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A GEOMETRIC APPROACH FOR TESTING REGULARITY OF MULTI-DIMENSIONAL POLYNOMIAL MATRICES AND A PENCIL OF n-MATRICES

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In this paper, a geometric approach will be derived for testing column regularity (CR) of multi-dimensional (m-D) polynomial matrices and a pencil of n-matrices "n-pencil" using some spaces defined by the coefficient matrices of the polynomial matrices. Assigning x^{L_i} to multi-variable x_i and using the form preserving polynomials concept it has been shown that CR of an n-pencil and m-D polynomial matrices can be stated as CR of a 1-D polynomial matrix. Defining an associated companion form for the 1-D polynomial matrix it has been proved that CR of 1-D polynomial matrix can be reduced to CR of the related 2-pencil. Thus CR of a m-D polynomial matrix and n-pencil has been stated in terms of CR of a pair of matrices.

1. INTRODUCTION

Motivation for studying the column regularity problem arises in different contexts: i) Specifically in the parameter identification problem [1, 2]; ii) in solving multi-dimensional systems with Z-transform techniques and in invertibility of a polynomial matrix of several independent variables [3].

i) Parameter Identification in Linear Networks:

The parameter identification problem can be formulated by writing network equations in terms of parameter coefficient matrices $A_i \in \mathbb{R}^{n \times n}$, $i = 1, 2, \ldots, n_e$ as follows

$$\Big(\sum_{i=1}^{n_e} g_i A_i + D\Big) X = Y \tag{1}$$

where g_i's are unknown parameters, $X \in \mathbb{R}^n$ is the measurement vector, $Y \in \mathbb{R}^n$ is the input vector; $D \in \mathbb{R}^{n \times n}$ and n_e is the number of parameters which is usually greater than the number of measurements. In [1, 2], it has been shown that the parameter set $\{g_i\}_{i=1}^{m}$ can uniquely be determined in terms of $\{g_i\}_{i=m+1}^{n_e}$ if and only if there exist $X \in \mathbb{R}^{n \times 1}$ such that

$$E(X) := [A_1 X \; A_2 X \ldots A_m X] = \sum_{i=1}^{n} x_i E_i \quad m \leqq n \tag{2}$$

is of full column rank where x_i is the ith element of X and E_i consists of corresponding columns of A_i.

Definition 1. A linear map $E(\cdot)$: $\mathbb{R}^n \to \mathbb{R}^{n\times m}$ defined by (2) is called an n-pencil.

Definition 2. An n-pencil is column-regular if and only if there exists $X \in \mathbb{R}^n$ such that $E(X)$ is of full column rank.

Thus, based on these definitions it has been shown that: The unique solvability of the parameters is equivalent to column regularity of the related n-pencil; a Toeplitz matrix which depends only on the matrices E_i's can be iteratively constructed for testing column-regularity of the n-pencil (2) [1].

ii) Solution of Multidimensional Systems:

Although the motivation is quite different, the n-pencil defined in (2) arises when the Z-transform of the following specific multidimensional system

$$E_1 y(i_1 + 1, i_2, \dots, i_n) + E_2 y(i_1, i_2 + 1, \dots, i_n) + \dots$$
$$\dots + E_n y(i_1, i_2, \dots, i_n + 1) = Bu(i_1, i_2, \dots, i_n) \tag{3}$$

is taken as

$$\left(\sum_{i=1}^{n} z_i E_i\right) Y(z_1, z_2, \dots, z_n) = BU(z_1, z_2, \dots, z_n) \tag{4}$$

where $Y(z_1, z_2, \dots, z_n)$ and $U(z_1, z_2, \dots, z_n)$ are the Z-transforms of the sequences $y(i_1, i_2, \dots, i_n)$ and $u(i_1, i_2, \dots, i_n)$, respectively. Uniqueness of the solution to (4) can be investigated by considering the column regularity of the n-pencil in (4).

