2. NOMENCLATURE

MFD ... matrix fraction description
DCF ... doubly coprime factorization
$\Gamma_c[\cdot]$ ... highest column degree coefficient matrix
$\Gamma_r[\cdot]$ ... highest row degree coefficient matrix
$\delta_{cj}[\cdot]$ ... jth column degree
$\delta_{rj}[\cdot]$ ... jth row degree
$\Pi[\cdot]$ ... polynomial part
$SP[\cdot]$ ... strictly proper part
l.s.v.f. ... linear state variable feedback

3. PRELIMINARIES

We consider linear, time invariant, completely controllable and observable systems

$$\dot{x} = Ax + Bu$$
$$y = Cx \mathrel{\hat{=}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mathrel{\hat{=}} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x \tag{1}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^p$, $y_1 \in \mathbb{R}^{m-\varkappa}$, $y_2 \in \mathbb{R}^{\varkappa}$. The control $u = -\tilde{u} + r$ consists of the l.s.v.f. $\tilde{u} = Kx$ and of the reference input $r \in \mathbb{R}^p$. The l.s.v.f. is parametrized by the $p \times n$ matrix K which can be chosen to assign stable closed loop poles. The $(n - \varkappa)$-dimensional state observer

$$\dot{z} = Fz + [D_1 \; D_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + TBu \tag{2}$$

yields $z = Tx$ in the steady state if $TA - FT = DC$ holds and F is stable. If the state estimate

$$\hat{x} = \begin{bmatrix} C_2 \\ T \end{bmatrix}^{-1} \begin{bmatrix} y_2 \\ z \end{bmatrix} = [\Psi_2 \; \Theta] \begin{bmatrix} y_2 \\ z \end{bmatrix} \tag{3}$$

is substituted in the l.s.v.f. a dynamic compensator of order $n - \varkappa$ results. For the following it will be of importance that (3) implies

$$C_2 \Psi_2 = I_\varkappa , \quad C_2 \Theta = 0 ; \quad T \Psi_2 = 0 ; \quad T\Theta = I_{n-\varkappa} ; \quad \Psi_2 C_2 + \Theta T = I_n . \tag{4}$$

An alternative representation of the observer (2) is

$$\dot{z} = T(A - L_1 C_1)\, \Theta z + [TL_1 \; T(A - L_1 C_1)\, \Psi_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + TBu \tag{5}$$

where the so far undefined matrix L_1 must meet $TL_1 = D_1$ which is true for $L_1 = \Theta D_1$. This, on the other hand, implies

$$C_2 L_1 = 0 . \tag{6}$$

In the sequel the abbreviation $F = T(A - L_1 C_1)\,\Theta$ will be used which follows from a comparison of (2) and (5). Using the observer representation (5) the dynamic compensator is described by its transfer function behaviour $u(s) = -F_c(s)\,y(s) + F_r(s)\,r(s)$ with

$$F_c(s) = K\Theta[sI - F + TBK\Theta]^{-1}\,[TL_1\; T(A - L_1 C_1 - BK)\,\Psi_2] + [0\;\; K\Psi_2] \tag{7}$$

The system transfer matrix $F(s) = C(sI - A)^{-1}\,B$ can be represented in a right coprime MFD $F(s) = N(s)\,D^{-1}(s)$ with $D(s)$ column proper or in a left coprime MFD $F(s) = \bar{D}^{-1}(s)\,\bar{N}(s)$ with $\bar{D}(s)$ row proper. Likewise the compensator (7) can be described by the left coprime MFD $F_c(s) = D_c^{-1}(s)\,N_c(s)$ with $D_c(s)$ row proper and by the right coprime MFD $F_c(s) = \bar{N}_c(s)\,\bar{D}_c^{-1}(s)$ with $\bar{D}_c(s)$ column proper.

