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ON ESTIMABLE AND LOCALLY-ESTIMABLE FUNCTIONS IN THE NON-LINEAR REGRESSION MODEL

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The non-linear regression model $y = \eta(\vartheta) + \varepsilon$ with an error vector ε having the zero mean and the covariance matrix $\delta^2 I$ (δ^2 unknown) is considered. Some sufficient conditions of estimability and local estimability of the function of the parameter ϑ are obtained, whilst the regularity of the model (i.e. the regularity of Jacobi matrix of the function $\eta(\vartheta)$ is not required). Consequently, there are given – in addition – precisions of A. H. Bird's and G. A. Milliken's research [1] concerning local reparameterization of a singular model onto a regular model.

1. INTRODUCTION

Let

$$y = \eta(\vartheta) + \varepsilon$$

be a non-linear regression model. Here $y := (y_1, \dots, y_N)^{\mathrm{T}}$ is a vector of the observed data, $\vartheta := (\vartheta_1, \dots, \vartheta_m)$ is a vector of unknown parameters, $m \le N$, $\vartheta \in \Theta$, where Θ is a given parameter space being an open non-empty subset of $\mathbb{R}^m$. The mapping $\eta : \vartheta \in \Theta \mapsto \eta(\vartheta) \in \mathbb{R}^N$ is supposed to be known and to have continuous second order derivatives on Θ. The vector ε is the zero mean random vector with the covariance matrix $\delta^2 I$ (δ^2 unknown, I = identity matrix). Finally, it is supposed that the probability distribution of ε is absolutely continuous with respect to the Lebesgue measure λ on $\mathbb{R}^N$. We shall use the following notations:

$$J(\vartheta) := \left(\frac{\partial \eta(\vartheta)}{\partial \vartheta_1}, \dots, \frac{\partial \eta(\vartheta)}{\partial \vartheta_m} \right) := \frac{\mathrm{D}(\eta_1, \dots, \eta_N)}{\mathrm{D}(\vartheta_1, \dots, \vartheta_m)},$$

$$\mathcal{M}_y := \operatorname*{Arg\,min}_{\vartheta \in \Theta} \| y - \eta(\vartheta) \|^2,$$

where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^N$,

$$\mathcal{E} := \{ \eta(\vartheta) : \vartheta \in \Theta \},$$

$$\mathcal{N}_y := \operatorname*{Arg\,min}_{\eta \in \mathcal{E}} \| y - \eta \|^2 .$$

By $\mathcal{O}(\vartheta^0)$ we denote a neighbourhood of the point $\vartheta^0 \in \Theta$. $R[J(\vartheta)]$ and $\mathcal{L}[J(\vartheta)]$ denotes the rank of Jacobi matrix $J(\vartheta)$ and the linear space generated by the columns of the matrix $J(\vartheta)$. By $C^n(A)$ we denote the class of the functions that have continuous partial derivatives of the n-th order on open set A.

Definition 1.1. The estimate $\hat{\vartheta} := \hat{\vartheta}(.)$ of the parameter ϑ is that by the least squares method if $\hat{\vartheta}(y) \in \mathcal{M}_y$ for $\mathcal{M}_y \neq \emptyset$.

Definition 1.2. (cf. [5]) The function $f(.) : \Theta \mapsto \mathbb{R}$ is said to be an estimable function of the parameter ϑ iff

$$\lambda \left\{ y \in \mathbb{R}^N : \underset{\hat{\vartheta},\tilde{\vartheta} \in \mathcal{M}_y}{\exists} \; f(\hat{\vartheta}) \neq f(\tilde{\vartheta}) \right\} = 0.$$

The class of estimable functions of the parameter ϑ shall be denoted by $\mathcal{F}$.

Note 1.1. This definition is equivalent to the definition of estimability of a linear function in a linear regression model (cf. C. R. Rao [7]).

