SUPPLEMENT TO KYBERNETIKA VOLUME 28 (1992), PAGES 41–44

A NONSTANDARD APPROACH TO FUZZY SET THEORY

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The nonstandard approach to fuzzy sets [1] is based on a Boolean generalization of Infinitesimal Analysis [2], [4], [6].

This paper, gives a short review of this approach, describes some applications to mathematical structures and indicates the way for an extension using fuzzy partitions.

In addition, we prove that the theory is general, since for any ordinary fuzzy set $f : X \to [0,1]$ there exists a unique Boolean probability algebra $(\mathbb{B}, p)$ and a $\mathbb{B}$-possibility distribution $\pi : X \to \mathbb{B}$, such that

$$f = p \circ \pi.$$

0. INTRODUCTION

The transition from standard ZFC model of set theory to a nonstandard one, can be done in two formally equivalent ways: the extensional and the intentional one [1].

In the Cantorian absolute framework, concepts have an ideal Platonic and absolute character, and their exactness and sharpness are attained through the use of absolute reference space and of actual infinite. Seeing the absolute ZFC framework with a local and non-Cantorian way we get a fuzzy deformation of it, which gives "a realism with a human face" and constitutes "A Theory of Fuzzy Sets" which of course depends on the way we introduce the local concepts [1], [3].

1. $\mathbb{B}$-FUZZY REALS, SETS, AND FUNCTIONS

In the following $(\mathbb{B}, p)$ is a probability algebra. We construct $\mathbb{R}^{\#}$ $(:= \mathbb{R}[\mathbb{B}])$ the Boolean power of the reals [2], which is a Boolean valued model of $\mathbb{R}$. The elements of this set are called $\mathbb{B}$-*fuzzy reals* and can be written as a countable "mixing" of elements of $\mathbb{R}$ with various "weights" ($f = \sum_{i \in I} x_i \cdot t_i$, where t_i's form a partition of unity in $\mathbb{B}$ and x_i's are real numbers). Actually such a mixing is a function $f \in \mathbb{B}^{\mathbb{R}}$ of the form :

$$f(x_i) = t_i \qquad \text{and} \qquad f(x) = 0_{\mathbb{B}} \qquad \text{for} \qquad x \neq x_i$$

The "weight" $\|\phi\|$ of any statement ϕ about real numbers is an element of $\mathbb{B}$ and

$$\mathbb{R}^{\#} \models \phi \iff \|\phi\| = 1_{\mathbb{B}}$$

Especially $\|f = x\| = f(x)$. The extensional analog of a $\mathbb{B}$-fuzzy real is a randomly variable real in discrete time [3]. Taking the superstructures $V(\mathbb{R})$ and $V(\mathbb{R}^{\#})$ we establish a function [4] $\#(\cdot)$ between them satisfying:

(i) **$\mathbb{B}$-*Extension Principle.***

The set $\mathbb{R}^{\#}$ is a $\mathbb{B}$-totally ordered Archimedean field, which is a proper extension of $\mathbb{R}$, i.e. $\mathbb{R}^{\#} \supset \mathbb{R}$ and $r^{\#} = r \equiv \widehat{r}$ for all $r \in \mathbb{R}$.

(ii) ***Transfer Principle.***

For every $a_1, \ldots a_n \in V(\mathbb{R})$ and every bounded statement ϕ then $\phi(a_1 \ldots a_n)$ holds in $V(\mathbb{R})$ iff its $\#$-transform, $\phi(a_1^{\#} \ldots a_n^{\#})$ holds in $V(\mathbb{R}^{\#})$.

The sets $\#(A) := A^{\#}$ are called ***standard.*** Elements of standard sets are called *internal.* Internal elements of the non standard superstructure are the $\mathbb{B}$-fuzzy analogues of the elements of the standard one; i.e. **$\mathbb{B}$-*fuzzy subsets of* A** are elements of $[\mathcal{P}(A^{\#})]$ (countable mixings of sets), $\mathbb{B}$-fuzzy functions from A to B are elements of $(A^{B\#})$ (mixings of functions), etc. The membership function can be defined recursively for all the elements of the non standard superstructure and clearly any fuzzy element can contain also fuzzy elements of a lower type. This membership function can be completely determined by its restriction to the standard elements [3]. Details on the above notions, as well as the $\mathbb{B}$-development of Zadeh's extension principle can be found in [3]. Next we describe briefly some of possible mathematical applications of the above theory.

2. $\mathbb{B}$-FUZZY TOPOLOGICAL AND MEASURE SPACES

Let $(X, \mathcal{T})$ be a topological space. Then $(X^{\#}, \mathcal{T}^{\#})$ is a $\mathbb{B}$-fuzzy topological space, where the open subsets of X are the elements of $\mathcal{T}^{\#}$ (mixings of standard open sets). Clearly $\mathcal{T}^{\#} \subseteq [\mathcal{P}(X)]^{\#}$ and to any $\mathbb{B}$-fuzzy subset f of X a *degree of openness* (*op*) can be assigned by

$$op(f) := \|f \in \mathcal{T}^{\#}\| \in \mathbb{B}.$$

This degree of openess is with respect to the internal observer (cf. [3]). We also have a degree of oppeness with respect to the absolute external observer:

