SOME RECENT RESULTS IN SINGULAR 2-D SYSTEMS THEORY

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Solvability conditions for the general singular model of 2-D linear systems are established. The general response formula for the general singular model is derived. The concepts of local reachability and local controllability are extended for the singular model. Necessary and sufficient conditions for the local reachability and local controllability are established. The minimum energy control problem for the singular model is solved.

1. INTRODUCTION

The most popular models of two-dimensional (2-D) linear systems are the models presented by Attasi [1], Fornasini and Marchesini [2, 3], Roesser [20], and Kurek [17]. The Kurek model has been extended for 2-D linear systems with variable coefficients by Kaczorek in [4]. Singular models of 2-D linear systems have been introduced by Kaczorek [5–7]. In this paper some recent results for the general singular model of 2-D linear systems will be presented. Solvability conditions and the general response formula for the singular model will be established.

2. SINGULAR MODELS OF 2-D LINEAR SYSTEMS

Consider the general singular model of 2-D linear systems [8]

$$Ex_{i+1,j+1} =$$
$$= A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \quad (1)$$
$$y_{ij} = Cx_{ij} + Du_{ij} \quad (2)$$

where i, j are integer-valued vertical and horizontal coordinates, respectively, x_{ij} is the n-dimensional local semistate vector at (i, j), u_{ij} is the m-dimensional input vector, $y(i, j)$ is the p-dimensional output vector and A_k, B_k $(k = 0, 1, 2)$, C, D, E are real matrices of appropriate dimensions and E may be singular and nonsquare.

Boundary conditions for (1) are given by

$$x_{i0},\ i \geqq 0\,, \quad x_{0j},\ j \geqq 0\,. \tag{3}$$

From (1) and (2) for $B_1 = 0$, $B_2 = 0$ we obtain the first singular Fornasini-Marchesini model (FSF-MM) and for $A_0 = 0$, $B_0 = 0$ we obtain the second singular Fornasini-Marchesini model (SSF-MM). Similarly, for $-A_0 = A_1 A_2 = A_2 A_1$ the singular Attasi model (SAM).

The singular Roesser model (SRM) is given by

$$E\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} u_{ij} \tag{4}$$

$$y_{ij} = [C_{11}\ \ C_{22}]\begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + Du_{ij} \tag{5}$$

where x^h_{ij} is the n_1-dimensional horizontal semistate vector, x^v_{ij} is the n_2-dimensional vertical semistate vector, u_{ij} is the m-dimensional input vector, y_{ij} is the p-dimensional output vector. A_{ij}, B_{ij}, C_{ij}, $(i, j = 1, 2)$, D, E are real matrices of appropriate dimensions and E may be singular and nonsquare. In a similar way as for regular ($\det E \neq 0$) models it can be shown that FSF-MM is a particular case of SSF-MM and SAM is a particular case of SRM.

Defining $x^h_{ij} = Ex_{i,j+1} - A_1 x_{ij}$ and $x^v_{ij} = x_{ij}$ we may write (1) in the form

$$\begin{bmatrix} I, & -A_2 \\ 0 & E \end{bmatrix}\begin{bmatrix} x^h_{i+1,j} \\ x_{i,j+1} \end{bmatrix} = \begin{bmatrix} 0 & A_0 \\ I & A_1 \end{bmatrix}\begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + \begin{bmatrix} B_0 & B_1 & B_2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{ij} \\ u_{i+1,j} \\ u_{i,j+1} \end{bmatrix}$$

Therefore, SGM is a special case of SRM.

3. SOLVABILITY CONDITIONS

Consider the equation (1) in the rectangle

$$[0, N_1] \times [0, N_2] := \{(i, j)\colon 0 \leqq i \leqq N_1, 0 \leqq j \leqq N_2\}$$

Let us denote

$$\bar{x}_{N_1 N_2} := \{x_{00}, x_{01}, \ldots, x_{0N_2}, x_{10}, x_{11}, \ldots, x_{1N_2} x_{20}, \ldots, x_{N_1 N_2}\}$$

$$\bar{u}'_{N_1 N_2} := \bar{u}_{N_1 N_2} - u_{N_1 N_2}$$

where $\bar{u}_{N_1 N_2}$ is defined in a similar way as $\bar{x}_{N_1 N_2}$.

