KYBERNETIKA — VOLUME 28 (1992), NUMBER 6, PAGES 454–471

ADAPTIVE MAXIMUM–LIKELIHOOD–LIKE ESTIMATION IN LINEAR MODELS

Part 2. Asymptotic normality

JAN ÁMOS VÍŠEK

An adaptive estimator of regression model coefficients based on maximization of kernel estimate of likelihood was proposed and its consistency proved in the Part 1. Asymptotic normality is shown in the Part 2. An asymptotic representation of the estimate implies also its asymptotic efficiency. .

1. INTRODUCTION

This paper is a continuation of the paper "Adaptive maximum-lkelihood-like estimation in linear models. Part 1. Consistency". The reasons and discussions about the adaptive estimation may be found there and also in [8] and [9]. The notation of the present paper is the same as in the Part 1 and numeration of Theorems and Lemmas continues.

The proof of consistency of the maximum-likelihood-like adaptive estimator has shown that the basic technique is simple application of classical tools of stochastic approximation. This technique overcomes difficulty caused by the fact that residuals in regression model are (weakly) dependent. The same technique is used here. That is why some proofs were omitted. On the other hand at some places we have left also details to facilitate reading where some hesitations could occur.

The conditions under which all results will be given are the same as in Part 1. Since the conditions are rather complicated to write down we will not recall them now and we refer to Chapter 3 (Assumptions of Part 1 (i.e. to Conditions A, B, C and D)). We shall recall them just before the Theorem 2. Nevertheless to prove asymptotic normality we will need one additional condition.

Condition E. Let Fisher information $I(g)$ exist and be finite. Moreover let for any $j = 1, \ldots, n$, $i = 1, \ldots, n$, $t \in \mathcal{R}$ and $s \in \mathcal{R}$

$$P(\tilde{e}_i - \mathsf{E}\{\tilde{e}_i | e_j = t\} < -s | e_j = t) = P(\tilde{e}_i - \mathsf{E}\{\tilde{e}_i | e_j = t\} > s | e_j = t).$$

Remark 7. The requirement to use as a preliminary estimator $\tilde{\beta}^n$ such estimator which implies "symmetry" of distribution of $\tilde{e}_i$ may seem – at a first glance – rather restrictive. But under assumption what e_i's are symmetrically distributed only estimator $\tilde{\beta}^n$ which prefers in some way positive values of residuals before the negative ones (or vice versa; for instance, estimators based on asymmetric trimming) may yield $\tilde{e}_i$ "asymmetrically" distributed.

2. PRELIMINARIES

Lemma 5 (Csörgö, Révész [2].) If $g(y)$ has bounded derivative on an interval $-\infty \leq A < B \leq +\infty$ then for any $\varepsilon > 0$ we have

$$\sup_{A+\varepsilon \leq y \leq B-\varepsilon} \left| \mathsf{E} g_n(y, Y, \beta^0) - g(y) \right| = O(c_n).$$

Further we have

$$\sup_{A+\varepsilon \leq y \leq B-\varepsilon} \; \operatorname*{var}_{A+\varepsilon \leq z \leq B-\varepsilon} \; c_n^{-1} w\left(c_n^{-1}(y-z)\right) = c_n^{-1}.$$

(Clearly we mean $-\infty + \varepsilon = -\infty$, $+\infty - \varepsilon = +\infty$.)

Proof. See [2], Lemma 6.1.1. □

Lemma 6 (Csörgö, Révész [2].) Let for any $y \in \mathcal{R}$

$$\begin{aligned} &\lim_{z \to -\infty} c_n^{-1} w\left(c_n^{-1}(y-z)\right) \left[G(z) \log\log G^{-1}(z)\right]^{\frac{1}{2}} \\ =\ &\lim_{z \to \infty} c_n^{-1} w\left(c_n^{-1}(y-z)\right) \left[(1-G(z)) \log\log(1-G(z))^{-1}\right]^{\frac{1}{2}} = 0 \end{aligned}$$

then

$$\lim_{n \to \infty} \sup_{y \in \mathcal{R}} \left| g_n(y, Y, \beta^0) - g(y) \right| = 0 \qquad \text{a.e. } g.$$

Proof. See [2], Theorem 6.2.1. □

Lemma 7. We have

$$\frac{1}{nc_n^3} \sum_{i=1}^{n} \int \left[w''(c_n^{-1}(y-e_i)) - \mathsf{E} w''(c_n^{-1}(y-e_i)) \right] b_n(y) \mathrm{d}y = o_p(1).$$

Proof. For any $\varepsilon > 0$ we have

$$P\left\{ \left| \frac{1}{nc_n^3} \sum_{i=1}^{n} \int \left[w''(c_n^{-1}(y-e_i)) - \mathsf{E} w''(c_n^{-1}(y-e_i)) \right] b_n(y) \mathrm{d}y \right| > \varepsilon \right\}$$

$$\leq \frac{1}{\varepsilon^2 n^2 c_n^6} \mathsf{E} \left\{ \sum_{i=1}^{n} \left[\int w''(c_n^{-1}(y - e_i)) b_n(y) \mathrm{d}y - \int \mathsf{E} w''(c_n^{-1}(y - e_i)) b_n(y) \mathrm{d}y \right] \right\}^2$$

$$\leq \frac{1}{\varepsilon^2 n c_n^6} \mathsf{E} \left[\int w''(c_n^{-1}(y - e_1)) b_n(y) \mathrm{d}y \right]^2 \leq$$

$$\leq \frac{1}{\varepsilon^2 n c_n^6} \int b_n(y) \mathrm{d}y \int \int \left[\frac{w''(c_n^{-1}(y - z))}{w(c_n^{-1}(y - z))} \right]^2 w^2(c_n^{-1}(y - z)) b_n(y) g(z) \mathrm{d}z \mathrm{d}y$$

$$\leq \frac{a_n^2 K_3^2 \cdot K_1}{\varepsilon^2 n c_n^6}. \qquad \square$$

Since in what follows we shall prepare only technical steps for proving Theorems 2 we shall assume that Conditions A, B, C, D and E hold not stating it explicitly.

Lemma 8. We also have

$$\int \left[\frac{d^2 \mathsf{E} g_n(z, Y, \beta^0)}{dz^2} \right]_{z=y} b_n(y) \mathrm{d}y = o(1).$$

Proof may be carried out nearly along the same lines as the proof of Lemma 6 of [10].

The absolute value of the integral in the assertion of lemma is bounded from above by

$$\frac{1}{n c_n^3} \sum_{i=1}^{n} \left| \int \int w''(c_n^{-1}(y - r)) g(r) b_n(y) \mathrm{d}r \mathrm{d}y \right|$$

and in fact the first part of proof of Lemma 6 in [10] is devoted to proving that

$$\frac{1}{n c_n^2} \left| \int \int_{-a_n}^{a_n} w''(t) g(y - t c_n) \mathrm{d}t \mathrm{d}y \right| \xrightarrow[n \to \infty]{} 0$$

and it is nothing else than the fact that

$$\frac{1}{n c_n^3} \int \int w''(c_n^{-1}(y - r)) g(r) b_n(y) \mathrm{d}r \mathrm{d}y \xrightarrow[n \to \infty]{} 0$$

which is equivalent to the assertion of the Lemma. □

Lemma 9. We have

$$\int \left[\frac{d g_n^{\frac{1}{2}}(z, Y, \beta^0)}{dz} - \frac{d \mathsf{E}^{\frac{1}{2}} g_n(z, Y, \beta^0)}{dz} \right]_{z=y}^2 b_n^2(y) \mathrm{d}y = o_p(1).$$

For the proof see [1], Lemma 3 or [10], Lemma 3. □

Lemma 10 (Beran [1].)

$$\lim_{n\to\infty} c_n^{-3} \int \frac{\left[\int w'(c_n^{-1}(y-z))g(z)\mathrm{d}z\right]^2}{\int w(c_n^{-1}(y-z))g(z)\mathrm{d}z}\mathrm{d}y = I(g).$$

For the proof see [1] - lemma is not isolated there - or [10], Lemma 9. □

We are going to prove one of lemmas which are basic for establishing asymptotic normality. Although the proof of lemma is rather long we have decided to give it in details because it illustrates technique of (simple but unfortunately tedious chain of) approximations. Results of the approximations enable us, however, to substitute kernel estimates (of density and its derivatives) by corresponding integrals. On the other hand since the proofs of the rest of lemmas are very similar, using precisely this technique, they will be omitted.