A polynomial matrix of p independent variables arising in more general multidimensional systems, $E(x_1, \dots, x_p)$ can be written as

$$E(x_1, \dots, x_p) = \sum_{i_1=0}^{n_1} \dots \sum_{i_p=0}^{n_p} E_{i_1,\dots,i_p} x_1^{i_1} \dots x_p^{i_p}\,, \tag{5}$$

where $n_i = \deg_{x_i}$ in E for $(i = 1, \dots, p)$ and $E_{i_1 \dots i_p} \in \mathbb{R}^{n\times n}$ are coefficient matrices.

In [3], necessary and sufficient conditions for the invertibility of a p-D polynomial matrix (5) has been given. The invertibility matrix for testing the regularity of p-D polynomial matrix given in [3] is quite involved, therefore is omitted here.

The purpose of this paper is to formulate the column regularity of n-pencil and a MDP matrix in terms of column regularity of a pair of matrices (i.e., 2-pencil) arising while investigating uniqueness of solutions of implicit linear systems [4–6]. Therefore, some definitions related to 2-pencil existing in the literature will be given.

2. PENCILS OF 2-MATRICES

Consider systems defined by the following form

$$E\,y(i + 1) = A\,y(i) + B\,u(i) \tag{6}$$

where $y(i) \in \mathbb{R}^m$ is the output of the system and $u(i) \in \mathbb{R}^r$ is the input of the system. The pair E, A are $n \times m$ $(m \leqq n)$ constant matrices, B is $n \times r$ constant matrix, i is the dependent variable.

The matrix pencil relating to the above implicit linear system "$zE - A$" where z is the independent variable is called as 2-pencil.

In order to investigate uniqueness of the solution to equation (6) for any $u(\cdot)$ the following definitions are given.

Definition 3. The characteristic subspace of the pair (E, A) is the largest subspace $\mathbf{V}^*$ satisfying the following relation for a linear subspace $\mathbf{V}$ of $\mathbb{R}^n$.

$$A\mathbf{V} \subset E\mathbf{V}\,. \tag{7}$$

Definition 4. Characteristic kernel of the pair (E, A) is the subspace N defined by

$$N = \ker E \cap \mathbf{V}^* \tag{8}$$

In the literature column regularity of the pair (E, A) is usually defined as: "The pair (E, A) is C-regular (column regular) if

$$\dim N = 0''\,. \tag{9}$$

Note that the definition above is different than Definition 2. But the following theorem will show that there is no discrepancy between these definitions.

Theorem 1. The generalized spectrum of the pair (E, A) is finite if and only if this pair is C-regular [4].

In [4], it has been proved that when the matrix pencil "$zE - A$" is column regular then the solution to the system defined by (6) is unique.

Above definitions and theorem can be used when the multidimensional column regularity problem is stated in terms of column-regularity of 2-pencils.

3. FORM PRESERVING POLYNOMIALS

In order to link the multidimensional Z-transform with one dimensional Z-transform the following definition which is a generalization of the definition in [7] is needed.

Definition 5. A 1-D polynomial

$$p(x) = \sum_{k=0}^{d} a_k x^k \tag{10}$$

is a form preserving polynomial with respect to r-dimensional "r-D" polynomial

$$p(x_1 \dots x_r) = \sum_{i_1=0}^{m_1} \dots \sum_{i_r=0}^{m_r} a_{i_1 \dots i_r} x_1^{i_1} \dots x_r^{i_r} \quad (m_1 \leqq m_2 \leqq \dots \leqq m_r) \tag{11}$$

if for every integer set $(i_1, i_2, \ldots, i_r)$ in $p(x_1, x_2, \ldots, x_r)$ there exists a unique k in $p(x)$ such that $a_k = a_{i_1 i_2, \ldots, i_r}$.

As a consequence of this definition, the number of distinct terms in $p(x_1, x_2, \ldots, x_r)$ is equal to the number of distinct terms in $p(x)$.

