &= f^{[-1]}\left(n \cdot f\left(L\left(\frac{a - z}{\alpha}\right)\right)\right) = \\ &= f^{[-1]}\Big(n \cdot f(M(z))\Big)\end{aligned}$$

The proof will be complete if we show that

$$\begin{aligned}(M \odot ... \odot M)(z) &= \sup_{x_1 \cdot ... \cdot x_n = z} T\Big(M(x_1), ..., M(x_n)\Big) = \qquad (2) \\ &= T\Big(M(\sqrt[n]{z}), ..., M(\sqrt[n]{z})\Big) = \\ &= f^{[-1]}\Big(n \cdot f(M(\sqrt[n]{z}))\Big)\end{aligned}$$

We shall justify it by induction:

(i) for $n = 1$ (2) is obviously valid.

(ii) Let us suppose that (2) holds for some $n = k$ i.e.

$$\begin{aligned} (M^k)(z) &= \sup_{x_1 \cdot \ldots \cdot x_k = z} T\Big(M(x_1), \ldots, M(x_k)\Big) = \\ r &= T\Big(M(\sqrt[k]{z}), \ldots, M(\sqrt[k]{z})\Big) = \\ &= f^{[-1]}\Big(k \cdot f(M(\sqrt[k]{z}))\Big) \end{aligned}$$

and verify the case $n = k + 1$. It is clear that

$$\begin{aligned} (M^{k+1})(z) &= \sup_{x \cdot y = z} T\Big(M^k(x), M(y)\Big) = \\ &= \sup_{x \cdot y = z} T\Big(M(\sqrt[k]{x}), \ldots, M(\sqrt[k]{x}), M(y)\Big) = \\ &= f^{[-1]}\left(\inf_{x \cdot y = z} \Big(k \cdot f(M(\sqrt[k]{x})) + f(M(y))\Big)\right) = \\ &= f^{[-1]}\Big(\inf_x \Big(k \cdot f(M(\sqrt[k]{x})) + f(M(z/x))\Big)\Big) \end{aligned}$$

The support and the peak of M^{k+1} are

$$\begin{aligned} [M^{k+1}]^1 &= [M]^{1^{k+1}} = [a^{k+1}, b^{k+1}] \\ Supp(M^{k+1}) &\subset \Big(Supp(M)\Big)^{k+1} = [(a-\alpha)^{k+1}, (a+\beta)^{k+1}] \end{aligned}$$

According to the decomposition rule we can consider only the left hand side of M, that is let $z \in [(a-\alpha)^{k+1}, a^{k+1}]$. We need to find the minimum of the mapping

$$x \mapsto k \cdot f(M(\sqrt[k]{x})) + f(M(z/x))$$

in the interval $[(a-\alpha)^k, a^k]$. Let us introduce the auxiliary variable $t = \sqrt[k]{x}$ and look for the minimum of the function

$$t \mapsto \varphi(t) := k \cdot f(M(t)) + f(M(z/t^k))$$

in the interval $[a - \alpha, a]$. Dealing with the left hand side of M we have

$$M(t) = L\left(\frac{a-t}{\alpha}\right) \qquad \text{and} \qquad M(z/t^k) = L\left(\frac{a - z/t^k}{\alpha}\right)$$

The derivative of φ is equal to zero when

$$\begin{aligned} \varphi'(t) &= k \cdot f'(M(t)) \cdot L'\left(\frac{a-t}{\alpha}\right) \cdot \frac{-1}{\alpha} + \\ &\quad + f'(M(z/t^k)) \cdot L'\left(\frac{a - z/t^k}{\alpha}\right) \cdot \frac{-1}{\alpha} \cdot \left(-k \cdot \frac{z}{t^{k+1}}\right) = 0 \end{aligned}$$

i.e.

$$t \cdot f'(M(t)) \cdot L'\left(\frac{a-t}{\alpha}\right) = \frac{z}{t^k} \cdot f'(M(z/t^k)) \cdot L'\left(\frac{a - z/t^k}{\alpha}\right) \tag{3}$$

which obviously holds taking $t = z/t^k$. So $t_0 = \sqrt[k+1]{z}$ is a solution of (3), furthermore, from the strict monotony of

$$t \mapsto t \cdot f'(M(t)) \cdot L'\left(\frac{a-t}{\alpha}\right)$$

follows that there are no other solutions.

It is easy to check, that $\varphi''(t_0) > 0$, which means that φ attains its absolute minimum at t_0. Finally, from the relations $\sqrt[k]{x_0} = \sqrt[k+1]{z}$ and $z/x_0 = \sqrt[k+1]{z}$, we get

$$\begin{aligned}(M^{k+1})(z) &= T\Big(M(\sqrt[k+1]{z}),\dots,M(\sqrt[k+1]{z}),M(\sqrt[k+1]{z})\Big) = \\ &= f^{[-1]}\Big(k \cdot f(M(\sqrt[k+1]{z})) + f(M(\sqrt[k+1]{z}))\Big) = \\ &= f^{[-1]}\Big((k+1) \cdot f(M(\sqrt[k+1]{z}))\Big)\end{aligned}$$

which ends the proof. □

Remark 1. As an immediate consequence of Theorem 1 we can easily calculate the exact possibility distribution of expressions of the form $e_n^*(M) := \frac{M \oplus \dots \oplus M}{n}$ and the limit distribution of $e_n^*(M)$ as $n \to \infty$. Namely, from (1) we have

$$(e_n^*(M))(z) = \left(\frac{M \oplus \dots \oplus M}{n}\right)(z) = (M \oplus \dots \oplus M)(n \cdot z) = f^{[-1]}\Big(n \cdot f(M(z))\Big)$$

therefore, from $f(x) > 0$ for $0 \le x < 1$ and $\lim_{x \to \infty} f^{[-1]}(x) = 0$ we get

$$\begin{aligned}\Big(\lim_{n\to\infty} e_n^*(M)\Big)(z) &= \lim_{n\to\infty} (e_n^*(M))(z) = \\ &= \lim_{n\to\infty} f^{[-1]}\Big(n \cdot f(M(z))\Big) = \\ &= \begin{cases} 1 & \text{if } z \in [a,b] \\ 0 & \text{if } z \notin [a,b] \end{cases}\end{aligned}$$

that is

$$\lim_{n\to\infty} e_n^*(M) = [a,b] \tag{4}$$

which is the peak of M.

It can be shown [4] that (4) remains valid for the (non-Archimedean) weak t-norm. Other results along this line have appeared in [1, 2, 8].

Remark 2. It is easy to see [7] that, for instance, when $T(x,y) = x \cdot y$:

$$(M_1 \oplus \dots \oplus M_n)(n \cdot z) = (M_1 \odot \dots \odot M_n)(z^n) = \big(M(z)\big)^n$$

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