Note 1.2. Since the probability distribution of the random vector ε is absolutely continuous with respect to the Lebesgue measure λ for $f(.) \in \mathcal{F}$ the following is valid:

$$\underset{\vartheta \in \Theta}{\forall} \; P_\vartheta \left\{ y \in \mathbb{R}^N : \underset{\hat{\vartheta},\tilde{\vartheta} \in \mathcal{M}_y}{\exists} \; f(\hat{\vartheta}) \neq f(\tilde{\vartheta}) \right\} = 0.$$

Definition 1.3. (cf. [5]) The function $f(.) : \Theta \mapsto \mathbb{R}$ is said to be a locally-estimable function of the parameter ϑ in the point $\vartheta^0 \in \Theta$ iff

$$\underset{\mathcal{O}(\vartheta^0):=U}{\exists} \; \lambda \left\{ y \in \mathbb{R}^N : \underset{\hat{\vartheta},\tilde{\vartheta} \in U \cap \mathcal{M}_y}{\exists} \; f(\hat{\vartheta}) \neq f(\tilde{\vartheta}) \right\} = 0.$$

The function $f(.)$ is said to be a locally-estimable function of the parameter ϑ (in Θ) if $f(.)$ is a locally-estimable function of the parameter ϑ for every $\vartheta \in \Theta$.

The class of locally-estimable functions of the parameter ϑ shall be denoted by $\mathcal{F}_0$.

Note 1.3. If $f(.)$ is an estimable function of the parameter ϑ, then $f(.)$ is a locally-estimable function of the parameter ϑ. The reverse implication is invalid.

Theorem 1.1. (See A. Pázman [6], Theorem 3.) Let $\mathcal{E} = \bigcup_{k=0}^{s} \mathcal{E}_k$, where $\mathcal{E}_k$ ($k = 0, \ldots, s$) are differentiable C^2-manifolds, $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for $i \neq j$ $\quad (i, j = 0, \ldots, s)$. Then

$$\lambda \left\{ y \in \mathbb{R}^N : \underset{\hat{\eta},\tilde{\eta} \in \mathcal{N}_y}{\exists} \; \hat{\eta} \neq \tilde{\eta} \right\} = 0.$$

Theorem 1.2. (cf. [5]) Let $\mathcal{E} = \bigcup_{k=0}^{\infty} \mathcal{E}_k$, where $\mathcal{E}_k$ $(k = 0, \ldots)$ are differentiable C^2-manifolds. Then

$$\lambda \left\{ y \in \mathbb{R}^N : \underset{\hat{\eta}, \tilde{\eta} \in \mathcal{N}_y}{\exists} \hat{\eta} \neq \tilde{\eta} \right\} = 0.$$

Proof. The proof is analogous to the proof of Theorem 1.1, because the presumption of the disjunction of the C^2-manifolds has not been used in this proof. The presumption of the countable union is then a trivial generalization of this proof. □

2. ESTIMABLE AND LOCALLY-ESTIMABLE FUNCTIONS IN THE NON-LINEAR REGRESSION MODEL

Assumption A. It is supposed that every $\vartheta \in \Theta$ is the regular point of the function $\eta(\vartheta)$, which is not constant on any open subset of Θ, i. e.

$$\underset{\vartheta^0 \in \Theta}{\forall} \quad \underset{\mathcal{O}(\vartheta^0) := U}{\exists} \quad \underset{\vartheta \in U}{\forall} \; R[J(\vartheta)] = R[J(\vartheta^0)] \neq 0.$$

Lemma 2.1. For every $\vartheta^0 \in \Theta$ there is a neighbourhood $\mathcal{O}(\vartheta^0) := U$ such that $\eta(U)$ is C^2-manifold.