Theorem 2. Let $p(x)$ be the 1-D polynomial obtained from the r-D polynomial by

$$p(x) := p(x_1, x_2, \ldots, x_r)\Big|_{\substack{x_1 = x^{L_1} \\ x_2 = x^{L_2} \\ \vdots \\ x_r = x^{L_r}}} \tag{12}$$

then for $L_1 := 1$, $L_2 := (m_1 + 1)$, $L_3 := (m_1 + 1)(m_2 + 1), \ldots, L_r := (m_1 + 1) \cdot (m_2 + 1) \ldots (m_{r-1} + 1)$, $p(x)$ is form preserving.

Proof. Consider the distinct monomials $x_1^{i_1} x_2^{i_2} x_3^{i_3} \ldots x_r^{i_r}$ and $x_1^{j_1} x_2^{j_2} x_3^{j_3} \ldots x_r^{j_r}$ of $p(x_1, x_2, \ldots, x_r)$. Suppose that with the assignment given above these monomials are transformed to the same monomial in x, that is

$$x^{i_1} x^{L_2 i_2} x^{L_3 i_3} \ldots x^{L_r i_r} = x^{j_1} x^{L_2 j_2} x^{L_3 i_3} \ldots x^{L_r j_r} \tag{13}$$

(13) implies that

$$(i_1 - j_1) + (m_1 + 1)(i_2 - j_2) + (m_1 + 1)(m_2 + 1)(i_3 - j_3) + \ldots$$
$$\ldots + (m_1 + 1)(m_2 + 1) \ldots (m_{r-1} + 1)(i_r - j_r) = 0$$

which can also be written as

$$\prod_{k=1}^{r-1} (m_k + 1)(i_r - j_r) = (i_1 - j_1) + (m_1 + 1)(i_2 - j_2) + \ldots$$
$$\ldots \prod_{k=1}^{r-2} (m_k + 1)(i_{r-1} - j_{r-1}) \, .$$

Taking the absolute value of each side and using the triangle inequality the following inequality is obtained.

$$\prod_{k=1}^{r-1} (m_k + 1) |i_r - j_r| \leqq |i_1 - j_1| + (m_1 + 1) |i_2 - j_2| + \ldots$$
$$\ldots + \prod_{k=1}^{r-2} (m_k + 1) |i_{r-1} - j_{r-1}| \, . \tag{14}$$

as $|i_k - j_k| \leqq m_k$ (14) becomes

$$\prod_{k=1}^{r-1} (m_k + 1) \leqq m_1 + (m_1 + 1) m_2 + \ldots + \prod_{k=1}^{r-2} (m_k + 1) m_{r-1} \, . \tag{15}$$

Since $\prod_{k=1}^{r-1} (m_k + 1) = \prod_{k=1}^{r-2} (m_k + 1) m_{r-1} + \prod_{k=1}^{r-2} (m_k + 1)$, (14) is reduced to

$$\prod_{k=1}^{r-2} (m_k + 1) \leqq m_1 + (m_1 + 1) m_2 + \ldots + \prod_{k=1}^{r-3} (m_k + 1) m_{r-2} \tag{16}$$

Reducing (16) successively $(m_1 + 1) \leqq m_1$ is obtained which is a contradiction. Therefore, the assumption made in the proof is not true and the theorem holds. □

Theorem 3. Let the coefficients $a_{i_1 i_2 \dots i_r}$ in (11) be all non-zero. Then the values for L_i in (12) are minimal values to preserve the form.