Lemma 11. For any $k, \ell = 1, \ldots, p$ we have

$$n^{-1}c_n^{-2}\sum_{j=1}^{n} x_{jk}\, x_{j\ell}\left[\frac{\sum_{i=1}^{n} w''(c_n^{-1}(e_j-\tilde{e}_i))}{\sum_{r=1}^{n} w(c_n^{-1}(e_j-\tilde{e}_r))}b_n(e_j) - \right.$$
$$\left. - \; c_n^{-1}\int\int w''(c_n^{-1}(y-z))g(z)b_n(y)\mathrm{d}z\mathrm{d}y\right] = o_p(1)$$

and

$$n^{-1}c_n^{-2}\sum_{j=1}^{n} x_{jk}\, x_{j\ell}\left[\left\{\frac{\sum_{i=1}^{n} w'(c_n^{-1}(e_j-\tilde{e}_i))}{\sum_{r=1}^{n} w(c_n^{-1}(e_j-\tilde{e}_r))}\right\}^2 b_n(e_j) - \right.$$
$$\left. - \; c_n^{-1}\int \frac{\left[\int w'(c_n^{-1}(y-z))g(z)dz\right]^2}{\int w(c_n^{-1}(y-z))g(z)\mathrm{d}z}b_n(y)\mathrm{d}y\right] = o_p(1).$$

Proof. At first we shall show that the following difference is small in probability.

$$n^{-1}c_n^{-2}\sum_{j=1}^{n} x_{jk}\, x_{j\ell}\left[\frac{\sum_{i=1}^{n} w''(c_n^{-1}(e_j-\tilde{e}_i))}{\sum_{r=1}^{n} w(c_n^{-1}(e_j-\tilde{e}_r))} - \frac{\sum_{i\neq j} w''(c_n^{-1}(e_j-\tilde{e}_i))}{\sum_{r\neq j} w(c_n^{-1}(e_j-\tilde{e}_r))}\right] b_n(e_j)$$
$$= n^{-1}c_n^{-2}\sum_{j=1}^{n} x_{jk}\, x_{j\ell}\left[\frac{\sum_{i=1}^{n} w''(c_n^{-1}(e_j-\tilde{e}_i))\left\{\sum_{r\neq j} w(c_n^{-1}(e_j-\tilde{e}_r)) - \sum_{r=1}^{n} w(c_n^{-1}(e_j-\tilde{e}_r))\right\}}{\sum_{r=1}^{n} w(c_n^{-1}(e_j-\tilde{e}_r))\sum_{t\neq j} w(c_n^{-1}(e_j-\tilde{e}_t))} + \right.$$
$$\left. + \frac{\sum_{i=1}^{n} w(c_n^{-1}(e_j-e_i))\left\{\sum_{i=1}^{n} w''(c_n^{-}(e_j-\tilde{e}_i)) - \sum_{i\neq j} w''(c_n^{-1}(e_j-\tilde{e}_i))\right\}}{\sum_{r=1}^{n} w(c_n^{-1}(e_j-\tilde{e}_r))\sum_{t\neq j} w(c_n^{-1}(e_j-e_t))}\right] b_n(e_j). \quad (22)$$

Let us consider the first member of the right-hand-side. Since $\sum_{i=1}^{n} |w''(c_n^{-1}(e_j-\tilde{e}_i)| \cdot [\sum_{r=1}^{n} w(c_n^{-1}(e_j-\tilde{e}_r))]^{-1} \leq K_3$ it is sufficient to show that

$$K_3 n^{-1}c_n^{-2}\sum_{j=1}^{n} x_{jk}\, x_{j\ell}\frac{\frac{1}{n}w(c_n^{-1}(e_j-\tilde{e}_j))}{\frac{1}{n}\sum_{r\neq j} w(c_n^{-1}(e_j-\tilde{e}_r))} \quad (23)$$

is small in probability. $\sup w(z) < K_1$ and also $\sup_{\substack{j=1,\dots,n \\ k,\ell=1,\dots,n}} |x_{jk}\, x_{j\ell}| < K_5^2$, hence the last expression is not greater than

$$n^{-2} c_n^{-2} K_3 \cdot K_1 \cdot K_5^2 \sum_{j=1}^{n} \frac{1}{\frac{1}{n}\sum_{r\neq j} w(c_n^{-1}(e_j - \tilde{e}_r))}.$$

Finally, using Assertion 2 (Part 1) we find an upper bound for (23) in the form

$$n^{-3} c_n^{-2} K_3 \cdot K_1 \cdot K_5^2 \sum_{j=1}^{n} \sum_{r\neq j} \frac{1}{w(c_n^{-1}(e_j - \tilde{e}_r)}. \tag{24}$$

Now for any $\varepsilon > 0$

$$P\left\{ n^{-3} c_n^{-2} \sum_{j=1}^{n} \sum_{r\neq j} \frac{1}{w(c_n^{-1}(e_j - e_r)} > \frac{\varepsilon}{2} \right\} \leq \tag{25}$$
$$\leq \; 2\varepsilon^{-1} n^{-1} c_n^{-2} \int w^{-1}(c_n^{-1}(z-t)) g(z) g(t) \mathrm{d}z \mathrm{d}t$$

which converges to zero for $n \to \infty$ due to Condition B. Let us denote by

$$B_n = \left\{ \omega \in \Omega : \; n^{-3} c_n^{-2} \sum_{j=1}^{n} \sum_{r\neq j} \frac{1}{w(c_n^{-1}(e_j - e_r))} > \frac{\varepsilon}{2} \right\}$$

and fix a $\Delta > 0$. Then find n_0 so that for any $n \geq n_0$ $P(B_n) < \Delta$. For $w(c_n^{-1}(e_j - \tilde{e}_r))$ write

$$w(c_n^{-1}(e_j - \tilde{e}_r)) = w(c_n^{-1}(e_j - e_r)) + c_n^{-1} w'(\xi_{jrn}) \cdot X_r^{\mathrm{T}}(\beta^0 - \tilde{\beta}^n)$$

where

$$\xi_{jnr} \in \left[c_n^{-1} \min\{e_j - e_r,\, e_j - \tilde{e}_r\},\, c_n^{-1} \max\{e_j - e_r, e_j - \tilde{e}_r\} \right].$$

Due to the assumption about the order of consistency of $\tilde{\beta}^n$ we may find $n_1 \in \mathcal{N}$ and $L \in \mathcal{R}$ such that for any $n \geq n_1$

$$P\left\{ n^{\delta} \,\| \beta^0 - \tilde{\beta}^n \| > L \right\} < \Delta.$$

Denote by $C_n = \left\{ \omega \in \Omega : \; n^{\delta} \|\beta^0 - \tilde{\beta}^n\| > L \right\}$ and

$$E_n = \left\{ \omega \in \Omega : \; n^{-3} c_n^{-2} \sum_{j=1}^{n} \sum_{r\neq j} w^{-1}(c_n^{-1}(e_j - \tilde{e}_r)) > \varepsilon \right\}.$$

Find $n_2 \in \mathcal{N}$, $n_2 \geq n_1$ such that for any $n \geq n_2$ $n^{-\delta} c_n^{-1} K_2 \cdot K_5 \cdot D \cdot L \cdot p < \frac{1}{2}$ and $n^{-\delta} c_n^{-1} L \cdot K_5 \cdot p < \nu$ (remember that D and ν were introduced in Condition B). Let us recall that we have

$$|e_j - \tilde{e}_r - e_j + e_r| = |e_r - \tilde{e}_r| = \left| X_r^{\mathrm{T}}(\beta^0 - \tilde{\beta}^n) \right|.$$