$$op_{ex}(f) := \|f \in \hat{\mathcal{T}}\| \in \mathbb{B}.$$

Similarly, if $(\Omega, \mathcal{A}, P)$ is a measure space then we construct the fuzzy measure space $(\Omega^{\#}, \mathcal{A}^{\#}, P^{\#})$, where the $\mathbb{B}$-fuzzy measurable subsets of $\Omega^{\#}$ are mixings of standard measurable subsets of Ω and $P^{\#}$ is a $\mathbb{B}$-fuzzy measure defined for any $A \in \mathcal{A}^{\#}$, $A = \sum A_i \cdot t_i$ by

$$P^{\#}(A) := \sum P(A_i) \cdot t_i \in [0, 1]^{\#}$$

We may also assign *degrees of measurabilities* (*me*) in a similar way, to any $\mathbb{B}$-fuzzy subset of $\Omega^{\#}$ by

$$me(f) := \|f \in \mathcal{A}^{\#}\| \in \mathbb{B} \quad \text{and} \quad me_{ex}(f) := \|f \in \hat{\mathcal{A}}\| \in \mathbb{B} \quad \text{correspondingly.}$$

Consequently, $\mathbb{B}$-fuzzy random variables (r. v.) are internal measurable functions of the form $X : (\Omega^{\#}, \mathcal{A}^{\#}) \to (\mathbb{R}^{\#}, \mathcal{B}^{\#})$, where $\mathcal{B}$ is the Borel σ-algebra of the real line. The rationale behind the concept of fuzzy r.v. is as follows: Suppose that we have a random experiment and at the same time there is an "observational" limitation about the values of the r. v., which results to a second kind of uncertainty, namely "vagueness". We would like to combine the two types of uncertainties into a new concept, i. e. the concept of fuzzy random variable. Our proposal for the concept of fuzzy r.v. captures all the above.

3. THE GENERALITY OF $\mathbb{B}$-FUZZY SETS

In this section we give a brief description of a theorem which proves that $\mathbb{B}$-fuzzy sets are general enough and give a factorization of a fuzzy set to a qualitative $\mathbb{B}$-valued component and a quantitative one, which is a probability.

Theorem. Let $f : X \to [0,1]$ be an arbitrary fuzzy set. Then there is a probability Boolean Algebra $(\mathbb{B}, p)$ and a $\mathbb{B}$-valued function, such that,

$$f = p \circ \pi. \qquad (\star)$$

Proof. The proof of the Theorem, essentially is a consequence of the results of Störmer [9], and the general theory of $\mathbb{B}$-fuzzy sets [1] – [6].

However one can give a more direct proof using the theory of Cantor spaces, $Y = 2^X$ and defining $\pi(x) := \{y \in Y : y(x) = 1\}$. For the measure p, let $(Z_x)_{x \in X}$ be a stochastic process defined on Y. If we define the finite dimensional distributions as in [9], then the Kolmogorov's consistency theorem is satisfied, giving a probability on Y, such that $(\star)$ holds. □

4. FUZZY PARTITIONS

For all concepts not defined here the reader is referred to [7] and other works of Piasecki. In the definition of *soft σ-algebras* there is always the nuisance of constant fuzzy set $\frac{1}{2}$. To overcome this problem we propose the following construction: Let $\mathbb{F} = \mathbb{F}(\Omega)$ be the class of all fuzzy subsets of Ω. Using an idea of Yager-Sgarro [8], we may classify fuzzy subsets $\mathbb{F}$ as follows:

Let $\mathcal{U}$ be a free ultrafilter on Ω. We may say that a property p holds *a.s.* with respect to $\mathcal{U}$ if $\{\omega \in \Omega : p(\omega) \text{ holds}\} \in \mathcal{U}$. For each $f \in \mathbb{F}$, let *the degree of fuzziness* $\hat{f}$ be defined as the essential supremum of $f^{\star} = f \wedge (1 - f)$, i.e. $\hat{f} := ess.sup\ f^{\star}$. Thus for every fuzzy set, $\hat{f} \leq \frac{1}{2}$ and $f = 0$ iff f is crisp a. s. If $\hat{f} = \frac{1}{2}$ then the degree of fuzziness of f is *unlimited or infinite*.

Let,

$$\mathbb{F}_{\infty} := \bigcup_{0 \leq \alpha < \frac{1}{2}} \mathbb{F}_{\alpha} \qquad \text{where } \mathbb{F}_{\alpha} := \{f \in \mathbb{F} : \hat{f} \leq \alpha\}$$

Then $\frac{1}{2}(\cdot) \notin \mathbb{F}_\infty$, and thus a fuzzy σ-algebra can be defined as $\sigma \subseteq \mathbb{F}_\infty$ satisfying the usual axioms. The above constructed fuzzy superstructure over Ω has a non-standard flavor and may be proven useful in general.

Now if (Ω, σ, P) is a strong fuzzy P-measure [7] then the following relation defines an equivalence relation:

$$f \simeq g \iff P[(f \wedge g') \vee (f' \wedge g)] = 0, \qquad \text{where} \qquad f' := 1 - f.$$

The $\mathbb{B} := \sigma / \simeq$ is a Boolean algebra and the canonical mapping is a σ-homomorphism. Using this probability Boolean algebra $(\mathbb{B}, p)$ in our general theory of $\mathbb{B}$-fuzzy sets [1], [2], [3], [4], we get a direct stochastic approach to fuzzy sets theory [5].

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