Boundary conditions are called admissible for (1) in $[0, N_1] \times [0, N_2]$ for a given input sequence $\bar{u}'_{N_1 N_2}$ iff there exists a sequence $\bar{x}_{N_1 N_2}$ satisfying (1) for $0 \leqq i \leqq N_1$ and $0 \leqq j \leqq N_2$. The sequence $\bar{x}_{N_1 N_2}$ will be called a solution to (1) in $[0, N_1] \times \times [0, N_2]$ for $\bar{u}'_{N_1 N_2}$. We shall say that (1) has a solution for $\bar{u}'_{N_1 N_2}$ and (3) if there exists $\bar{x}_{N_1 N_2}$ satisfying (1) for all N_1 and N_2.

Theorem 1. The equation (1) has a solution for any sequence $\{u_{ij}\}$ and any boundary conditions (3) iff

$$\operatorname{rank} E = \operatorname{rank}[E, A_0, A_1, A_2, B_0, B_1, B_2] \tag{6}$$

Proof. Let rank $E = r$. It is well known that there exists a nonsingular matrix P such that

$$PE = \begin{bmatrix} \bar{E} \\ 0 \end{bmatrix} \tag{7}$$

where $\bar{E}$ is $r \times n$ full row rank matrix. Premultiplying (1) by P and using (7) we obtain

$$\bar{E}x_{i+1,j+1} = \bar{A}_0 x_{ij} + \bar{A}_1 x_{i+1,j} + \bar{A}_2 x_{i,j+1} + \bar{B}_0 u_{ij} + \bar{B}_1 u_{i+1,j} + \\ + \bar{B}_2 u_{i,j+1} \tag{8a}$$

$$0 = \hat{A}_0 x_{ij} + \hat{A}_1 x_{i+1,j} + \hat{A}_2 x_{i,j+1} + \hat{B}_0 u_{ij} + \hat{B}_1 u_{i+1,j} + \hat{B}_2 u_{i,j+1} \tag{8b}$$

where $PA_k = \begin{bmatrix} \bar{A} \\ \hat{A}_k \end{bmatrix}$, $PB_k = \begin{bmatrix} \bar{B}_k \\ \hat{B}_k \end{bmatrix}$ for $k = 0, 1, 2$.

Note that $\hat{A}_k = 0$ and $\hat{B}_k = 0$ for $k = 0, 1, 2$ and (8b) is satisfied for any $\{u_{ij}\}$ and any (3) iff (6) holds.

Solving (8a) we get

$$x_{i+1,j+1} = A'_0 x_{ij} + A'_1 x_{i+1,j} + A'_2 x_{i,j+1} + B'_0 u_{ij} + B'_1 u_{i+1,j} + B'_2 u_{i,j+1} \tag{8c}$$

where

$$A'_k = \bar{E}^{\mathrm{T}}[\bar{E}\ \bar{E}^{\mathrm{T}}]^{-1}\,\bar{A}_k\,, \quad B'_k = \bar{E}^{\mathrm{T}}[\bar{E}\ \bar{E}^{\mathrm{T}}]^{-1}\,\bar{B}_k \quad \text{for} \quad k = 0, 1, 2\,.$$

The equation (8c) has a solution for any $\{u_{ij}\}$ and any (3). □

It can be shown [11] that (1) has the unique solution for any $\{u_{ij}\}$ and admissible boundary conditions (3) if

$$\operatorname{rank} G(z_1, z_2) = n \quad \text{for some} \quad z_1, z_2 \in \mathbb{C} \tag{9}$$

where $G = G(z_1, z_2) := [Ez_1 z_2 - A_0 - A_1 z_1 - A_2 z_2]$

and $\mathbb{C}$ is the field of complex members.

4. GENERAL RESPONSE FORMULA

Following [8] we may write the expansion

$$G^{-1} = \sum_{p=-n_1}^{\infty} \sum_{q=-n_2}^{\infty} T_{pq} z_1^{-p} z_2^{-q} \tag{10}$$

where T_{pq} are real matrices defined by

$$ET_{pq} = \begin{cases} A_0 T_{00} + A_1 T_{10} + A_2 T_{01} + I & \text{for} \quad p = q = 1 \\ A_0 T_{p-1,q-1} + A_1 T_{p,q-1} + A_2 T_{p-1,q} & \text{for} \quad p \neq 1 \quad \text{and/or} \quad q \neq 1 \\ 0 \quad \text{for} \quad p < -n_1 \quad \text{and/or} \quad q < -n_2 \end{cases} \tag{11}$$

I is the identity matrix.