Let us consider any $\omega \in C_n^c$. Then $n^\delta \|\beta^0 - \tilde{\beta}^n\| < L$ and therefore

$$c_n^{-1}|e_r - \tilde{e}_r| = c_n^{-1}\left|X_r^{\mathrm{T}}(\beta^0 - \tilde{\beta}^n)\right| < \nu,$$

and hence

$$|\xi_{jrn} - c_n^{-1}(e_j - e_r)| < \nu.$$

It implies (see Condition B) that

$$\left|\frac{w(\xi_{jrn})}{w(c_n^{-1}(e_j - e_r))}\right| < D.$$

Hence it holds for any $\omega \in C_n^c$

$$\begin{aligned}|w'(\xi_{jrn})| &= \left|\frac{w'(\xi_{jrn})}{w(\xi_{jrn})} \cdot \frac{w(\xi_{jrn})}{w(c_n^{-1}(e_j - e_r))} \cdot w(c_n^{-1}(e_j - e_r))\right| \\ &\leq K_2 \cdot D \cdot w(c_n^{-1}(e_j - e_r)).\end{aligned}$$

It again implies that for $\omega \in C_n^c$ and $n \geq n_2$

$$\begin{aligned}w(c_n^{-1}(e_j - \tilde{e}_r)) &\geq w(c_n^{-1}(e_j - e_r))\left[1 - n^{-\delta}c_n^{-1} \cdot p \cdot K_2 \cdot K_5 \cdot D \cdot L\right] \\ &\geq \frac{1}{2}w(c_n^{-1}(e_j - e_r)).\end{aligned}$$

Now for any $\omega \in E_n \cap C_n^c$ we have

$$\varepsilon < n^{-3}c_n^{-2}\sum_{j=1}^{n}\sum_{r\neq j} w^{-1}(c_n^{-1}(e_j - \tilde{e}_r)) < 2n^{-3}c_n^{-2}\sum_{j=1}^{n}\sum_{r\neq j} w^{-1}(c_n^{-1}(e_j - e_r))$$

which means that $\omega \in B_n$. But it gives

$$P(E_n) = P(E_n \cap C_n) + P(E_n \cap C_n^c) \leq P(C_n) + P(B_n) \leq 2\Delta.$$

It proves that the first member of the right-hand-side of (22) is small in probability. For the second one we obtain instead of (23) an expression

$$n^{-1}c_n^{-2}\sum_{j=1}^{n} x_{jk}\, x_{j\ell}\frac{\frac{1}{n}w''(c_n^{-1}(e_j - \tilde{e}_j))}{\frac{1}{n}\sum_{r\neq j} w(c_n^{-1}(e_j - \tilde{e}_r))}$$

which is not greater than (compare (24))

$$n^{-3}c_n^{-2}K_3 \cdot K_1 \cdot K_5^2 \sum_{j=1}^{n}\sum_{r\neq j}^{n}\left[w^{-1}(c_n^{-1}(e_j - \tilde{e}_r))\right]$$

and hence the second member of (22) is also small in probability.

Now let us show that also

$$n^{-1}c_n^{-2}\sum_{j=1}^{n} x_{jk}\, x_{j\ell}\left[\frac{\sum_{i\neq j} w''(c_n^{-1}(e_j - \tilde{e}_i))}{\sum_{r\neq j} w(c_n^{-1}(e_j - \tilde{e}_r))} - \frac{\sum_{i\neq j} w''(c_n^{-1}(e_j - e_i))}{\sum_{r\neq j} w(c_n^{-1}(e_j - e_r))}\right\} b_n(e_j) \qquad (26)$$

is small in probability. Similarly as above we obtain that (26) is equal to

$$n^{-1}c_n^{-2}\sum_{j=1}^{n} x_{jk}\,x_{j\ell}\left\{\frac{\sum_{t\neq j} w(c_n^{-1}(e_j-e_t))\sum_{i\neq j}[w''(c_n^{-1}(e_j-\tilde{e}_i)-w''(c_n^{-1}(e_j-e_i))]}{\sum_{r\neq j} w(c_n^{-1}(e_j-\tilde{e}_r))\sum_{t\neq j} w(c_n^{-1}(e_j-e_t))}\right.$$
$$+\left.\frac{\sum_{i\neq j} w''(c_n^{-1}(e_j-e_i))\sum_{t\neq j}[w(c_n^{-1}(e_j-e_t)-w(c_n^{-1}(e_j-\tilde{e}_t))]}{\sum_{r\neq j} w(c_n^{-1}(e_j-\tilde{e}_r))\sum_{t\neq j} w(c_n^{-1}(e_j-e_t))}\right\} b_n(e_j). \tag{27}$$

For the first member we obtain an upper bound in the form

$$n^{-1}c_n^{-2}K_5^2\sum_{j\neq 1}\frac{\sum_{i\neq j}\left|w'''(\xi_{jin})\cdot c_n^{-1}X_i^{\mathrm T}(\tilde{\beta}^n-\beta^0)\right|}{\sum_{r\neq j} w(c_n^{-1}(e_j-\tilde{e}_r))} \tag{28}$$
$$\leq\; n^{-1}c_n^{-3}K_5^3\cdot p\cdot\|\tilde{\beta}^n-\beta^0\|\sum_{j=1}^{n}\frac{\sum_{i\neq j}|w'''(\xi_{jin})|}{\sum_{r\neq j} w(c_n^{-1}(e_j-\tilde{e}_r))}$$

where again $\xi_{jin}\in[c_n^{-1}\min\{e_j-e_i,\,e_j-\tilde{e}_i\},\,c_n^{-1}\max\{e_j-e_i,\,e_j-\tilde{e}_i\}]$. Now fix $\varepsilon>0,\Delta>0$ and find $n_1\in\mathcal{N}$ and $L>0$ so that for any $n\geq n_1$ we have

$$P\left\{n^{\delta}\|\tilde{\beta}^n-\beta^0\|>L\right\}<\Delta$$

and denote by $C_n=\left\{\omega\in\Omega:\ n^{\delta}\|\tilde{\beta}^n-\beta^0\|>L\right\}$. Moreover find $n_2\in\mathcal{N},\ n_2\geq n_1$ such that for any $n\geq n_2$ we have

$$n^{-\delta}c_n^{-1}K_5\cdot p\cdot L<\nu.$$

(Remember that ν stays in Condition B and keep its role in mind.) Then we have again for any $\omega\in C_n^c$ and $n\geq n_2$

$$\begin{aligned}&\left|\xi_{jin}-c_n^{-1}(e_j-\tilde{e}_i)\right|\leq c_n^{-1}\left|e_j-e_i-e_j+\tilde{e}_i\right|\\ =\;& c_n^{-1}|e_j-\tilde{e}_i|=c_n^{-1}\left|X_i^{\mathrm T}(\tilde{\beta}^n-\beta^0)\right|\\ \leq\;& c_n^{-1}\cdot K_5\cdot p\,\|\tilde{\beta}^n-\beta^0\|=n^{-\delta}c_n^{-1}\cdot K_5\cdot p\cdot n^{\delta}\|\tilde{\beta}^n-\beta^0\|\leq\nu\end{aligned}$$

and hence

$$\begin{aligned}|w'''(\xi_{jin})|&\leq\frac{|w'''(\xi_{jin})|}{w(\xi_{jin})}\cdot\frac{w(\xi_{jin})}{w(c_n^{-1}(e_j-\tilde{e}_i))}\,w(c_n^{-1}(e_j-\tilde{e}_i))\\&\leq K_4\cdot D\cdot w(c_n^{-1}(e_j-\tilde{e}_i)).\end{aligned}$$