The l.s.v.f. can be parametrized in the frequency domain by the $p \times p$ polynomial matrix $\tilde{D}(s)$. It is well known that

$$\tilde{D}(s)\,D^{-1}(s) = I + K(sI - A)^{-1}\,B \tag{8}$$

holds and consequently

$$\Gamma_c[\tilde{D}(s)] = \Gamma_c[D(s)]\;; \quad \text{and} \quad \delta_{cj}[\tilde{D}(s)] = \delta_{cj}[D(s)]\,, \quad j = 1, 2, \ldots, p\,. \tag{9}$$

This implies that $\tilde{D}(s)$ contains the same number of free parameters as the $p \times n$ feedback matrix K, namely pn [1]. A choice of the pn degrees of freedom in $\tilde{D}(s)$ such that $\det \tilde{D}(s)$ becomes a stable polynomial corresponds to a pole placing design of the state feedback matrix K. An optimal l.s.v.f. results by solving the corresponding polynomial matrix equation for $\tilde{D}(s)$ [7]. It should be noted, however, that the optimal solution $\tilde{D}(s)$ has to be aligned to meet the restrictions (9). Equation (8) can be used to compute the corresponding $\tilde{D}(s)$ for a given K and vice versa.

The right doubly coprime fractional representations (DCF) of the plant transfer matrix are given by [1], [8]

$$D(s)\,\tilde{D}^{-1}(s) = I - K(sI - A + BK)^{-1}\,B \tag{10}$$

and

$$N(s)\,\tilde{D}^{-1}(s) = C(sI - A + BK)^{-1}\,B \tag{11}$$

The frequency domain parametrization of the reduced order observer of order $n - \varkappa$ with $0 \leqq \varkappa \leqq m$ can be derived with the aid of a nonminimal observer representation introduced in [4].

It has been shown that the reduced order observer of order $n - \varkappa$ with $0 \leqq \varkappa \leqq m$ is parametrized in the frequency domain by the $m \times m$ polynomial matrix $\hat{D}(s)$ which meets the following relation [4]

$$\bar{D}^{-1}(s)\,\hat{D}(s) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (sI - A)^{-1}\,[L_1\;\; \Psi_2] + \begin{bmatrix} I_{m-\varkappa} & 0 \\ 0 & 0_\varkappa \end{bmatrix} \tag{12}$$

For $\varkappa = 0$ this coincides with the well known result for full order observers and for $\varkappa = m$ the minimal order observer results [2] are obtained.

Introducing the row proper $m \times m$ polynomial matrix

$$\bar{D}^{\varkappa}(s) = \Gamma_r^{\cdot}\left\{\bar{D}(s)\begin{bmatrix} I_{m-\varkappa} & 0 \\ 0 & s^{-1}I_{\varkappa}\end{bmatrix}\right\}, \tag{13}$$

the relation (12) implies [3]

$$\Gamma_r[\hat{D}(s)] = \Gamma_r[\bar{D}^{\varkappa}(s)]\ ; \quad \text{and} \quad \delta_{rj}[\hat{D}(s)] = \delta_{rj}[\bar{D}^{\varkappa}(s)]\ ; \quad j = 1, 2, \ldots, m\,. \tag{14}$$

As a consequence of this $\hat{D}(s)$ contains $m(n - \varkappa)$ free parameters which also exist in the time domain if only y_2 is used to reconstruct x via (3). A choice of these free parameters such that det $\hat{D}(s)$ is a stable polynomial corresponds to the pole placement problem. An optimal linear estimator results when solving the corresponding polynomial matrix equation for $\hat{D}(s)$ by spectral factorization. This polynomial equation has the same simple structure for any filter order n_0 within the limits $n - m \leqq n_0 \leqq n$ [3].

Equation (12) can be used to compute $\hat{D}(s)$ from a given time domain parametrization of the filter and vice versa [3]. Thus together with (8) one can establish a one to one relationship between the time and the frequency domain design of observer based compensators related to arbitrary observer orders.

The left coprime DCFs of the plant are given by [4]

$$\hat{D}^{-1}(s)\,\bar{D}(s) =$$
$$= \begin{bmatrix} I - C_1\Theta(sI - F)^{-1}\,TL_1 & C_1[-I - \Theta(sI - F)^{-1}\,T(A - L_1C_1)]\,\Psi_2 \\ -C_2A\Theta(sI - F)^{-1}TL_1 & C_2[sI - A - A\Theta(sI - F)^{-1}\,T(A - L_1C_1)]\,\Psi_2 \end{bmatrix} \tag{15}$$

and

$$\hat{D}^{-1}(s)\,\bar{N}(s) = \begin{bmatrix} C_1\Theta(sI - F)^{-1}\,TB \\ C_2B + C_2A\Theta(sI - F)^{-1}\,TB \end{bmatrix}. \tag{16}$$

