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SUM AND PRODUCT OF THE MODIFIED REAL FUZZY NUMBERS

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The aim of the paper is to analyse some possible applications of the convolution principle for defining a calculus between fuzzy numbers. Such approach is essentially different from the classical one, published e. g. in [1], [2]; howewer it is very similar to the principles contained in [3], [4], where the convolution principle was probably used for the first time. The specificity of this paper is the notion of the *modified fuzzy number*. All the submitted considerations can be extended to the more general algebraical and topological structures, which will be the subject of further investigations.

1. MODIFIED FUZZY NUMBER

Let $\mathcal{R}$ be the set of the real numbers endowed with usual topology. Let us denote $\mathcal{F} = \{f \in \langle 0, +\infty)^{\mathcal{R}}; f$ – piecewise continuous, $\overline{\operatorname{supp}(f)}$ – compact, $0 < \operatorname{essup}(f) < +\infty\}$, $\mathcal{F}^+ = \{f \in \mathcal{F};\ \operatorname{supp}(f) \subset \langle 0, +\infty)\}$. Let us define the equivalence relation on the sets $\mathcal{F}$ and $\mathcal{F}^+$ in the following way:

$$fEg \Leftrightarrow^{\mathrm{def}} \exists \alpha \in (0, +\infty) : f = \alpha g.$$

The elements of the factor set $\Phi = \mathcal{F}/E$ resp. $\Phi^+ = \mathcal{F}^+/E$ will be called the *modified real fuzzy numbers* or *nonnegative modified real fuzzy numbers* respectively. Let $f \in \mathcal{F}$ be an arbitrary representant of the decomposition class F. Let $\varphi = f/\operatorname{essup}(f)$. Evidently φ is the function uniquelly determined by the class F. Such a function φ satisfies the condition $0 = \operatorname{esinf}(\varphi) < \operatorname{essup}(\varphi) = 1$, and can be regarded as an essential generalisation of the notion of the fuzzy number in the sense of [1], [2]. Because of every modified fuzzy number F is uniquelly determined by an arbitrary representant $f \in F$, we can without fear of being confused, write only f instead of F. Instead of the notion "modified fuzzy number" we can use its shortened form – "fuzzy number".

2. SUM AND PRODUCT OF THE FUZZY NUMBERS

Definition 1. Let f and g be fuzzy numbers. As a sum of these numbers we shall consider the function

$$(f \dot{+} g)(x) = \int_{-\infty}^{+\infty} f(u)\, g(x-u)\, \mathrm{d}u \tag{1}$$

Definition 2. Let f and g be nonnegative fuzzy numbers. As a product of these numbers we shall consider the function

$$(f \dot{\cdot} g)(x) = \int_0^{+\infty} f(u)\, g\Big(\frac{x}{u}\Big)\, \frac{\mathrm{d}u}{u} \tag{2}$$

if an integral on the right-hand side of (2) exists.

The motivation of introducing these definitions is the fact that in the case of the sum, the sum of arguments of integrands in (1) is equal to the value x for every $x \in (-\infty, +\infty)$. Analogically it is in the case of product in the relation (2). Such a principle can be realised in infinitely many ways. The next part of the paper is devoted to the analysis of some fundamental properties. At the same time we can see why the relations (1) and (2) can by assumed as a base for reasonable definitions of the above operations.

3. FUNDAMENTAL PROPERTIES OF THE SUM AND PRODUCT

Theorem 1. Let $\dot{+} : \Phi \times \Phi \to \Phi$ be the operation from Definition 1. This operation is commutative and associative.

The contents of this assertion is the re-formulation of the well known properties of the "ordinary" convolution. Hence the proof will be omitted.