Proof. When $x_i^{L_i}$ is assigned as in (12), the polynomial in (11) can be written as

$$p(x, x^{L_1}, x^{L_2}, \dots, x^{L_r}) = \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \dots \sum_{i_r=0}^{m_r} a_{i_1 i_2 \dots i_r} x^{i_1} x^{L_1 i_2} \dots x^{L_{r-1} i_r} \tag{17}$$

Let $a_k := a_{i_1 i_2 \dots i_r}$ and $k := i_1 + L_1 i_2 + \dots + L_{r-1} i_r$ then (17) is

$$p(x) = \sum_{k=0}^{d} a_k x^k$$

where $d = m_1 + (m_1 + 1) m_2 + (m_1 + 1)(m_2 + 1) m_3 + \dots + (m_1 + 1) \dots \dots (m_{r-1} + 1) m_r$. The total number of a_k's is $d + 1$ which is equal

$$\begin{aligned} d + 1 &= m_1 + (m_1 + 1) m_2 + (m_1 + 1)(m_2 + 1) m_3 + \dots \\ &\dots + (m_1 + 1)(m_2 + 1) \dots (m_{r-1} + 1) m_r + 1 = \\ &= (m_1 + 1)[(m_2 + 1) + (m_2 + 1) m_3 + \dots + (m_2 + 1) \dots \\ &\dots (m_{r-1} + 1) m_r] = (m_1 + 1)(m_2 + 1) \dots (m_{r-1} + 1)(m_r + 1) = \\ &= \prod_{i=1}^{r} (m_i + 1) . \end{aligned}$$

Hence, the number of terms of the polynomials (10) is exactly equal to the number of terms of the polynomial in (11) and therefore the theorem holds. □

Example 1. Let $p(x_1, x_2) = a_{00} + a_{01}x_2 + a_{02}x_2^2 + a_{10}x_1 + a_{11}x_1x_2$, then 1-D polynomial

$$p_1(x) := p(x_1, x_2)|_{\substack{x_1 = x \\ x_2 = x^2}} = a_{00} + a_{01}x^2 + a_{02}x^4 + a_{10}x + a_{11}x^3$$

is a form preserving polynomial with respect to 2-D polynomial $p(x_1, x_2)$.

Suppose that some monomials in (11) have zero coefficients. In this case, the assignment given in theorem (2) is not minimal. The following example will illustrate this point.

Example 2. Let $p(x_1, x_2) = a_{00} + a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2$. Then according to Theorem 2,

$$p_1(x) := p(x_1. x_2)|_{\substack{x_1 = x \\ x_2 = x^3}} = a_{00} + a_{20}x^2 + a_{11}x^4 + a_{02}x^6$$

is a form preserving polynomial with respect to $p(x_1, x_2)$. But

$$p_2(x) := p(x_1, x_2)|_{\substack{x_1 = x \\ x_2 = x^2}} = a_{00} + a_{20}x^2 + a_{11}x^3 + a_{02}x^4$$

is another form preserving polynomial with minimal assignment.

When a homogenous polynomial is considered the assignment given in Theorem 2 as Example 2 shows is not necessarily the minimal one in degree. Because there is no one-to-one correspondence between the monomials of a homogeneous polynomial and the ones of a polynomial in the form (11). The following assignment will ensure that the resulting 1-D polynomial is the form preserving polynomial with respect to the homogenous polynomial.

Notation. $R_h^i[x_1. x_2, \ldots, x_r]$ denotes the set of all polynomials of degree i, homogenous in $x_1, x_2, \ldots, x_r$ over $\mathbb{R}$.

Fact 1. Number of terms in a polynomial in $r - 1$ variables of degree equal to n is equal to the number of terms of monomials of degree n in r variables.

Theorem 4. $p(x) := p(x_1, x_2, x_2, \ldots, x_r)\, x_1 = 1, x_2 = x, x_3 = x^{(m+1)}, x_4 = x^{(m(m+1)+1)}, \ldots, x_r = x^q$ is the form preserving polynomial with respect to $p(x_1, x_2, \ldots, x_r) \in R_h^m(x_1, x_2, \ldots, x_r)$ where $q = m^{r-2} + m^{r-3} + \ldots + 1$.