The pair of positive integers (n_1, n_2) is called the index of the model. In general n_1 and n_2 are not finite. It is easy to show that n_1 and n_2 are finite if the coefficient $d_{m_1m_2}$ of the polynomial

$$\det G = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} d_{ij} z_1^i z_2^j$$

is not zero.

Theorem 2. If (9) holds then the unique solution to (1) with admissible boundary conditions (3) is given by

$$\begin{aligned} x(i, j) &= \sum_{p=1}^{i+n_1} \sum_{q=1}^{j+n_2} T_{i-p,j-q} B_0 u_{pq} + \sum_{p=1}^{i+n_1+1} \sum_{q=1}^{j+n_2} T_{i-p+1,j-q} B_1 u_{pq} + \\ &+ \sum_{p=1}^{i+n_1} \sum_{q=1}^{j+n_2+1} T_{i-p,j-q+1} B_2 u_{pq} + \sum_{p=1}^{i+n_1+1} T_{i-p+1,j} [A_1\ B_1] \begin{bmatrix} x_{p0} \\ u_{p0} \end{bmatrix} + \\ &+ \sum_{p=1}^{i+n_1} T_{i-p,j} [A_0\ B_0] \begin{bmatrix} x_{p0} \\ u_{p0} \end{bmatrix} + \sum_{q=1}^{j+n_2+1} T_{i,j-q+1} [A_2\ B_2] \begin{bmatrix} x_{0q} \\ u_{0q} \end{bmatrix} + \\ &+ \sum_{q=1}^{j+n_2} T_{i,j-q} [A_0\ B_0] \begin{bmatrix} x_{0q} \\ u_{0q} \end{bmatrix} + T_{ij} [A_0\ B_0] \begin{bmatrix} x_{00} \\ u_{00} \end{bmatrix} \end{aligned} \tag{12}$$

The proof is given in [8]. □

The desidered response formula for GSM can be obtained by substitution of (12) into (2). T_{pq} can be found from the series expansion (10) or from (11). The general response formula given by Kurek [17] is a particular case of (12). Note that the set of admissible boundary conditions for GSM is specified by (12) for $i = 0$ and $j = 0$.

Now let us assume that

$$\det G = 0 \quad \text{for all} \quad z_1, z_2 \in \mathbb{C} \tag{13}$$

and rank $G = r < n$. In this case there is a nonsingular matrix M of row elementary operations such that

$$MG = \begin{bmatrix} G_1 \\ 0 \end{bmatrix} \tag{14}$$

where G_1 is $r \times n$ matrix of full row rank for some $z_1, z_2 \in C$. Premultiplying the equation [8]

$$\begin{aligned} GX(z_1, z_2) &= (B_0 + B_1 z_1 + B_2 z_2) U(z_1, z_2) - B_1 z_1 U(0, z_2) + \\ &- B_2 z_2 U(z_1, 0) - A_1 z_1 X(0, z_2) - A_2 z_2 X(z_1, 0) + E z_1 z_2 (X(z_1, 0) + \\ &+ X(0, z_2) - x(0, 0)) \end{aligned}$$

by M and using (14) we obtain

$$\begin{aligned} G_1 X(z_1, z_2) &= (\bar{B}_0 + \bar{B}_1 z_1 + \bar{B}_2 z_2) U(z_1, z_2) - \bar{B}_1 z_1 U(0, z_2) + \\ &- \bar{B}_2 z_2 U(z_1, 0) - \bar{A} z_1 X(0, z_2) - \bar{A}_2 z_2 X(z_1, 0) + \bar{E} z_1 z_2 (X(z_1, 0) + \\ &+ X(0, z_2) - x(0, 0)) \end{aligned} \tag{15a}$$

and

$$0 = (\hat{B}_0 + \hat{B}_1 z_1 + \hat{B}_2 z_2) U(z_1, z_2) - \hat{B}_1 z_1 U(0, z_2) - \hat{B}_2 z_2 U(z_1, 0) + \\ - \hat{A}_1 z_1 X(0, z_2) - \hat{A}_2 z_2 X(z_1, 0) + \hat{E} z_1 z_2 (X(z_1, 0) + X(0, z_2) + -x(0,0)) \tag{15b}$$

where

$$MA_k = \begin{bmatrix} \bar{A}_k \\ \hat{A}_k \end{bmatrix}, \quad MB_k = \begin{bmatrix} \bar{B}_k \\ \hat{B}_k \end{bmatrix}, \quad k = 0, 1, 2, \quad ME = \begin{bmatrix} \bar{E} \\ \hat{E} \end{bmatrix}$$