Thus both the l.s.v.f. and the observer of order $n - \varkappa$ can directly be parametrized in the frequency domain without recurrence to time domain parameters. In order to give the complete set of relations with the time domain approach, we here repeat the connections between the time domain representation of the compensator and its DCFs. Hippe [4] derived the following relations

$$\Delta^{-1}(s)\,N_c(s) = K\Theta(sI - F)^{-1}\,[TL_1\ \ T(A - L_1C_1)\,\Psi_2] + [0\ \ K\Psi_2] \tag{17}$$

$$\Delta^{-1}(s)\,D_c(s) = I + K\Theta(sI - F)^{-1}\,TB \tag{18}$$

and

$$\bar{N}_c(s)\,\bar{\Delta}^{-1}(s) = K(sI - A + BK)^{-1}\,[L_1\ \ \Psi_2] \tag{19}$$

$$\bar{D}_c(s)\,\bar{\Delta}^{-1}(s) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}(sI - A + BK)^{-1}\,[L_1\ \ \Psi_2] + \begin{bmatrix} I_{m-\varkappa} & 0 \\ 0 & 0_{\varkappa} \end{bmatrix} \tag{20}$$

and it was shown that these DCFs directly describe the compensator transfer matrix (7). The stable $p \times p$ polynomial matrix $\Delta(s)$ contains the observer dynamics, i.e.

$\det \Delta(s) = \det \hat{D}(s)$, and the $m \times m$ polynomial matrix $\bar{\Delta}(s)$ contains the controlled plant dynamics, i.e. $\det \bar{\Delta}(s) = \det \tilde{D}(s)$. We now establish the solution procedure for the DCFs (17)–(20) on the basis of the parametrizing matrices $\tilde{D}(s)$ for the l.s.v.f. and of $\hat{D}(s)$ for the (reduced order) observer in the frequency domain.

4. COMPUTATION OF THE LEFT COPRIME COMPENSATOR MFD

Consider any polynomial solutions $Y(s)$ and $X(s)$ of the linear diophantine equation

$$Y(s)\,N(s) + X(s)\,D(s) = \tilde{D}(s) \tag{21}$$

where the $p \times p$ polynomial matrix $\tilde{D}(s)$ characterizes the controlled plant dynamics. Since $D(s)$ and $N(s)$ are relatively coprime, such polynomial matrices exist [6]. With $\Pi[\cdot]$ denoting the polynomial part and $SP[\cdot]$ denoting the strictly proper part of a rational matrix it is obvious that for a given transfer matrix $P(s)$ we have $P(s) = \Pi[P(s)] + SP[P(s)]$.

Lemma 1. Consider the plant transfer matrix $F(s) = \bar{D}^{-1}(s)\,\bar{N}(s)$, a solution $Y(s)$ of (21) and the $m \times m$ polynomial matrix $\hat{D}(s)$ characterizing the observer dynamics. Then the strictly proper part of $Y(s)\,\bar{D}^{-1}(s)\,\hat{D}(s)$ is given by

$$SP[Y(s)\,\bar{D}^{-1}(s)\,\hat{D}(s)] = K(sI - A)^{-1}[L_1\ \Psi_2]\,. \tag{22}$$

Proof. Using the basic relation (12) we can write

$$Y(s)\,\bar{D}^{-1}(s)\,\hat{D}(s) = Y(s)\,C(sI - A)^{-1}\,[L_1\ \Psi_2] + Y(s)\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $Y(s)$ is a polynomial matrix it remains to be shown that

$$SP\{Y(s)\,C(sI - A)^{-1}\,[L_1\ \Psi_2]\} = K(sI - A)^{-1}\,[L_1\ \Psi_2]$$

or equivalently that

$$W(s) = [Y(s)\,C - K]\,(sI - A)^{-1}\,[L_1\ \Psi_2]$$

is a polynomial matrix. Right multiplication of (21) by $D^{-1}(s)$ yields

$$Y(s)\,N(s)\,D^{-1}(s) + X(s) = \tilde{D}(s)\,D^{-1}(s)\,,$$

or with the right coprime plant MFD and (8)

$$[Y(s)\,C - K]\,(sI - A)^{-1}\,B = I - X(s)\,. \tag{23}$$

The right hand side of (23) is a polynomial matrix. As we have assumed a completely controllable plant, $(sI - A)^{-1}\,B$ is a coprime pair and consequently, $[Y(s)\,C - K]$ has the form $\hat{N}(s)\,(sI - A)$ with $\hat{N}(s)$ being a polynomial matrix. Therefore $[Y(s)\,C - K]\,(sI - A)^{-1}$ constitutes a polynomial matrix which completes the proof. □

With this preliminary result we can formulate the solution procedure for the left coprime compensator factorization.