Proof. Let $\vartheta^0 \in \Theta$, then according to the rank theorem [2], exists:

i) $\mathcal{O}(\vartheta^0) := U$ and a homeomorphism φ of U onto the set $I^m := \{(x_1, \ldots, x_m) \in \mathbb{R}^m : |x_j| < 1, j = 1, \ldots, m\}$, for which is valid: $\varphi \in C^2(U), \varphi^{-1} \in C^2(I^m)$;

ii) $\mathcal{O}[\eta(\vartheta^0)] := V \supset \eta(U)$ and a homeomorphism ψ of the set $I^N := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : |x_j| < 1, j = 1, \ldots, N\}$ onto the set V, for which is valid: $\psi \in C^2(I^N), \psi^{-1} \in C^2(V)$;

– so that $\eta(\vartheta) = \psi \circ \pi \circ \varphi(\vartheta)$ for every $\vartheta \in U$, where π is the projection

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0).$$

Then $\eta(U) = \psi \circ \pi \circ \varphi(U), \varphi(U) = I^m$ and $\pi(I^m) = \{x \in I^N : x_{k+1} = \ldots = x_N = 0\}$. Hence it is implied that $\eta(U)$ is a C^2-manifold (see [3]). □

Theorem 2.1. The function $\eta_i(.)$ is an estimable function for every $i = 1, \ldots, N$.

Proof. From Lemma 2.1 it follows that a system of open sets $\{U_\gamma\}$ exists, so that $\Theta = \bigcup_{\gamma \in \Gamma} U_\gamma$ and $\mathcal{E} = \bigcup_{\gamma \in \Gamma} \eta(U_\gamma)$ hold, where (U_γ) is a C^2-manifold for every $\gamma \in \Gamma$. As Θ is a separable metric space, it is possible to select a finite or countable covering from every open covering of Θ. It is thus possible to express $\mathcal{E}$ as an at most countable union of the C^2-manifolds. According to Theorem 1.2, then

$$\lambda \left\{ y \in \mathbb{R}^N : \underset{\hat{\vartheta}, \tilde{\vartheta} \in \mathcal{M}_y}{\exists} \eta(\hat{\vartheta}) \neq \eta(\tilde{\vartheta}) \right\} = 0.$$

Obviously

$$\lambda\left\{y \in \mathbb{R}^N : \exists_{\hat{\vartheta},\tilde{\vartheta}\in\mathcal{M}_y} \eta(\hat{\vartheta}) \neq \eta(\tilde{\vartheta})\right\} = 0 \Leftrightarrow \forall_{i=1,\dots,N} \lambda\left\{y \in \mathbb{R}^N : \exists_{\hat{\vartheta},\tilde{\vartheta}\in\mathcal{M}_y} \eta_i(\hat{\vartheta}) \neq \eta_i(\tilde{\vartheta})\right\} = 0.$$

Thus $\eta_i(\cdot) \in \mathcal{F}$ for every $i = 1, \dots, N$. □

Theorem 2.2. a) The function $f(\cdot) \in \mathcal{F}$ iff

$$\lambda\left\{y \in \mathbb{R}^N : \exists_{\hat{\vartheta},\tilde{\vartheta}\in\mathcal{M}_y} f(\hat{\vartheta}) \neq f(\tilde{\vartheta}),\ \eta(\hat{\vartheta}) = \eta(\tilde{\vartheta})\right\} = 0.$$

b) The function $f(\cdot) \in \mathcal{F}_0$ iff for every $\vartheta \in \Theta$ there is $\mathcal{O}(\vartheta) := U$ so that

$$\lambda\left\{y \in \mathbb{R}^N : \exists_{\hat{\vartheta},\tilde{\vartheta}\in U\cap\mathcal{M}_y} f(\hat{\vartheta}) \neq f(\tilde{\vartheta}),\ \eta(\hat{\vartheta}) = \eta(\tilde{\vartheta})\right\} = 0.$$

P r o o f. It is apparently valid that

$$\begin{aligned}\left\{y \in \mathbb{R}^N : \exists_{\hat{\vartheta},\tilde{\vartheta}\in\mathcal{M}_y} f(\hat{\vartheta}) \neq f(\tilde{\vartheta})\right\} &= \left\{y \in \mathbb{R}^N : \exists_{\hat{\vartheta},\tilde{\vartheta}\in\mathcal{M}_y} f(\hat{\vartheta}) \neq f(\tilde{\vartheta}), \eta(\hat{\vartheta}) \neq \eta(\tilde{\vartheta})\right\} \cup \\ &\cup \left\{y \in \mathbb{R}^N : \exists_{\hat{\vartheta},\tilde{\vartheta}\in\mathcal{M}_y} f(\hat{\vartheta}) \neq f(\tilde{\vartheta}), \eta(\hat{\vartheta}) = \eta(\tilde{\vartheta})\right\}.\end{aligned}$$

The statement of Theorem 2.2a) results from Theorem 2.1. The proof of the statement of Theorem 2.2b) is quite analogous. □

Theorem 2.3. Let $f(\vartheta) = F[\eta(\vartheta)]$ for every $\vartheta \in \Theta$, then $f(\cdot) \in \mathcal{F}$.