Proof. Since the number of monomials of degree m in r variables is equal to the number of monomials of degree $\leqq m$ in $r - 1$ variables (Fact 1), there is one-to-one correspondence between the terms of the homogenous polynomial $p(x_1, x_2, \ldots, x_r)$ and the polynomial $\hat{p}(x_2, x_3, \ldots, x_r) := p(x_1, x_2, \ldots, x_r)|_{x_1=1}$. The degree of monomials of $\hat{p}(x_2, \ldots, x_r)$ is less than or equal to m and

$$\hat{p}(x_2, x_3, \ldots, x_r) = \sum_{j_2=0} \ldots \sum_{j_r=0} a_{j_1 j_2 \ldots j_r} x_2^{j_2} \ldots x_r^{j_r} \tag{18}$$

where $\sum_{i=2}^{r} j_i \leq$ and $j_1 = m - \sum_{i=2}^{r} j_i$.

With the above assignment a polynomial in two variables $\hat{p}(x_2, x_3)$ can be written as

$$\begin{aligned} \hat{p}(x_2, x_3)|_{\substack{x_2=x \\ x_3=x^{m+1}}} &= (a_{m,0,0} + a_{m-1,0,1}x^{1(m+1)} + \ldots a_{0,0,m}x^{m(m+1)}) + \\ &+ x(a_{m-1,1,0} + a_{m-2,1,1}x^{(m+1)} + \ldots + a_{0,1,m-1}x^{(m-1)(m+1)}) + \\ &\vdots \\ &+ x^{m-1}(a_{1,m-1,0} + a_{0,m-1,1}x^{(m+1)}) + x^m a_{0,m,0} \end{aligned} \tag{19}$$

whose terms are distinct. Now assume that the above assignment for the polynomial in $r - 1$ variable "$\hat{p}(x_2, x_3, \ldots, x_{r-1})$" with $x_2 = x$, $x_3 = x^{m+1}, \ldots, x_{r-1} = x^{m^{r-3}+m^{r-4}\ldots+1}$ is form preserving. A polynomial in r variables "$\hat{p}(x_2, x_3, \ldots, x_r)$" can be written as

$$\hat{p}(x_2, x_3, \ldots, x_r) = \sum_{i=0}^{n} x_r^i\, \hat{p}_{m-i}(x_2, x_3, \ldots, x_{r-1}) \tag{20}$$

where $\hat{p}_{m-i}(x_2, \ldots, x_{r-1})$ is an $r - 1$ variable polynomial with degree $m - i$. By induction hypothesis each terms of $\hat{p}_{m-i}(x, x^{m+1}, \ldots, x^{q_1})$ where $q_1 = m^{r-3} + m^{r-4} + \ldots + 1$ will be distinct and highest degree term will be $x^{m \cdot q_1}$. Therefore the assignment for x_r should be x^{mq_1+1} where $mq_1 + 1 = m^{r-2} + m^{r-3} + \ldots + 1$.

□

This result can be used to state column-regularity problem of an n-pencil in terms of column-regularity of minimal degree one-dimensional polynomial matrix. Minimality is needed since the associated companion form will have a minimum dimension.

Theorem 5. The pencil in (2) is C-regular if and only if $E(x) = \sum_{i=1}^{n} x^{q_i} E_i$ is C-regular where $q_1 = 0$, $q_i = \sum_{k=2}^{i} m^{i-k}$ $2 \leqq i \leqq n$.

Proof. (If part) is obvious.

(Only if part): Assume that $E(x)$ is not C-regular which implies all $m \times m$ determinants of $E(x)$ are identically equal to zero which means all coefficients are zero. Since the assignment $x^{q_i} = x_i$ is form preserving, by Theorem 4 all coefficients of any $m \times m$ subdeterminant of $E(x)$ are zero; hence $E(x)$ can not be C-regular. □

Theorem 6. The p-D polynomial matrix defined in (5) is invertible (regular) if $E(x) := E(x_1, x_2, \ldots, x_p)|_{\substack{x_1 = x^{r_1} \\ x_2 = x^{r_2} \\ \vdots \\ x_p = x^{r_p}}}$ where $r_1 = 1$, $r_2 = (n_1 n + 1), \ldots, r_p = (n_1 n + 1) \cdot (n_2 n + 1) \ldots (n_{p-1} n + 1)$ is regular.