So expression (28) is for $n\geq n_2$ and $\omega\in C_n^c$ bounded by

$$c_n^{-3}K_4^3\cdot K_5\cdot p\cdot\|\tilde{\beta}^n-\beta^0\|\cdot D.$$

Therefore finding $n_3\in\mathcal{N},\ n_3\geq n_2$ so that for any $n\geq n_3$

$$n^{-\delta}c_n^{-3}K_4^3\cdot K_5\cdot D\cdot p\cdot L<\varepsilon,$$

we have for any $n \geq n_3$ and $\omega \in C_n^c$ (remember that for $\omega \in C_n^c$ and $n \geq n_2$ (and hence also for $n \geq n_3$) we have $\|\tilde{\beta}^n - \beta^0\| < Ln^{-\delta}$)

$$\left| n^{-1} c_n^{-2} \sum_{j=1}^{n} x_{jk}\, x_{j\ell} \frac{\sum_{i \neq j} [w''(c_n^{-1}(e_j - \tilde{e}_i)) - w''(c_n^{-1}(e_j - e_i))]}{\sum_{r \neq j} w(c_n^{-1}(e_j - e_r))} \right| < \varepsilon$$

and it implies that for any $n \geq n_3$ and for any $\omega \in \Omega$ for which

$$\left| n^{-1} c_n^{-2} \sum_{j=1}^{n} x_{jk}\, x_{j\ell} \frac{\sum_{i \neq j} [w''(c_n^{-1}(e_j - \tilde{e}_i)) - w''(c_n^{-1}(e_j - e_i))]}{\sum_{r \neq j} w(c_n^{-1}(e_j - e_r))} \right| > \varepsilon$$

we have $\omega \in C_n$. But it means that

$$P \left\{ \left| n^{-1} c_n^{-2} \sum_{j=1}^{n} x_{jk} x_{j\ell} \left\{ \frac{\sum_{i \neq j} [w''(c_n^{-1}(e_j - \tilde{e}_i)) - w''(c_n^{-1}(e_j - e_i))]}{\sum_{r \neq j} w(c_n^{-1}(e_j - \tilde{e}_r))} \right\} \right| > \varepsilon \right\} < \Delta.$$

The second member of (27) may be treated along the similar lines as the first one.

The next step will be to show that also

$$S_{nk\ell} = n^{-1} c_n^{-2} \sum_{j=1}^{n} x_{jk}\, x_{j\ell} \left[\frac{\sum_{i \neq j} w''(c_n^{-1}(e_j - e_i))}{\sum_{r \neq j} w(c_n^{-1}(e_j - e_r))} b_n(e_j) - \right.$$
$$\left. - \;\mathsf{E} \left\{ \frac{\sum_{i \neq j} w''(c_n^{-1}(e_j - e_i))}{\sum_{r \neq j} w(c_n^{-1}(e_j - e_r))} b_n(e_j) \middle| e_1 = z_1, \ldots, e_{j-1} = z_{j-1}, e_{j+1} = z_{j+1}, \ldots, e_n = z_n \right\} \right]$$

is small in probability. Let us fix again some $\varepsilon > 0$. Then using Chebyshev's inequality one obtains

$$P\left(|S_{nk\ell}| > \varepsilon\right) \leq \frac{1}{\varepsilon^2} \mathsf{E} S_{nk\ell}^2 = \frac{1}{\varepsilon^2} \sum_{h=1}^{4} \mathcal{E}_h$$

where

$$\mathcal{E}_1 = \mathsf{E} n^{-2} c_n^{-4} \sum_{j=1}^{n} x_{jk}^2\, x_{j\ell}^2 \left[\frac{\sum_{i \neq j} w''(c_n^{-1}(e_j - e_i))}{\sum_{r \neq j} w(c_n^{-1}(e_j - e_r))} b_n(e_j) - \right.$$
$$\left. - \;\mathsf{E} \left\{ \frac{\sum_{i \neq j} w''(c_n^{-1}(e_j - e_i))}{\sum_{r \neq j} w(c_n^{-1}(e_j - e_r))} b_n(e_j) \middle| e_1 = z_1, \ldots, e_{j-1} = z_{j-1}, e_{j+1} = z_{j+1}, \ldots, e_n = z_n \right\} \right]^2,$$

$$\mathcal{E}_2 = 2\mathsf{E} n^{-2} c_n^{-4} \sum_{j=1}^{n} \sum_{s > j} x_{jk}\, x_{j\ell}\, x_{sk} x_{s\ell} \left[\frac{\sum_{i \neq j} w''(c_n^{-1}(e_j - e_i))}{\sum_{r \neq j} w(c_n^{-1}(e_j - e_r))} b_n(e_j) - \right.$$
$$\left. - \;\mathsf{E} \left\{ \frac{\sum_{i \neq j} w''(c_n^{-1}(e_j - e_i))}{\sum_{r \neq j} w(c_n^{-1}(e_j - e_r))} b_n(e_j) \middle| e_1 = z_1, \ldots, e_{j-1} = z_{j-1}, e_{j+1} = z_{j+1}, \ldots, e_n = z_n \right\} \right]$$
$$\left[\frac{\sum_{\substack{t \neq s \\ t \neq j}} w''(c_n^{-1}(e_s - e_t))}{\sum_{\substack{v \neq s \\ v \neq j}} w(c_n^{-1}(e_s - e_v))} b_n(e_s) - \right.$$

$$- \mathsf{E}\left\{\frac{\sum_{\substack{t\neq s\\ t\neq j}} w''(c_n^{-1}(e_s-e_t))}{\sum_{\substack{v\neq s\\ v\neq j}} w(c_n^{-1}(e_s-e_v))}b_n(e_s)\right| e_1=z_1,\ldots,e_{s-1}=z_{s-1},e_{s+1}=z_{s+1},\ldots,e_n=z_n\left.\vphantom{\frac{1}{1}}\right\}\Bigg]\Bigg],$$

$$\mathcal{E}_3 = 2\mathsf{E}n^{-2}c_n^{-4}\sum_{j=1}^{n}\sum_{s>j} x_{jk}x_{j\ell}x_{sk}x_{s\ell}\left[\frac{\sum_{i\neq j} w''(c_n^{-1}(e_j-e_i))}{\sum_{r\neq j} w(c_n^{-1}(e_j-e_r))}b_n(e_j)-\right.$$

$$- \mathsf{E}\left\{\frac{\sum_{i\neq j} w''(c_n^{-1}(e_j-e_i))}{\sum_{r\neq j} w(c_n^{-1}(e_j-e_r))}b_n(e_j)\right| e_1=z_1,\ldots,e_{j-1}=z_{j-1},e_{j+1}=z_{j+1},\ldots,e_n=z_n\left.\vphantom{\frac{1}{1}}\right\}\Bigg]$$

$$\left[\frac{\sum_{\substack{t\neq s\\ t\neq j}} w''(c_n^{-1}(e_s-e_t))}{\sum_{\substack{v\neq s\\ v\neq j}} w(c_n^{-1}(e_s-e_v))}b_n(e_s) - \frac{\sum_{\substack{t\neq s\\ t\neq j}} w''(c_n^{-1}(e_s-e_t))}{\sum_{\substack{v\neq s\\ v\neq j}} w(c_n^{-1}(e_s-e_v))}b_n(e_s)\right]$$

and finally

$$\mathcal{E}_4 = 2\mathsf{E}n^{-2}c_n^{-4}\sum_{j=1}^{n}\sum_{s>j} x_{jk}x_{j\ell}x_{sk}x_{s\ell}\left[\frac{\sum_{i\neq j} w''(c_n^{-1}(e_j-e_i))}{\sum_{r\neq j} w(c_n^{-1}(e_j-e_r))}b_n(e_j)-\right.$$

$$- \mathsf{E}\left\{\frac{\sum_{i\neq j} w''(c_n^{-1}(e_j-e_i))}{\sum_{r\neq j} w(c_n^{-1}(e_j-e_r))}b_n(e_j)\right| e_1=z_1,\ldots,e_{j-1}=z_{j-1},e_{j+1}=z_{j+1},\ldots,e_n=z_n\left.\vphantom{\frac{1}{1}}\right\}\Bigg]$$

$$\Bigg[\mathsf{E}\left\{\frac{\sum_{t\neq s} w''(c_n^{-1}(e_s-e_t))}{\sum_{v\neq s} w(c_n^{-1}(e_s-e_v))}b_n(e_s)\right| e_1=z_1,\ldots,e_{s-1}=z_{s-1},e_{s+1}=z_{s+1},\ldots,e_n=z_n\left.\vphantom{\frac{1}{1}}\right\}$$

$$- \mathsf{E}\left\{\frac{\sum_{\substack{t\neq s\\ t\neq j}} w''(c_n^{-1}(e_s-e_t))}{\sum_{\substack{v\neq s\\ v\neq j}} w(c_n^{-1}(e_s-e_v))}b_n(e_s)\right| e_1=z_1,\ldots,e_{s-1}=z_{s-1},e_{s+1}=z_{s+1},\ldots,e_n=z_n\left.\vphantom{\frac{1}{1}}\right\}\Bigg].$$