The p r o o f follows immediately from Theorem 2.2 a).

Note 2.1. The precondition $f(\vartheta) = F[\eta(\vartheta)]$ for every $\vartheta \in \Theta$ is a sufficient one (not a necessary one) for validity of $f(.) \in \mathcal{F}$, as the following example shows.
Let

$$\eta_1(\vartheta) = \frac{3\vartheta}{1+\vartheta^3}, \qquad \eta_2(\vartheta) = \frac{3\vartheta}{1+\vartheta^3}, \qquad \vartheta \in \mathbb{R}^1 - \{-1\}.$$

Take $f(\vartheta) = \vartheta$. Evidently, $f(\vartheta)$ is not possible to express as a function of $\eta(\vartheta)$ and $f(\cdot) \in \mathcal{F}$.

Notations. By $\vartheta_i(\cdot)$ we denote the ith coordinate function, i. e.

$$\vartheta_i(\vartheta) = \vartheta_i \text{ for every } \vartheta \in \Theta, \quad i = 1, \dots, m.$$

Theorem 2.4. Let $R[J(\vartheta)] = m$ for every $\vartheta \in \Theta$.

a) Then $\vartheta_i(.) \in \mathcal{F}_0$ for every $i = 1, \dots, m$.

b) If $\eta(\vartheta)$ is a one-to-one mapping, then $\vartheta_i(\cdot) \in \mathcal{F}$ for every $i = 1, \dots, m$.

Proof. These assertions are the consequences of Theorem 2.2. □

Note 2.2. i) If $R[J(\vartheta^0)] = k < m$ for $\vartheta^0 \in \Theta$, then $\eta(\vartheta)$ is not a one-to-one mapping (nor locally a one-to-one mapping).

ii) It follows from Theorem 2.2 a) that the precondition of a one-to-one character of the function $\eta(\vartheta)$ is not a necessary one but only a sufficient one for validity of

$$\vartheta_i(\cdot) \in \mathcal{F} \quad (i = 1, \dots, m).$$

iii) If $R[J(\vartheta)] = m$ for every $\vartheta \in \Theta$, then it need not be valid that $\vartheta_i(\cdot) \in \mathcal{F}$ for every $i = 1, \dots, m$.

Theorem 2.5. If

$$g(\vartheta) \in C^1(\Theta) : \left(\frac{\partial g(\vartheta)}{\partial \vartheta_1}, \dots, \frac{\partial g(\vartheta)}{\partial \vartheta_m} \right)^{\mathrm{T}} \in \mathcal{L}\left[J(\vartheta)^{\mathrm{T}} \right]$$

for every $\vartheta \in \Theta$, then $g(\cdot) \in \mathcal{F}_0$.

Proof. Let $\vartheta^0 \in \Theta$ and let $R[J(\vartheta^0)] = k \quad (1 \le k \le m)$. Then a neighbourhood $\mathcal{O}(\vartheta^0) := V$ exists such that there is a submatrix of the kth order from the matrix $J(\vartheta)$, which is regular on V and $R[J(\vartheta)] = k$ for every $\vartheta \in V$. Let for instance

$$\det \frac{\mathrm{D}(\eta_1, \dots, \eta_k)}{\mathrm{D}(\vartheta_1, \dots, \vartheta_k)} \neq 0 \text{ for every } \vartheta \in V.$$

Under the assumption of the theorem we have

$$R\left[\frac{\mathrm{D}(\eta_1, \dots, \eta_k, g)}{\mathrm{D}(\vartheta_1, \dots, \vartheta_m)} \right] = k \text{ for every } \vartheta \in V.$$