Theorem 1. With the polynomial matrix

$$V(s) = \Pi[Y(s)\,\bar{D}^{-1}(s)\,\hat{D}(s)] \tag{24}$$

the doubly coprime left factorization of the compensator is given by

$$N_c^*(s) = \Delta^{-1}(s)\,N_c(s) = Y(s) - V(s)\,\hat{D}^{-1}(s)\,\bar{D}(s) \tag{25}$$

and

$$D_c^*(s) = \Delta^{-1}(s)\,D_c(s) = X(s) + V(s)\,\hat{D}^{-1}(s)\,\bar{N}(s)\,. \tag{26}$$

The left coprime compensator MFD and the observer matrix $\Delta(s)$ can easily be computed from (25) and (26) by prime factorization of $[N_c^*(s)\ D_c^*(s)] = \Delta^{-1}(s)\,[N_c(s)\ D_c(s)]$ (cf. [6]).

The proof of the Theorem basically goes along the lines in [1] and [2]. For the sake of brevity it is omitted here.

5. COMPUTATION OF THE RIGHT COPRIME COMPENSATOR MFD

Consider any polynomial solutions $\bar{Y}(s)$ and $\bar{X}(s)$ of the linear diophantine equation

$$\bar{N}(s)\,\bar{Y}(s) + \bar{D}(s)\,\bar{X}(s) = \hat{D}(s) \tag{27}$$

where the $m \times m$ polynomial matrix $\hat{D}(s)$ characterizes the observer dynamics. Since $\bar{D}(s)$ and $\bar{N}(s)$ are relatively coprime such polynomial matrices exist [6].

Lemma 2. Consider the plant transfer matrix $F(s) = N(s)\,D^{-1}(s)$, a solution $\bar{Y}(s)$ of (27) and the $p \times p$ polynomial matrix $\tilde{D}(s)$ parametrizing the linear state feedback control in the frequency domain. Then the strictly proper part of $\tilde{D}(s)\,.$ $.\,D^{-1}(s)\,\bar{Y}(s)$ is given by

$$SP[\tilde{D}(s)\,D^{-1}(s)\,\bar{Y}(s)] = K(sI - A)^{-1}\,[L_1\ \Psi_2]\,. \tag{28}$$

The proof of Lemma 2 goes along the lines of the proof of Lemma 1 and it uses the fact that we have assumed complete observability which implies that $C(sI - A)^{-1}$ is a coprime pair.

Now we can formulate the solution procedure for the right coprime compensator factorization.

Theorem 2. With the polynomial matrix

$$\bar{V}(s) = \Pi[\tilde{D}(s)\,D^{-1}(s)\,\bar{Y}(s)] \tag{29}$$

the doubly coprime right factorization of the compensator is given by

$$\bar{N}_c^*(s) = \bar{N}_c(s)\,\bar{\Delta}^{-1}(s) = \bar{Y}(s) - D(s)\,\tilde{D}^{-1}(s)\,\bar{V}(s) \tag{30}$$

and

$$\bar{D}_c^*(s) = \bar{D}_c(s)\,\bar{\Delta}^{-1}(s) = \bar{X}(s) + N(s)\,\tilde{D}^{-1}(s)\,\bar{V}(s)\,. \tag{31}$$

The right coprime compensator MFD and the matrix $\bar{\Delta}(s)$, containing the controlled plant dynamics, can easily be computed from (30) and (31) by prime factorization of

$$\begin{bmatrix} \bar{N}_c^*(s) \\ \bar{D}_c^*(s) \end{bmatrix} = \begin{bmatrix} \bar{N}_c(s) \\ \bar{D}_c(s) \end{bmatrix} \bar{\Delta}^{-1}(s)\,.$$

Again the proof is omitted. It basically goes along the lines presented in [1] for $\varkappa = 0$ and in [2] for $\varkappa = m$.