It is assumed that (15b) is consistent and it is satisfied by the admissible boundary conditions for a given sequence $\{u_{ij}\}$. In a similar way as in [8] solving (15a) we obtain

$$X(z_1, z_2) = G_1^+ [(\bar{B}_0 + \bar{B}_1 z_1 + \bar{B}_2 z_2) U(z_1, z_2) - \bar{B}_1 z_1 U(0, z_2) + \\ - \bar{B}_2 z_2 U(z_1, 0) - \bar{A}_1 z_1 X(0, z_2) - \bar{A}_2 z_2 X(z_1, 0) + E z_1 z_2 (X(z_1, 0) + \\ + X(0, z_2) - x(0, 0))]$$

where

$$G_1^+ = G_1^{\mathrm{T}} [G_1 \; G_1^{\mathrm{T}}]^{-1}$$

Note that G_1^+ plays the same role as G^{-1} in the regular case $\det G \neq 0$. Let

$$G_1^+ = \sum_{p=-\bar{n}_1}^{\infty} \sum_{q=-\bar{n}_2}^{\infty} \bar{T}_{pq} z_1^{-p} z_2^{-q}$$

where $\bar{n}_1, \bar{n}_2$ are positive integers.

Substituting T_{pq}, A_0, A_1, A_2, B_0, B_1, B_2 and E by $\bar{T}_{pq}$, $\bar{A}_0$, $\bar{A}_1$, $\bar{A}_2$, $\bar{B}_1$, $\bar{B}_2$, $\bar{E}$, respectively we may use also (12) for finding a solution (if it exists) to (1) when $\det G = 0$.

5. CAYLEY-HAMILTON THEOREM

Let

$$d(z_1, z_2) = \det G = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} d_{ij} z_1^i z_2^j \tag{16a}$$

and

$$\operatorname{Adj} G = \sum_{i=0}^{m_1'} \sum_{j=0}^{m_2'} H_{ij} z_1^i z_2^j \quad (m_1' \leqq n - 1, m_2' \leqq n - 1) \tag{16b}$$

Theorem 3. The matrices T_{pq} for GSM satisfy the equation

$$\sum_{i=0}^{m_1} \sum_{j=0}^{m_2} d_{ij} T_{i-p, j-q} = 0 \quad \text{for} \begin{cases} p < 0 & \text{and} \quad m_1' < p < m_1 + n_1 \\ p < 0 & \text{and} \quad m_2' < q < m_2 + n_2 \end{cases} \tag{17}$$

Proof. From (10) and (16) we have

$$\sum_{i=0}^{m_1'} \sum_{j=0}^{m_2'} H_{ij} z_1^i z_2^j = (\sum_{i=0}^{m_1} \sum_{j=0}^{m_2} d_{ij} z_1^i z_2^j)(\sum_{p=n_1}^{\infty} \sum_{q=n_2}^{\infty} T_{pq} z_1^{-p} z_2^{-q})$$

Equating the coefficient matrices at the same powers of z_1 and z_2 we obtain (17). □

6. LOCAL REACHABILITY AND LOCAL CONTROLLABILITY

The following partial ordering of 2-tuple integers will be used

$$(h, k) \leqq (i, j) \quad \text{iff} \quad h \leqq i \quad \text{and} \quad k \leqq j$$

$$(h, k) = (i, j) \quad \text{iff} \quad h = i \quad \text{and} \quad k = j$$

$$(h, k) < (i, j) \quad \text{iff} \quad (h, k) \leqq (i, j) \quad \text{and} \quad (h, k) \neq (i, j) .$$

For $(h, k) < (p, q)$ we define the rectangle $[(h, k), (p, q)]$ as follows $[(h, k), (p, q)] :=$ $:= \{(h, k) \leqq (i, j) \leqq (p, q)\}$

Definition 1. GSM is called locally reachable in the rectangle $[(0, 0), (h, k)]$ if for admissible boundary conditions (3) and every vector $x_f \in \mathbb{R}^n$ there exists a sequence of input vectors u_{ij} for $(0, 0) \leqq (i, j) \leqq (h + n_1 + 1, k + n_2 + 1)$ such that $x_{hk} = x_f$.