Denote $\eta^1(\vartheta) := (\eta_1(\vartheta), \dots, \eta_k(\vartheta))$. Then there is $\mathcal{O}(\vartheta^0) := A$ and $\mathcal{O}[\eta^1(\vartheta^0)] := B$ and a function $F(\eta^1) \in C^1(B)$, so that $g(\vartheta) = F[\eta(\vartheta)]$ holds for every $\vartheta \in A$ (see [4], Theorem 213). Since ϑ^0 has been chosen arbitrarily, it is implied from Theorem 2.2 b) that $g(\cdot) \in \mathcal{F}_0$. □

Theorem 2.6. Denote by

$$J^1(\vartheta) := \left[\frac{\partial \eta(\vartheta)}{\partial \vartheta_1}, \dots, \frac{\partial \eta(\vartheta)}{\partial \vartheta_p} \right], \qquad J^2(\vartheta) := \left[\frac{\partial \eta(\vartheta)}{\partial \vartheta_{p+1}}, \dots, \frac{\partial \eta(\vartheta)}{\partial \vartheta_m} \right]$$

the decomposition of the matrix $J(\vartheta)$ for which is valid:

i) $R[J^1(\vartheta)] = p$ for every $\vartheta \in \Theta$,

ii) $\mathcal{L}[J^1(\vartheta)] \cap \mathcal{L}[J^2(\vartheta)] = \{0\}$ for every $\vartheta \in \Theta$.

Then $\vartheta_i(\cdot) \in \mathcal{F}_0$ for every $i = 1, \ldots, p$.

Proof. Let $\vartheta_0 \in \Theta$ and let $R[J(\vartheta_0)] = p+r$, where $0 \le r \le m-p$. Then $\mathcal{O}(\vartheta_0) := W$ exists such that there is at least one subdeterminant of the $(p+r)$-th order from the matrix $J(\vartheta)$, which is non-zero on W and $R[J(\vartheta)] = p+r$ for every $\vartheta \in W$. Let us suppose for instance that

$$\det \frac{D(\eta_1, \ldots, \eta_{p+r})}{D(\vartheta_1, \ldots, \vartheta_{p+r})} \neq 0 \quad \text{for every } \vartheta \in W. \tag{2.1}$$

If $r = 0$, resp. $r = m-p$, then it follows from Theorem 2.4 a) that $\vartheta_i(.)$ is a locally-estimable function in the point ϑ_0 for every $i = 1, \ldots, p$. Consequently, let $0 < r < m-p$. Then

$$\frac{\partial \eta(\vartheta)}{\partial \vartheta_j} = \sum_{k=p+1}^{p+r} H_k^j(\vartheta) \frac{\partial \eta(\vartheta)}{\partial \vartheta_k} \qquad \text{for every } \vartheta \in W \quad (j = p+r+1, \ldots, m). \tag{2.2}$$

Let us introduce the notations:

$$\begin{aligned} \vartheta^1 &:= (\vartheta_1, \ldots, \vartheta_p),\ \vartheta^2 := (\vartheta_{p+1}, \ldots, \vartheta_{p+r}), \\ \vartheta^3 &:= (\vartheta_{p+r+1}, \ldots, \vartheta_m),\ \eta^1 := (\eta_1, \ldots, \eta_{p+r}). \end{aligned}$$

Further, consider the function

$$F(\vartheta, \eta^1) := \eta^1(\vartheta) - \eta^1.$$

Take $\eta_0 = \eta(\vartheta_0)$. Then, according to Assumption 2.1, we obtain

$$\det \left. \frac{D(F_1, \ldots, F_{p+r})}{D(\vartheta_1, \ldots, \vartheta_{p+r})} \right|_{(\vartheta_0, \eta_0^1)} \neq 0.$$