Since

$$\frac{\sum_{i\neq j} |w''(c_n^{-1}(e_j-e_i))|}{\sum_{r\neq j} w(c_n^{-1}(e_j-e_r))} < K_3,$$

$\mathcal{E}_1$ may be bounded by $n^{-1}c_n^{-4}\cdot K_5^2\cdot K_3^2$. $\mathcal{E}_2$ may be rewritten into the form (remember that $\mathsf{E}\{\mathcal{Z}_1\mathcal{Z}_2\} = \mathsf{E}\{\mathcal{Z}_1\mathsf{E}[\mathcal{Z}_2|\mathcal{Z}_1]\}$)

$$2\mathsf{E}\left\{n^{-2}c_n^{-4}\sum_{j=1}^{n}\sum_{s>j} x_{jk}x_{j\ell}x_{sk}x_{s\ell}\left[\frac{\sum_{\substack{t\neq s\\ t\neq j}} w''(c_n^{-1}(e_s-e_t))}{\sum_{\substack{v\neq s\\ v\neq j}} w(c_n^{-1}(e_s-e_v))}b_n(e_s)\right.\right.$$

$$- \mathsf{E}\left\{\frac{\sum_{\substack{t\neq s\\ t\neq j}} w''(c_n^{-1}(e_s-e_t))}{\sum_{\substack{v\neq s\\ v\neq j}} w(c_n^{-1}(e_s-e_v))}b_n(e_s)\right| e_1=z_1,\ldots,e_{s-1}=z_{s-1},e_{s+1}=z_{s+1},\ldots,e_n=z_n\left.\vphantom{\frac{1}{1}}\right\}\Bigg]$$

$$\mathsf{E}\left\{\left[\frac{\sum_{i\neq j} w''(c_n^{-1}(e_j-e_i))}{\sum_{r\neq j} w(c_n^{-1}(e_j-e_r))}b_n(e_j)-\right.\right.$$

$$- \mathsf{E}\left\{\frac{\sum_{i\neq j} w''(c_n^{-1}(e_j-e_i))}{\sum_{r\neq j} w(c_n^{-1}(e_j-e_r))}b_n(e_j)\right| e_1=z_1,\ldots,e_{j-1}=z_{j-1},e_{j+1}=z_{j+1},\ldots,e_n=z_n\left.\vphantom{\frac{1}{1}}\right\}\Bigg]$$

$$\left| e_1=z_1,\ldots,e_{j-1}=z_{j-1},e_{j+1}=z_{j+1},\ldots,e_n=z_n\right\}\Bigg\}.$$

The modification is possible due to fact that the expression $\frac{\sum_{\substack{t\neq s\\ t\neq j}} w''(c_n^{-1}(e_s-e_t))}{\sum_{\substack{v\neq s\\ v\neq j}} w(c_n^{-1}(e_s-e_v))} b_n(e_s)$ as well as its conditional mean value depends only on random variables which are "fixed" by the set in condition, namely $\{e_1 = z_1, \ldots, e_{j-1} = z_{j-1}, e_{j+1} = z_{j+1}, \ldots, e_n = z_n\}$. But

$$\mathsf{E}\left\{\left[\frac{\sum_{i\neq j} w''(c_n^{-1}(e_j - e_i))}{\sum_{r\neq j} w(c_n^{-1}(e_j - e_r))} b_n(e_j) - \right.\right.$$
$$\left. - \mathsf{E}\left\{\frac{\sum_{i\neq j} w''(c_n^{-1}(e_j - e_i))}{\sum_{r\neq j} w(c_n^{-1}(e_j - e_r))} b_n(e_j) \middle| e_1 = z_1, \ldots, e_{j-1} = z_{j-1}, e_{j+1} = z_{j+1}, \ldots, e_n = z_n\right\}\right]$$
$$\left. \middle| e_1 = z_1, \ldots, e_{j-1} = z_{j-1}, e_{j+1} = z_{j+1}, \ldots, e_n = z_n\right\} = 0.$$

Hence $\mathcal{E}_2 = 0$. The expression $\mathcal{E}_3$ may be bounded by (remember again Assertion 2)

$$n^{-2}c_n^{-4}K_5^4 \cdot 2K_3 \sum_{j=1}^{n} \sum_{s>j} \mathsf{E}\left\{\frac{\left|\sum_{\substack{t\neq s\\ t\neq j}} w''(c_n^{-1}(e_s - e_t)) - \sum_{t\neq s} w''(c_n^{-1}(e_s - e_t))\right|}{\sum_{\substack{t\neq s\\ t\neq j}} w(c_n^{-1}(e_s - e_v))}\right.$$
$$\left. + \frac{\left|\sum_{t\neq s} w''(c_n^{-1}(e_s - e_t)) \left[\sum_{\substack{t\neq s\\ t\neq j}} w(c_n^{-1}(e_s - e_t)) - \sum_{t\neq s} w(c_n^{-1}(e_s - e_t))\right]\right|}{\sum_{\substack{t\neq s\\ t\neq j}} w(c_n^{-1}(e_s - e_t) \sum_{v\neq s} w(c_n^{-1}(e_s - e_v))}\right\} \leq$$
$$\leq 2n^{-4}c_n^{-4}K_5^4 \cdot 2K_3^2 \cdot K_1 \sum_{j=1}^{n} \sum_{s>j} \sum_{t\neq s} \mathsf{E} w^{-1}(c_n^{-1}(e_s - e_t))$$

and this expression may be treated in the same way as (24). The expression $\mathcal{E}_4$ may be bounded in a similar way as $\mathcal{E}_3$.

Since

$$\mathsf{E}\left\{\frac{\sum_{i\neq j} w''(c_n^{-1}(e_j - e_i))}{\sum_{r\neq j} w(c_n^{-1}(e_j - e_r))} b_n(e_j) \middle| e_1 = z_1, \ldots, e_{j-1} = z_{j-1}, e_{j+1} = z_{j+1}, \ldots, e_n = z_n\right\}$$
$$= \left[\int \frac{\sum_{i\neq j} w''(c_n^{-1}(y - e_i))}{\sum_{r\neq j} w(c_n^{-1}(y - e_r))} b_n(y) g(y) \mathrm{d}y\right]_{e_1 = z_1, \ldots, e_{j-1} = z_{j-1}, \ldots, e_{j+1} = z_{j+1}, \ldots, e_n = z_n}$$

we have proved that

$$n^{-1}c_n^{-2} \sum_{j=1}^{n} x_{jk}\, x_{j\ell} \left[\frac{\sum_{i=1} w''(c_n^{-1}(e_j - \tilde{e}_i))}{\sum_{r=1} w(c_n^{-1}(e_j - \tilde{e}_r))} b_n(e_j) - \right.$$
$$\left. - \int \frac{\sum_{i\neq j} w''(c_n^{-1}(y - e_i))}{\sum_{r\neq j} w(c_n^{-1}(y - e_r))} b_n(y) g(y) \mathrm{d}y\right] = o_p(1).$$