Theorem 4. GSM is locally reachable in the rectangle $[(0, 0), (h, k)]$ iff

$$\operatorname{rank} R_{hk} = n \tag{18}$$

where

$$R_{hk} = [M_0, M_1^1, \ldots, M_{\bar{h}}^1, M_1^2, \ldots, M_{\bar{k}}^2, M_{11}, \ldots, M_{1\bar{k}}, M_{21}, \ldots, M_{\bar{h}\bar{k}}]$$

$$M_0 = T_{hk}B_0, M_p^1 = T_{h-p,k}B_0 + T_{h-p+1,k}B_1 ,$$

$$p = 1, \ldots, \bar{h}, \bar{h} = h + n_1 + 1$$

$$M_q^2 = T_{h,k-q}B_0 + T_{h,k-q+1}B_2 , \quad q = 1, \ldots, \bar{k}, \bar{k} = k + n_2 + 1$$

$$M_{pq} = T_{h-p,k-q}B_0 + T_{h-p+1,k-q}B_1 + T_{h-p,k-q+1}B_2$$

The proof is given in [8]. □

Definition 2. GSM is called locally controllable in the rectangle $[(0, 0), (h, k)]$ if for admissible boundary conditions (3) there exists a sequence of input vectors u_{ij} for $(0, 0) \leqq (i, j) < (\bar{h}, \bar{k})$ such that $x_{hk} = 0$.

A different definition of the local controllability for regular 2-D systems was given by Šebek, Bisiacco and Fornasini [21].

Theorem 5. GSM is locally controllable in the rectangle $[(0, 0) (h, k)]$ iff

$$\operatorname{rank} R_{hk} = \operatorname{rank} [R_{hk}, P_{hk}] \tag{19}$$

where

$$P_{hk} = [P_0, P_{11}, \ldots, P_{1\bar{h}}, P_{21}, \ldots, P_{2\bar{k}}]$$

$$P_0 = T_{hk}A_0 , \quad P_{1p} = T_{h-p,k}A_0 + T_{h-p+1,k}A_1 , \quad p = 1, \ldots, \bar{h}$$

$$P_{2q} = T_{h,k-q}A_0 + T_{h,k-q+1}A_2 , \quad q = 1, \ldots, \bar{k}$$

The proof is given in [8]. □

From (18) and (19) it follows that if GSM is locally reachable it is also locally controllable but if GSM is not locally reachable it may be locally controllable.

7. LOCAL OBSERVABILITY

Following [20] we may define the local observability of GSM as follows.

Definition 3. GSM is called locally observable in the rectangle $[(0,0),(h,k)]$ if there is no local initial semistate vector $x_{00} \neq 0$ such that for zero input vectors u_{ij}, $(0,0) \leqq (i,j) < (\bar{h},\bar{k})$ and zero boundary conditions: $x_{i0} = 0$, $0 < i \leqq \bar{h}$, $x_{0j} = 0$, $0 < j \leqq \bar{k}$, the output is also zero y_{ij} for $(0,0) \leqq (i,j) \leqq (h,k)$.

Theorem 6. GSM is locally observable in the rectangle $[(0,0),(h,k)]$ iff

$$\operatorname{rank} Q_{hk} = n \tag{20}$$

where

$$Q_{hk} = [q_{00}^{\mathrm{T}}, q_{10}^{\mathrm{T}}, \ldots, q_{h0}^{\mathrm{T}}, q_{01}^{\mathrm{T}}, \ldots, q_{0k}^{\mathrm{T}}, \ldots, q_{11}^{\mathrm{T}}, \ldots, q_{hk}^{\mathrm{T}}]\, T$$

$$q_{ij} = CT_{ij}A_0\,, \quad i = 0\,,\ 1, \ldots, h\,; \ \ j = 0, 1, \ldots, k$$

Proof. From (12) and (2) for $u_{ij} = 0$, $(0,0) \leqq (i,j) \leqq (\bar{h},\bar{k})$ and zero boundary conditions we have

$$y_{ij} = CT_{ij}A_0x_{00} = q_{ij}x_{00}$$

Taking into account that $y_{ij} = 0$ for $(0,0) \leqq (i,j) \leqq (h,k)$ we obtain

$$Q_{hk}x_{00} = 0 \tag{21}$$

From (21) it follows that GSM is locally observable in the rectangle $[(0,0),(h,k)]$ iff (20) holds. □

Following Kurek [18] necessary and sufficient conditions for strong observability and strong reconstructibility of GSM can be established. In [12] necessary and sufficient conditions for global and causal observability and causual reconstructibility of SSF-MM have been given. With slight modifications the conditions can be extended for GSM.