From the theorem on implicit function it follows that there is $\mathcal{O}(\vartheta_0^1, \vartheta_0^2) := A^1 \times A^2 \subset \mathbb{R}^p \times \mathbb{R}^r$ and $\mathcal{O}(\vartheta_0^3, \eta_0^1) := B^1 \times B^2 \subset \mathbb{R}^{m-p-r} \times \mathbb{R}^{p+r}$ such that to every point $(\vartheta, \eta^1) \in B^1 \times B^2$ there exists only one point $f(\vartheta^3, \eta^1) \in A^1 \times A^2$, for which is valid $F[f(\vartheta^3, \eta^1), \vartheta^3, \eta^1] = 0$, while $f \in C^2(B^1 \times B^2)$. Further, we have

$$\frac{\partial F[f(\vartheta^3, \eta^1), \vartheta^3, \eta^1]}{\partial \vartheta_j} = 0$$

for every $(\vartheta^3, \eta^1) \in B^1 \times B^2; j = p+r+1, \ldots, m$. Hence, using (2.2) we obtain

$$\frac{D(\eta_1, \ldots, \eta_{p+r})}{D(\theta_1, \ldots, \theta_p)} \frac{D(f_1, \ldots, f_p)}{D(\theta_j)} + \frac{D(\eta_1, \ldots, \eta_{p+r})}{D(\theta_{p+1}, \ldots, \theta_{p+r})} \left[\frac{D(f_{p+1}, \ldots, f_{p+r})}{D(\theta_j)} + \overline{H}_j(\vartheta) \right] = 0$$

for every $(\vartheta^3, \eta^1) \in B^1 \times B^2$; $j = p+r+1, \ldots, m$, where $\overline{H}_j(\vartheta) := (H_{p+1}^j(\vartheta), \ldots, H_{p+r}^j(\vartheta))^{\mathrm{T}}$ and the derivatives of the function $\eta^1(\vartheta)$ are computed at the point $\vartheta = (f(\vartheta^3, \eta^1), \vartheta^3)$. Now, it follows from the preconditions of the theorem and from (2.1) that $\frac{D(f_1, \ldots, f_p)}{D(\vartheta_j)} = 0$ for every $(\vartheta^3, \eta^1) \in B^1 \times B^2$; $j = p+r+1, \ldots, m$. Thus $\vartheta^1 = (f_1(\eta^1(\vartheta)), \ldots, f_p(\eta^1(\vartheta)))$ holds for $\vartheta \in A^1 \times A^2 \times B^1$. Then it follows from Theorem 2.2 b), that $\vartheta_i(\cdot)$ is a locally-estimable function in the point ϑ_0 for every $i = 1, \ldots, p$.

Since ϑ_0 has been chosen as an arbitrary one, $\vartheta_i(.) \in \mathcal{F}_0$ holds for every $i = 1, \ldots, p$.

□

Note 2.3. If instead of the assumption ii) at Theorem 2.6 it is only supposed that each column of $J^1(\vartheta)$ is not an element of $\mathcal{L}[J^2(\vartheta)]$, it does not have to be $\vartheta_i(\cdot) \in \mathcal{F}_0$ valid for $i = 1, \dots, p$, as the following counterexample shows.
Take $\eta(\vartheta) = (F^1, F^2)\vartheta$, $\Theta = \mathbb{R}^3$ where

$$F^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad F^2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then $R(F^1) = 2$ and each column of F^1 is not an element of $\mathcal{L}(F^2)$. Evidently, $\vartheta_i(\cdot) \notin \mathcal{F}$ for $i = 1, 2, 3$. As the considering model is linear, there is $\mathcal{F} = \mathcal{F}_0$ and thus $\vartheta_i(\cdot) \notin \mathcal{F}_0$ for $i = 1, 2, 3$.

Note 2.4. In [1] A. H. Bird and G. A. Milliken have presented a technique to determine the estimable functions in the non-linear regression model. They call the function $f(\vartheta)$ as an estimable one if it is invariant with respect to the least squares solution obtained from normal equations. The estimable function was proved to be a solution of a certain homogeneous partial differential equation in special case of a singular non-linear model, i.e. the model in which the matrix $J(\vartheta)$ is not full rank (see [1], Theorem 2.3). The proof of this statement is based on reparameterization of the singular non-linear model. But this reparameterization can generally be done only locally and the parameters are also only locally estimable in the reparameterized model, because nor regularity of the model, i. e. regularity of the matrix $J(\vartheta)$, does not ensure estimability, but only local estimability of the parameter. Consequently, the authors do not differentiate between estimability and local estimability of the parameter function. The next section gives precisions to results of this article.