Using once again Condition B we may prove – along the similar lines as at the start of this proof – that

$$n^{-1}c_n^{-2} \sum_{j=1}^{n} x_{jk}\, x_{j\ell} \left\{\int \left[\frac{\sum_{i\neq k} w''(c_n^{-1}(y - e_i))}{\sum_{r\neq j} w(c_n^{-1}(y - e_r))}\right.\right.$$

$$- \frac{\sum_{i=1} w''(c_n^{-1}(y - e_i))}{\sum_{r=1} w(c_n^{-1}(y - e_r))}\Bigg] b_n(y)g(y)\mathrm{d}y\Bigg\} = o_p(1),$$

i. e. that also

$$n^{-1}c_n^{-2}\sum_{j=1}^{n} x_{jk}\,x_{j\ell}\left[\frac{\sum_{i=1} w''(c_n^{-1}(e_j - \tilde{e}_i))}{\sum_{r=1} w(c_n^{-1}(e_j - \tilde{e}_r))} b_n(e_j) - \right. \tag{29}$$

$$\left. - \int \frac{\sum_{i=1} w''(c_n^{-1}(y - e_i))}{\sum_{r=1} w(c_n^{-1}(y - e_r))} b_n(y)g(y)\mathrm{d}y\right] = o_p(1).$$

Now let us prove that

$$n^{-1}c_n^{-2}\sum_{j=1}^{n} x_{jk}\,x_{j\ell}\int \frac{\sum_{i=1} w''(c_n^{-1}(y - e_i))}{\sum_{r=1} w(c_n^{-1}(y - e_r))}\left[\frac{1}{nc_n}\sum_{t=1}^{n} w(c_n^{-1}(y - e_t)) - g(y)\right] \tag{30}$$

$$b_n(y)\mathrm{d}y = o_p(1).$$

Since $n^{-1}\left|\sum_{j=1}^{n} x_{jk}\,x_{j\ell}\right| < K_5^2$ it is sufficient to show that

$$c_n^{-2}\left|\int \frac{\sum_{i=1} w''(c_n^{-1}(y - e_i))}{\sum_{r=1} w(c_n^{-1}(y - e_r))}\left[\frac{1}{nc_n}\sum_{k=1}^{n} w(c_n^{-1}(y - e_r)) - g(y)\right] b_n(y)\mathrm{d}y\right| = o_p(1). \tag{31}$$

Let us consider at first

$$\sup_{y\in\mathcal{R}} \frac{\left|\frac{1}{nc_n}\sum_{r=1}^{n} w(c_n^{-1}(y - e_r)) - g(y)\right|}{\frac{1}{nc_n}\sum_{s=1}^{n} w(c_n^{-1}(y - e_s))} b_n(y).$$

Now we shall use the Condition C. Let us fix some $\varepsilon > 0$ and $\Delta > 0$ and find $n_0 \in \mathcal{N}$ such that $d_{n_0} < \min\{\varepsilon^2, \Delta\}$ and $d_{n_0} < \frac{1}{2}d_{n_0}^{\frac{1}{2}}$. Further denote for any $n \ge n_0$

$$S_{\varepsilon,\Delta,n} = \left\{\omega \in \Omega,\ \sup_{y\in\mathcal{R}}\left|g_n(y, Y, \beta^0) - g(y)\right| < \frac{d_n}{2}\right\}.$$

Then for any $\omega \in S_{\varepsilon,\Delta,n}$ and $n \ge n_0$ we have (notice that supremum in what follows is taken in fact over $\left(-\frac{1}{2}a_n, \frac{1}{2}a_n\right)$ and hence $g(y) > d_n^{\frac{1}{2}}$)

$$\begin{aligned} &\sup_{y\in\mathcal{R}} \frac{|g_n(y, Y, \beta^0) - g(y)|}{g_n(y, Y, \beta^0)} b_n(y) \\ =\ &\sup_{y\in\mathcal{R}} \frac{|g_n(y, Y, \beta^0) - g(y)|}{g(y) + g_n(y, Y, \beta^0) - g(y)} b_n(y) \\ \le\ &\frac{\frac{d_n}{2}}{d_n^{\frac{1}{2}} - \frac{d_n}{2}} < \frac{d_n}{2}\cdot\frac{2}{d_n^{\frac{1}{2}}} = d_n^{\frac{1}{2}} < \varepsilon \end{aligned}$$

(remember that $\{d_n\}_{n=1}^{\infty}$ is decreasing to zero). But since $P\left(S_{\varepsilon,\Delta,n}^c\right) < d_n$ we have for any $n \ge n_0$

$$P\left(\sup_{y\in\mathcal{R}} \frac{|g_n(y, Y, \beta^0) - g(y)|}{g_n(y, Y, \beta^0)} b_n(y) > \varepsilon\right) < \Delta.$$

Lemma 13. The asymptotic distribution of

$$\left\{ n^{-\frac{1}{2}} c_n^{-1} \sum_{j=1}^{n} x_{jk} \frac{\int w'(c_n^{-1}(e_j - z))g(z)\mathrm{d}z}{\int w(c_n^{-1}(e_j - t))g(t)dt} \, b_n(e_j) \right\}_{k=1,\ldots,p} \tag{32}$$

is $N(\mathbf{0},\, Q \cdot I(g))$.

Proof. Let us remember the fact that

$$\mathsf{E} \frac{\int w'(c_n^{-1}(e_j - z))g(z)\mathrm{d}z}{\int w(c_n^{-1}(e_j - t))g(t)\mathrm{d}t} \, b_n(e_j) = \int \frac{\int w'(c_n^{-1}(y - z))g(z)\mathrm{d}z}{\int w(c_n^{-1}(y - t))g(t)\mathrm{d}t} g(y) b_n(y) \mathrm{d}y = 0,$$

since

$$\frac{\int w'(c_n^{-1}(y - z))g(z)\mathrm{d}z}{\int w(c_n^{-1}(y - t))g(t)\mathrm{d}t} b_n(y) g(y) = - \frac{\int w'(c_n^{-1}(-y - z))g(z)\mathrm{d}z}{\int w(c_n^{-1}(-y - t))g(t)\mathrm{d}t} \, g(-y) \, b_n(-y).$$

Moreover for any $k = 1, \ldots, p$ and any $\ell, j = 1, \ldots, n$ the summands

$$x_{jk} \frac{\int w'(c_n^{-1}(e_j - z))g(z)\mathrm{d}z}{\int w(c_n^{-1}(e_j - t))g(t)\mathrm{d}t} b_n(e_j\,) \text{ and } x_{\ell k} \frac{\int w'(c_n^{-1}(e_\ell - z))g(z)\mathrm{d}z}{\int w(c_n^{-1}(e_\ell - t))g(t)\mathrm{d}t} b_n(e_\ell)$$

are i. i. distributed r. v. and hence the variance - covariance matrix of (32) is equal to (for $k, r = 1, \ldots, p$)

$$\left\{ \sum_{j=1}^{n} x_{jk} x_{jr} \cdot \mathsf{E} \left[\frac{c_n^{-1} \int w'(c_n^{-1}(e_1 - z))g(z)\mathrm{d}z}{\int w(c_n^{-1}(e_1 - t))g(t)\mathrm{d}t} \, b_n(e_1) \right]^2 \right\}_{kr}.$$

Using the same steps as in proving (30) leads to the fact that this expression is – up to a member $o_p(1)$ – equal to

$$c_n^{-3} \sum_{j=1}^{n} x_{jk} \, x_{jr} \int \frac{\left[\int w'(c_n^{-1}(y - z))g(z)\mathrm{d}z\right]^2}{\int w(c_n^{-1}(y - t))g(t)\mathrm{d}t} \, b_n(y) \, \mathrm{d}y.$$

Taking into account that variances of random variables

$$x_{jk} \frac{c_n^{-1} \int w'(c_n^{-1}(e_j - z))g(z)\mathrm{d}z}{\int w(c_n^{-1}(e_j - t))g(t)\mathrm{d}t} \, b_n(e_j)$$

are uniformly in $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, p$ bounded by

$$K \cdot \mathrm{var} \frac{c_n^{-1} \int \int w'(c_n^{-1}(e_1 - z))g(z)\mathrm{d}z}{\int w(c_n^{-1}(e_1 - t))g(t)\mathrm{d}t}$$

(which exists due to existence of the Fisher information and due to some other technical assumptions as $d_n/c_n \uparrow \infty$) and applying Lemma 10 one concludes the proof. □

Remark 8. Let us mention that the proof shows that an endeavour to avoid the assumption of symmetry of $g(y)$ would lead to some assumption about the rate of convergence of the tails of $g(y)$ to zero with respect to increase of the interval generated by $b_n(y)$. See the assumption iii) of the paragraph 2 of [10]. In such case however it would be necessary at least to modify all points at proofs which utilizes Condition E. It may lead to a number of considerable technical problems.