8. MINIMUM ENERGY CONTROL

Consider GSM and the performance index

$$I(u) = \sum_{i=0}^{\bar{h}} \sum_{j=0}^{\bar{k}} u_{ij}^{\mathrm{T}} Q u_{ij} \tag{22}$$

where Q is an $m \times m$ symmetric and positive definite matrix. The minimum energy control problem of GSM can be stated as follows: given A_k, B_k for $k = 0, 1, 2$, admissible boundary conditions (3), Q, and h, k, find a sequence of input vectors u_{ij} for $0 \leqq i \leqq \bar{h}$, $0 \leqq j \leqq \bar{k}$ which transfers GSM from x_{00} to x_1, $x_{h,k} = x_f$ and minimizes (22).

To solve the problem we define the matrix

$$W_{hk} := \sum_{i=0}^{\bar{h}} \sum_{j=0}^{\bar{k}} M_{h-i,k-j} Q^{-1} M_{h-i,k-j}^{\mathrm{T}} =: R_{hk} Q_d R_{hk}^{\mathrm{T}} \tag{23}$$

where

$$Q_d = \text{diag}\,[Q^{-1}, \ldots, Q^{-1}]$$

$$M_{i-p,j-q} = \begin{cases} T_{ij}B_0\,, & p = q = 0 \\ T_{i-p,j}B_0 + T_{i-p+1,j}B_1\,, & p > 0\,, \quad q = 0 \\ T_{i,j-q}B_0 + T_{i,j-q+1}B_2\,, & p = 0\,, \quad q > 0 \end{cases}$$

$$M_{i-p,j-q} = T_{i-p,j-q}B_0 + T_{i-p+1,j-q}B_1 + T_{i-p,j-q+1}B_2$$

$$\text{for} \quad p > 0\,, \quad q > 0$$

It is easy to show that W_{hk} is nonsingular (positive definite) if GSM is locally reachable in the rectangle $[(0, 0), (h, k)]$.

Let us define

$$\hat{u}_{ij} := Q^{-1}M^{\mathrm{T}}_{h-1,k-j}W_{hk}^{-1}(x_f - x_0)\,, \quad 0 \leqq i \leqq \bar{h}\,, \quad 0 \leqq j \leqq \bar{k} \tag{24}$$

where

$$x_0 = T_{hk}A_0x_{00} + \sum_{p=1}^{\bar{h}}(T_{h-p,k}A_0 + T_{h-p+1,k}A_1)\,x_{p0} + \sum_{q=1}^{\bar{k}}(T_{h,k-q}A_0 + T_{h,k-q+1}A_2)\,x_{0q}$$

Theorem 7. Let assume that GSM is locally reachable in the rectangle $([0, 0, (h, k)]$. If $\bar{u}_{ij}$ is any sequence of input vectors for $0 \leqq i \leqq \bar{h}$, $0 \leqq j \leqq \bar{k}$ which transfers GSM from x_{00} to x_f, then the sequence (24) accomplishes the same task and

$$I(\hat{u}) \leqq I(\bar{u})$$

The minimum value of (22) is given by

$$I(\hat{u}) = (x_f - x_0)\,W_{hk}^{\mathrm{T}-1}(x_f - x_0)$$

The proof is given in [8]. □

Sufficient conditions for the existence of a solution to the linear-quadratic optimal regulator problem for GSM with variable coefficients have been established in [15].

9. CONCLUDING REMARKS

The general response formula for GSM of 2-D linear systems has been presented. The well-known Cayley-Hamilton theorem has been extended for GSM. Necessary and sufficient conditions for the local reachibility, the local controllability and the local observability of GSM have been established. It has been shown that the local reachability of GSM implies its local controllability. The inverse theorem is not valid in general case. The minimum energy control for GSM has been solved. The general response formula can be also extended for GSM with variable coefficients [9]. In [13] sufficient conditions for the existence of full order asymptotic and deadbeat observers for SSF-MM have been established and design procedure for finding observer matrices have been given. With slight modifications the conditions and design procedure can be extended for GSM. The Luenberger's shuffle algorithm has been extended for GSM in [14]. This algorithm can be used for decomposition of GSM into its dynamic and static parts. A method for eigenvalue assignment by state feedback of SRM has been presented in [16].

The eigenvalue assignment problem by state or output feedback of GSM is one of the nontrivial open problems for singular 2-D linear systems.

(Received October 8, 1990.)

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