Definition 2.1. The vectors $H^i(\vartheta) := (H_1^i(\vartheta), \dots, H_m^i(\vartheta))^{\mathrm{T}}$, $i = 1, \dots, l$, are said to be the l-fundamental system of solutions of the system

$$H_1(\vartheta)\frac{\partial\eta(\vartheta)}{\partial\vartheta_1} + \dots + H_m(\vartheta)\frac{\partial\eta(\vartheta)}{\partial\vartheta_m} = 0 \tag{2.3}$$

on an open set U, if the following is valid:
a) $H^i(\vartheta)$ is the solution of (2.3) for every $\vartheta \in U$, $i = 1, \dots, l$
b) $H_j^i(\vartheta) \in C^1(U)$ for every $i = 1, \dots, l$; $j = 1, \dots, m$
c) $\| H^i(\vartheta) \| \neq 0$ for every $\vartheta \in U$, $i = 1, \dots, m$
d) the vectors $H^1(\vartheta), \dots, H^l(\vartheta)$ are linearly independent for every $\vartheta \in U$
e) every solution $H(\vartheta)$ of (2.3) on U can be expressed as a linear combination of the vectors $H^1(\vartheta), \dots, H^l(\vartheta)$, where coefficients of the combination are functions of ϑ generally.

Lemma 2.2. Let $R[J(\vartheta)] = k$ $(1 \le k < m)$ for every $\vartheta \in \Theta$. Then there is $\mathcal{O}(\vartheta^0) := U$ for every $\vartheta^0 \in \Theta$ and an $(m-k)$-fundamental system of solutions of the system (2.3) on U.

Proof. The system (2.3) is the consistent system of linear equations for variables $H_1(\vartheta), \dots, H_m(\vartheta)$ for every $\vartheta \in \Theta$. Let $\vartheta^0 \in \Theta$. Then there is at least one submatrix of the kth order from the matrix $J(\vartheta)$ that is regular at the point ϑ^0 and consequently on a neighbourhood $\mathcal{O}(\vartheta^0) := U$. Let for instance

$$\det \frac{\mathrm{D}(\eta_1, \dots, \eta_k)}{\mathrm{D}(\vartheta_1, \dots, \vartheta_k)} \neq 0 \quad \text{for every } \vartheta \in U.$$

From it follows that

$$\begin{bmatrix} H_1(\vartheta) \\ \vdots \\ H_k(\vartheta) \end{bmatrix} = -\left[\frac{\mathrm{D}(\eta_1, \dots, \eta_k)}{\mathrm{D}(\theta_1, \dots, \theta_k)}\right]^{-1} \frac{\mathrm{D}(\eta_1, \dots, \eta_k)}{\mathrm{D}(\theta_{k+1}, \dots, \theta_m)} \begin{bmatrix} H_{k+1}(\vartheta) \\ \vdots \\ H_m(\vartheta) \end{bmatrix}$$

holds for every $\vartheta \in U$. Let us take $(H^1_{k+1}(\vartheta), \dots, H^1_m(\vartheta)) = (1, 0, \dots, 0), \dots$ $\dots, (H^{m-k}_{k+1}(\vartheta), \dots, H^{m-k}_m(\vartheta)) = (0, \dots, 0, 1)$, we obtain the system of solutions of (2.3) on U, for which the preconditions a) – e) of Definition 2.1 are evidently valid. □