Proof. Is similar to the proof of Theorem 5 when the assignment given in Theorem 2 is considered with $m_i = n_i n$ for $i = 1, 2, \ldots, p$. □

Thus, as results of Theorem 5 and 6, column regularity of the pencil in (2) and the invertibility of (5) can be investigated in terms of column regularity and invertibility of their related one dimensional polynomial matrices, respectively. Then using Theorem 7 given below, regularity of a one dimensional polynomial matrix can be converted to column regularity of a pair of matrices and the geometric condition given in (9) can also be applied to multidimensional polynomial matrices. In order to find this relation, the companion form of a polynomial matrix is needed.

Definition 6. Let $E(z)$ be an $n \times m$ polynomial matrix of degree d:

$$E(z) = E_0 + E_1 z + E_2 z^2 + \ldots + E_d z^d \tag{21}$$

where each E_i is real matrix. The associated companion form of $E(z)$ is $zE - A$ where E and A are

$$E := \begin{bmatrix} I & 0 & \ldots & 0 \\ 0 & I & \ldots & 0 \\ & & \ddots & I & 0 \\ 0 & 0 & \ldots & E_d \end{bmatrix} \quad A := \begin{bmatrix} 0 & +I & 0 & \ldots & 0 \\ 0 & 0 & +I & \ldots & 0 \\ & & & \ddots & +I \\ -E_0 & -E_1 & & & -E_{d-1} \end{bmatrix} \tag{22}$$

Theorem 7. $E(z)$ in (21) is column regular if and only if the pair (E, A) defined in (22) is column regular [2].

The following example clarifies the above procedure and the usage of Theorem 7.

Example 3. Let the pencil of 3-matrices be

$$E(x_1, x_2, x_3) = x_1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

With the assignment given in Theorem 5

$$E(x) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + x \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + x^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + x^3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

According to (22) E and A are

$$E = \begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & & & \ominus & \\ 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ & \ominus & & & 0 & 0 \\ & & & & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic subspace of (E, A) is found by the following iteration

$$\mathbf{V}^{(k+1)} = A^{-1}(E\mathbf{V}^{(k)}), \quad \mathbf{V}^{(0)} = \mathbb{R}^6$$

which yields $\mathbf{V}^*$ as $\mathbf{V}^* = \mathbf{V}^{(k+1)} = \mathbf{V}^{(k)}$ for the smallest k for which equality holds.

$$\mathbf{V}^{(1)} = \operatorname{Im} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{V}^{(2)} = \operatorname{Im} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{V}^{(3)} = \operatorname{Im} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad \mathbf{V}^{(4)} = \operatorname{Im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{V}^{(5)} = \mathbf{V}^{(6)} = \mathbf{V}^* = \operatorname{Im} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The characteristic kernel of (E, A) is

$$N = \ker E \cap \mathbf{V}^* = \operatorname{Im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cap \operatorname{Im} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0\,.$$

Hence, the pair (E, A) is C-regular and so is $E(x_1, x_2, x_3)$.

4. CONCLUSIONS

Uniqueness of solutions for multidimensional implicit linear systems can be investigated first by converting the multidimensional polynomial matrix in (5) to the one-dimensional polynomial matrix using the assignment given in Theorem 6 and then determining its associated companion form through Theorem 7 and finally finding the characteristic kernel of the related 2-pencil.

Using the assignment given in Theorem 5 and finding the characteristic kernel of the 2-pencil related to this assignment, the column regularity of n-pencil in (2) can be determined and hence it can be decided whether the related network parameters can be uniquely determined from the measurements or not.

Thus, the new approach which needs to determine some subspaces has been introduced to test CR of an n-pencil and a multidimensional polynomial matrix while the other methods existing in the literature [1, 3] use iterative techniques.

In [8], another method based on the multivariable polynomial interpolation to test regularity of n-pencil has been introduced. In this algebraic method specific data points are chosen to test CR of n-pencil.

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