Earlier than we give the main result of the Part 2 we sum up all assumptions we have made up to now:

Condition A. Let the kernel $w(y)$ be three times differentiable, positive everywhere and symmetric. Suppose that there are constants K_1, K_2, K_3 and K_4 such that

$$\sup_{y\in\mathcal{R}} w(y) < K_1, \qquad \sup_{y\in\mathcal{R}} \tfrac{|w''(y)|}{w(y)} < K_2,$$
$$\sup_{y\in\mathcal{R}} \tfrac{|w''(y)|}{w(y)} < K_3 \quad \text{and} \quad \sup_{y\in\mathcal{R}} \tfrac{|w''(y)|}{w(y)} < K_4.$$

Preliminary estimator $\tilde{\beta}^n$ is assumed to be such that for some $\delta > \frac{1}{4}$ we have

$$n^{\delta} \left\| \tilde{\beta}^n - \beta^0 \right\| \doteq O_p(1).$$

Moreover let

$$\lim_{n\to\infty} c_n = 0, \qquad \lim_{n\to\infty} n c_n^8 = \infty \tag{33}$$

and

$$\frac{\log w^{-1}(c_n^{-1})}{n^{\delta}} = o(1).$$

Further let g be symmetric, having continuous second derivative and for some M, $0 < M < \infty$ we have

$$\sup_{y\in\mathcal{R}} |g'(y)| < M.$$

Finally let $g(x)$ be decreasing for $x > 0$.

Condition B. Let for any $a \in \mathcal{R}$

$$\lim_{n\to\infty} n^{-\frac{1}{2}} c_n^{-2} \int \sup_{|b|<a} w^{-1}(c_n^{-1}(z + b - t))\, g(t)\, g(z) \mathrm{d}t\mathrm{d}z = 0.$$

Moreover let us assume that there are ν, D ($\nu > 0$, $D > 0$) such that for any z_1, $z_2 \in \mathcal{R}$ such that $|z_1 - z_2| < \nu$ we have $w(z_1)/w(z_2) < D$.

Condition C. Let

$$\lim_{n\to\infty} \frac{d_n}{c_n} = \infty.$$

Moreover let there be $K_5 < \infty$ such that $\max_{i \in \mathcal{N}, j=1,\ldots,p} |x_{ij}| < K_5$. We shall also assume that the density g is the element of $G(\{d_n\}_{n=1}^{\infty})$. To simplify the next text we need to require that there is a sequence $\{a_n\}_{n=1}^{\infty}$, $a_n > 0, a_n \nearrow \infty$ such that

$$(-a_n, a_n) \subset \left\{ y \in \mathcal{R} : \; g(y) > d_n^{-\frac{1}{2}} \right\}.$$

Then define $b_n(y) = 1$ for $|y| < \frac{1}{2}a_n$ and $b_n(y) = 0$ elsewhere. In addition to the requirement (33) we will assume that

$$\lim_{n \to \infty} n c_n^6 a_n^{-2} = \infty.$$

Condition D. Let us assume that there is K_6 such that

$$P\left(\| \operatorname*{argmax}_{\beta \in \mathcal{R}^p} \prod_{j=1}^{n} g_n(e_j(\beta), Y, \tilde{\beta}^n) b_n(\tilde{e}_j) \| > K_6 \right) \xrightarrow[n \to \infty]{} 0.$$

Condition E. Let Fisher information $I(g)$ exist and be finite. Moreover let for any $j = 1, \ldots, n,\ i = 1, \ldots, n,\ t \in \mathcal{R}$ and $s \in \mathcal{R}$

$$P\left(\tilde{e}_i - \mathsf{E}\{\tilde{e}_i | e_j = t\} < -s | e_j = t\right) = P\left(\tilde{e}_i = \mathsf{E}\{\tilde{e}_i | e_j = t\} > s | e_j = t\right).$$

Theorem 2. Under Conditions A, B, C, D and E we have following asymptotic representation $(k = 1, \ldots, p)$

$$\left\{ n^{-\frac{1}{2}} (\hat{\beta}^n - \beta^0) X^{\mathrm{T}} X \right\}_k = n^{-\frac{1}{2}} I^{-1}(g) \cdot \sum_{j=1}^{n} x_{jk} \frac{c_n^{-1} \int w'(c_n^{-1}(e_j - z)) g(z) \mathrm{d}z}{\int w(c_n^{-1}(e_j - t)) g(t) \mathrm{d}t} + o_p(1).$$

Proof. From the definition of $\hat{\beta}^n$ it follows for any $k = 1, \ldots, p$

$$\begin{aligned} & n^{-\frac{1}{2}} c_n^{-1} \left\{ \sum_{j=1}^{n} x_{jk} \frac{\sum_{i=1}^{n} w'(c_n^{-1}(e_j - \tilde{e}_i))}{\sum_{r=1}^{n} w(c_n^{-1}(e_j - \tilde{e}_r))} b_n(\tilde{e}_j) + \right. \\ & + c_n^{-1} \sum_{j=1}^{n} \sum_{\ell=1}^{p} x_{jk} x_{j\ell} \left\{ \frac{\sum_{i=1}^{n} w''(c_n^{-1}(e_j - \tilde{e}_i))}{\sum_{r=1}^{n} w(c_n^{-1}(e_j - \tilde{e}_r))} - \right. \\ & \left. \left. - \left[\frac{\sum_{i=1}^{n} w'(c_n^{-1}(e_j - \tilde{e}_i))}{\sum_{r=1}^{n} w(c_n^{-1}(e_j - \tilde{e}_r))} \right]^2 \right\} b_n(\tilde{e}_j)(\hat{\beta}_\ell^n - \beta_\ell^0) \right\} + \\ & + \sum_{t=1}^{p} (\hat{\beta}_t^n - \beta_t^0) \sum_{\ell=1}^{p} R_{t\ell}^k (\hat{\beta}_\ell^n - \beta_\ell^0) = 0 \end{aligned}$$

where $R_{t\ell}^k = O_p(n^{-\frac{1}{2}})$ for any k, t, $\ell = 1, \ldots$,p. Hence we may write for any $k = 1, \ldots, p$

$$n^{-\frac{1}{2}} c_n^{-1} \sum_{j=1}^{n} x_{jk} \frac{\sum_{i=1}^{n} w'(c_n^{-1}(e_j - \tilde{e}_i))}{\sum_{i=1}^{n} w(c_n^{-1}(e_j - \tilde{e}_i))} b_n(\tilde{e}_j) \tag{34}$$

$$= \sqrt{n}\sum_{\ell=1}^{p}(\hat{\beta}_\ell^n - \beta_\ell^0)\left\{\frac{1}{nc_n}\sum_{j=1}^{n} x_{jk}\, x_{j\ell}\left[\frac{\sum_{i=1}^n w''(c_n^{-1}(e_j - \tilde{e}_i))}{\sum_{i=1}^n w(c_n^{-1}(e_j - \tilde{e}_i))} - \right.\right.$$
$$\left.\left. - \left\{\frac{\sum_{i=1}^n w'(c_n^{-1}(e_j - \tilde{e}_i))}{\sum_{i=1}^n w(c_n^{-1}(e_j - \tilde{e}_i))}\right\}^2\right] b_n(\tilde{e}_j) + \sum_{t=1}^{p} S_{t\ell}^k(\hat{\beta}_t^n - \beta_t^0)\right\}$$

where $S_{\ell t}^k = O_p(1)$ for any $k,\ t,\ \ell = 1,\dots,p$. Due to fact that $\hat{\beta}^n$ is consistent and due to Lemma 11 and then Lemmas 8 and 10 we have

$$T_{k\ell}^n = \frac{1}{nc_n}\sum_{j=1}^{n} x_{jk}\, x_{j\ell}\left[\frac{\sum_{i=1}^n w''(c_n^{-1}(e_j - \tilde{e}_i))}{\sum_{i=1}^n w(c_n^{-1}(e_j - \tilde{e}_i))} - \right.$$
$$\left. - \left\{\frac{\sum_{i=1}^n w'(c_n^{-1}(e_j - \tilde{e}_i))}{\sum_{i=1}^n w(c_n^{-1}(e_j - \tilde{e}_i))}\right\}^2 b_n(\tilde{e}_j)\right]$$
$$+ \sum_{t=1}^{p} S_{t\ell}^k(\hat{\beta}_t^n - \beta_t^0) \xrightarrow[n\to\infty]{} I(g) q_{k\ell}$$

in probability. Using Lemma 14 (from the Appendix) we see that to assume that

$$\sqrt{n}\|\hat{\beta}_\ell^n - \beta_\ell^0\| \neq O_p(1) \tag{35}$$

would imply that

$$\sqrt{n}\sum_{\ell=1}^{p}(\hat{\beta}_\ell^n - \beta_\ell^0)\cdot T_{k\ell}^n \neq O_p(1)$$

at least for one $k \in \{1, 2, \dots, p\}$. But left hand side of (34) is bounded in probability and hence the assumption (35) leads to contradiction. But it directly implies the assertion of the Theorem.