Note 2.5. Let $R[J(\vartheta)] = k$, where $1 \leq k < m$, for every $\vartheta \in \Theta$ and let a submatrix of the kth order from the matrix $J(\vartheta)$ exist such that it is regular for every $\vartheta \in \Theta$. Then, according to Lemma 2.2, there is an $(m-k)$-fundamental system of solutions of the system (2.3) on Θ. The system

$$\sum_{i=1}^{m} H_i^j \frac{\partial \psi(\vartheta)}{\partial \vartheta_i} = 0; \qquad j = 1, \dots, m-k \tag{2.4}$$

is the complete system of the homogeneous linear partial differential equations (see [8]). A function $\psi(\vartheta)$ is said to be a solution of (2.4), if

$$\sum_{i=1}^{m} H_i^j \frac{\partial \psi(\vartheta)}{\partial \vartheta_i} = 0; \quad \text{for every } \vartheta \in \Theta; \ j = 1, \dots, m-k.$$

Obviously, the functions $\eta_1(\vartheta), \dots, \eta_N(\vartheta)$ are solutions of (2.4).

Theorem 2.7. Let $R[J(\vartheta)] = k$, where $1 \leq k < m$, for every $\vartheta \in \Theta$ and let a submatrix of the kth order from the matrix $J(\vartheta)$ exist, which is regular for every $\vartheta \in \Theta$. Then every solution of (2.4) on Θ is a locally estimable function of the parameter ϑ.

Proof. Let for instance

$$\det \frac{\mathrm{D}(\eta_1, \dots, \eta_k)}{\mathrm{D}(\theta_1, \dots, \theta_k)} \neq 0 \qquad \text{for every } \vartheta \in \Theta.$$

Let $g(\vartheta) \in C^1(\Theta)$ be a solution of (2.4) and let $\vartheta^0 \in \Theta$ exist such that

$$R\left[\left.\frac{\mathrm{D}(\psi_1, \dots, \psi_k, g)}{\mathrm{D}(\theta_1, \dots, \theta_m)}\right|_{\vartheta^0}\right] = k+1$$

Then $R[(H^1, \dots, H^{m-k})] \leq m - k - 1$ holds for every matrix $(H^1, \dots, H^{m-k})$, which satisfies the equality

$$\left[\frac{\mathrm{D}(\psi_1, \dots, \psi_k, g)}{\mathrm{D}(\theta_1, \dots, \theta_m)} \bigg|_{\vartheta^0} \right] (H^1, \dots, H^{m-k}) = 0.$$

It is contrary to the precondition, that $g(\vartheta)$ solves (2.4) on Θ. Thus

$$R \left[\frac{\mathrm{D}(\psi_1, \dots, \psi_k, g)}{\mathrm{D}(\theta_1, \dots, \theta_m)} \right] = k \quad \text{for every} \quad \vartheta \in \Theta.$$

Analogically as in the proof of Theorem 2.5, it can be proved that $g(\cdot) \in \mathcal{F}_0$. □

Note 2.6. Let the function $\eta(\vartheta)$ be not regular on any open non-empty subset of Θ. Then the model $y = \eta(\vartheta) + \varepsilon$ can be reparameterized onto the regular model locally. That means, there is $\mathcal{O}(\vartheta^0) := U$ for every $\vartheta^0 \in \Theta$ and a system $\psi(\vartheta) :=$ $:= (\psi_1(\vartheta), \dots, \psi_k(\vartheta))^{\mathrm{T}}$ $(1 \leq k < m)$ of solutions of (3.3) on U, such that

$$R \left[\frac{\mathrm{D}(\psi_1, \dots, \psi_k)}{\mathrm{D}(\theta_1, \dots, \theta_k)} \right] = k \quad \text{for every} \quad \vartheta \in U,$$

and a function $F(\psi) : \psi(U) \mapsto \mathbb{R}^N$, $F(\psi) \in C^2[\psi(U)]$ satisfying $\eta(\vartheta) = F[\psi(\vartheta)]$ for every $\vartheta \in U$ (see [4], Theorem 213). Thus $y = F(\psi) + \varepsilon$, $\psi \in \Psi(U)$, where

$$R \left[\frac{\mathrm{D}(F_1, \dots, F_N)}{\mathrm{D}(\psi_1, \dots, \psi_k)} \right] = k \quad \text{for every} \quad \psi \in U.$$

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