6. THE DISCRETE TIME CASE

When applying z-transform techniques the input output behaviour of discrete time systems is described by transfer matrices $F(z)$ which can also be represented in coprime matrix fraction descriptions $F(z) = N(z)\,D^{-1}(z) = \bar{D}^{-1}(z)\,\bar{N}(z)$. The same is true for the compensator. Keeping in mind that the stability region is inside the unit circle of the z-plane, all formulas derived above also hold in the discrete time case if s is substituted by z.

There are differences between the continuous and the discrete time solutions when applying optimal estimators in conjunction with an updated state estimate. These differences are discussed in [5] for the optimal filter and their influence on the compensator design as presented above will be investigated in a forthcoming paper. For pole placement compensators and when using the one step prediction estimate in a stochastic setting, the above compensator design holds both in the s- and in the z-domain.

7. AN EXAMPLE

As a demonstration example we use a simple second order system with one input and two outputs described by its transfer vector

$$F(z) = \frac{1}{z^2 - z + 0{\cdot}25} \begin{bmatrix} z \\ 1 \end{bmatrix}$$

so that the entities in the left and right coprime MFDs are given by

$$N(z) = \begin{bmatrix} z \\ 1 \end{bmatrix}; \quad D(z) = z^2 - z + 0{\cdot}25\,; \quad \bar{D}(z) = \begin{bmatrix} 1 & -z \\ z & -z + 0{\cdot}25 \end{bmatrix};$$

$$\bar{N}(z) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We look for a compensator which moves the system poles to the origin of the z-plane using a first order state observer ($\varkappa = 1$) having an eigenvalue at $z = 0{\cdot}1$. Obviously the state feedback control is parametrized by $\tilde{D}(z) = z^2 + \alpha z + \beta$ where by choice of the $pn = 2$ degrees of freedom as $\alpha = \beta = 0$ we obtain the desired closed loop poles.

In order to obtain the parametrization of the first order observer the polynomial matrix

$$\bar{D}^1(z) = \Pi\left\{\bar{D}(z)\begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}\right\} = \begin{bmatrix} 1 & -1 \\ z & -1 \end{bmatrix}$$

must be row proper; which it is. So following (14) the polynomial matrix $\hat{D}(z)$ parametrizing the (reduced order) observer is given by

$$\hat{D}(z) = \begin{bmatrix} 1 & -1 \\ z + \alpha & \beta \end{bmatrix}.$$

When choosing the $m(n - \varkappa) = 2$ free parameters α and β, e.g., as $\alpha = -0{\cdot}1$ and $\beta = 0$ the desired observer eigenvalue results. A solution of (21) is $X = 1$, $Y = [1 \;\; -0{\cdot}25]$. Now following Theorem 1 one obtains $V(z) = [1 \;\; 0]$ and hence

$$N_C^*(z) = [-0{\cdot}1 \;\; 0{\cdot}75z - 0{\cdot}225]\frac{1}{z - 0{\cdot}1}; \quad D_C^*(z) = (z + 0{\cdot}9)\frac{1}{z - 0{\cdot}1}.$$

Consequently the left coprime compensator MFD is given by

$$F_C(z) = (z + 0{\cdot}9)^{-1}[-0{\cdot}1 \;\; 0{\cdot}75z - 0{\cdot}225]$$

and the observer characteristic polynomial by $\Delta(z) = z - 0{\cdot}1$.

8. CONCLUSION

Using a new nonminimal representation of the reduced order observer of order $n - \varkappa$ with $0 \leqq \varkappa \leqq m$ in the time domain the direct parametrization of such observers in the frequency domain becomes possible [4]. If the polynomial matrices $\tilde{D}$ for the l.s.v.f. and $\hat{D}$ for the linear state estimator are given, the computation of the compensator DCFs and consequently also of the compensator MFDs can be carried out using standard software. This is true both for continuous time and discrete time systems with one exception, namely when using the updated (discrete) state estimate in a stochastic setting [5]. Thus the general equivalence of state space and frequency domain approaches to observer based compensator design has been established.

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