Corollary 1.

$$\mathcal{L}\left(n^{-\frac{1}{2}}(\hat{\beta}^n - \beta^0)X'X\right) \xrightarrow[n\to\infty]{} N\left(\mathbf{0},\, Q \cdot I^{-1}(g)\right).$$

Proof. The proof follows directly from Theorem 2 and Lemma 13. □

APPENDIX

Lemma 14. Let for any $n \in \mathcal{N}$ $\{\mathcal{T}_{ik}\}_{i=1}^{n}{}_{k=1}^{p}$ be a matrix and let for any $j,\ k = 1,\dots,p$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathcal{T}_{ij}\,\mathcal{T}_{ik} = q_{jk}$$

where $Q = \{q_{jk}\}_{j=1}^{p}{}_{k=1}^{p}$ is a regular matrix. Then for any $\{\gamma^{(n)}\}_{n=1}^{\infty}$, $\gamma^{(n)} = \left(\gamma_1^{(n)},\dots,\gamma_p^{(n)}\right)^{\mathrm{T}}$ such that $\lim_{n\to\infty}\|\gamma^{(n)}\| = \infty$ we have at least for one $k \in \{1,\dots,p\}$

$$\lim_{n\to\infty}\frac{1}{n}\left|\sum_{j=1}^{p}\sum_{i=1}^{n}\mathcal{T}_{ij}\,\mathcal{T}_{ik}\,\gamma_j^{(n)}\right| = \infty.$$

Proof. Let $\varepsilon > 0$ and find n_0 so that for any $n \geq n_0$ we have

$$\left| \frac{1}{n} \sum_{i=1}^{n} \mathcal{T}_{ij} \mathcal{T}_{ik} - q_{jk} \right| < \frac{\varepsilon}{2}.$$

Now let us assume that the assertion of the Lemma is not true. Then there is a subsequence $\{n_\ell\}_{\ell=1}^{\infty}$ such that

$$\limsup_{\ell \to \infty} \frac{1}{n_\ell} \left| \sum_{j=1}^{p} \sum_{i=1}^{n_\ell} \mathcal{T}_{ij} \mathcal{T}_{ik} \gamma_j^{(n_\ell)} \right| = |K_k| < \infty$$

for any $k = 1, \ldots, p$. Hence starting from some $n_1 \geq n_0$ for any $n_\ell \geq n_1$ we may write

$$\sum_{j=1}^{p} \{q_{kj} + \tau_{kj}\} \gamma_j^{(n_\ell)} = K_k$$

where $|\tau_{kj}| < \varepsilon$. Then (denoting q_{kj}^{-1} – for a while – members of inversion matrix Q^{-1}) we have for any $\ell = 1, \ldots, q$

$$\sum_{k=1}^{p} q_{\ell k}^{-1} \sum_{j=1}^{p} (q_{kj} + \tau_{kj}) \gamma_j^{(n_\ell)} = \sum_{k=1}^{p} q_{\ell k}^{-1} K_k,$$

i.e.

$$\gamma_r^{(n_\ell)} + \sum_{k=1}^{p} q_{rk}^{-1} \sum_{j=1}^{p} \tau_{kj} \gamma_j^{(n_\ell)} = \sum_{k=1}^{p} q_{rk}^{-1} K_k$$

and finally

$$\gamma_r^{(n_\ell)} = \sum_{k=1}^{p} q_{rk}^{-1} K_k - \sum_{k=1}^{p} q_{rk}^{-1} \sum_{j=1}^{p} \tau_{kj} \gamma_j^{(n_\ell)}$$

for any $r = 1, \ldots, p$. And hence for r_0 such that

$$\left|\gamma_{r_0}^{(n_\ell)}\right| = \max_{r=1,\ldots,p} \left|\gamma_r^{(n_\ell)}\right|$$

we have

$$\left|\gamma_{r_0}^{(n_\ell)}\right| \leq \max_{k=1,\ldots,p} |K_k| \cdot \max_{k=1,\ldots,p} \left|q_{r_0 k}^{-1}\right| + \varepsilon \cdot \max_{k=1,\ldots,p} \left|q_{r_0 k}^{-1}\right| \cdot \left|\gamma_{r_0}^{(n_\ell)}\right|,$$

i.e.

$$\left|\gamma_{r_0}^{(n_\ell)}\right| \left(1 - \varepsilon \cdot p \cdot \max_{k=1\ldots,p} \left|q_{r_0 k}^{-1}\right|\right) \leq p \cdot \max_{k=1,\ldots,p} |K_k| \cdot \max_{k=1,\ldots,p} \left|q_{r_0 k}^{-1}\right|.$$

Since ε was arbitrary and $\left|\gamma_{r_0}^{(n_\ell)}\right| \to \infty$ this is a contradiction. □

ACKNOWLEDGEMENT

The author expresses his cordial thanks to Václav Dupač for his very valuable comments on the manuscript of paper which led to correction of some omissions, to simplification of some steps in proofs and finally, to an improvement of comprehensibility of the complicated topic.

(Received March 7, 1991.)

REFERENCES

[1] R. Beran: An efficient and robust adaptive estimator of location. Ann. Statist. *6* (1978), 292–313.

[2] M. Csörgö and P. Révész: Strong Approximations in Probability and Statistics. Akadémiai Kiadó, Budapest 1981.

[3] E. Hewitt and K. Stromberg: Real and Abstract Analysis. Springer-Verlag, Berlin – Heidelberg – New York 1965.

[4] R. A. Maronna and V. J. Yohai: Asymptotic behaviour of general M-estimates for regression and scale with random carriers. Z. Wahrsch. verw. Gebiete *58* (1981), 7–20.

[5] R. C. Rao: Linear Statistical Inference and Its Applications. J. Wiley, New York 1973.

[6] J. Á. Víšek: Adaptive Estimation in Linear Regression Model. Technical Report No. 1654, Institute of Information Theory and Automation of Czechoslovak Academy of Sciences, Prague 1989.

[7] J. Á. Víšek: Adaptive Maximum-likelihood-like Estimation in Linear Model. Technical Report No. 1654, Institute of Information Theory and Automation of Czechoslovak Academy of Sciences, Prague 1989.

[8] J. Á. Víšek: What is adaptivity of regression analysis intended for? In: Proceedings of ROBUST'90 (J. Antoch, ed.), Union of Czechoslovak Mathematicians and Physicists, Prague 1990, pp. 160–181.

[9] J. Á. Víšek: Adaptive estimation in linear regression model. Part 1. Consistency. Kybernetika *28* (1991), 1, 26–36.

[10] J. Á. Víšek: Adaptive estimation in linear regression model. Part 2. Asymptotic normality. Kybernetika *28* (1991), 2, 100–119.

[11] J. Á. Víšek: Adaptive maximum-likelihood-like estimation in linear models. Part 1. Consistency. Kybernetika *28* (1991), 5, 357–382.

RNDr. Jan Ámos Víšek, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation – Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.