Theorem 2. Let us denote $\dot{\varphi}(.,.) : \Phi^+ \times \Phi^+ \to \langle 0, +\infty)^{\mathcal{R}}$,

$$\dot{\varphi}(f,g)(x) = \int_0^{+\infty} f(u)\, g\Big(\frac{x}{u}\Big)\, \varphi(x,u)\, \mathrm{d}u \tag{3}$$

where $\varphi(x,u)$ is a continuous function of the variable u. Then:

1) The binary operation $\dot{\varphi}$ is commutative if and only if the function $\varphi(x,u)$ satisfies the functional equation

$$\varphi\left(x, \frac{x}{u}\right) \frac{x}{u^2} = \varphi(x,u); \quad x \geq 0,\ u > 0 \tag{4}$$

2) If φ satisfies (4) and $\varphi(x,u) = p(x)q(u)$ where $q(u)$ is differentiable on $(0, +\infty)$, then $q(u) = c/u$, where c is an arbitrary constant.

P r o o f. Let $f, g \in \Phi^+$. Let us denote

$$\psi(x,u) = f(u)\, g\Big(\frac{x}{u}\Big) - g(u)\, f\Big(\frac{x}{u}\Big), \quad u > 0. \tag{5}$$

It is easy to see

$$\psi(x,u) = -\psi\left(x, \frac{x}{u}\right). \tag{6}$$

The commutativity of the operation $\dot{\varphi}$ is equivalent to the condition

$$\int_0^{+\infty} \psi(x,u)\, \varphi(x,u)\, \mathrm{d}u = 0. \tag{7}$$

Employing condition (4), relation (6) and substitution $u = x/t$ after an elementary computation we obtain

$$\int_{x/a}^{\sqrt{x}} \psi(x,u)\,\varphi(x,u)\,\mathrm{d}u = -\int_{\sqrt{x}}^{a} \psi(x,u)\,\varphi(x,u)\,\mathrm{d}u \tag{8}$$

hence

$$\int_{0}^{+\infty} \psi(x,u)\,\varphi(x,u)\,\mathrm{d}u = \lim_{a\to\infty}\int_{x/a}^{a} \psi(x,u)\,\varphi(x,u)\,\mathrm{d}u = 0$$

Sufficiency of the condition (4) is proved.

Conversly, let $\tilde{\varphi}$ be a commutative operation. It is easy to see that every function $\psi(x,u)$ which satisfies condition (6) can be expressed in the form (5) where f and g are suitable functions. Let us choose

$$\psi(x,u) = \chi_{\langle x/(w+\Delta w)\rangle}(u) - \chi_{\langle w, w+\Delta w\rangle}(u)$$

where χ_A is the characteristic function of the set A, $\Delta w > 0$, $x > 0$. The condition (7) is equivalent to the condition

$$\int_{x/(x+\Delta w)}^{x/w} \varphi(x,u)\,\mathrm{d}u = \int_{w}^{w+\Delta w} \varphi(x,u)\,\mathrm{d}u \tag{9}$$

Using the mean value theorem and by differentiation we have

$$\varphi(x,\xi_1)[x/w^2.\Delta w + o(\Delta w)] = \varphi(x,\xi_2)\Delta w \tag{10}$$

where

$$\xi_1 \in \langle x/(w+\Delta w), x/w\rangle,\ \xi_2 \in \langle w, w+\Delta w\rangle,\ \lim_{z\to 0} o(z)/z = 0$$

The limit process for $\Delta w \to 0$ applied on (10) implies

$$\varphi(x, x/w)\,x/w^2 = \varphi(x,w);\quad w > 0$$

The necessity of (4) is proved.

2) After a short arrangement we can show that (4) is equivalent to

$$\varphi(u, \sqrt{u}/\alpha) = \alpha^2\varphi(u, \alpha\sqrt{u}),\quad \alpha > 0,\ u \geq 0 \tag{11}$$

If $\varphi(x,u) = p(x)q(u)$ then

$$q(\sqrt{u}/\alpha) = \alpha^2 q(\alpha\sqrt{u}),\quad \alpha > 0 \tag{12}$$

Since q is differentiable, then

$$q'(t)|_{t=\sqrt{u}} = \lim_{\alpha\to 1}\{(q(\alpha\sqrt{u}) - q(\sqrt{u}/\alpha)\}\Big/ (\alpha\sqrt{u} - \sqrt{u}/\alpha)\} = -q(\sqrt{u})/\sqrt{u} \tag{13}$$

Solution of the differential equation (13) is the function $q(u)$ of the type $q(u) = c/u$, where c is an arbitrary constant. The second part of the theorem is proved. □

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