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A SIMPLE DEFINITION OF HIDDEN MODES, POLES AND ZEROS

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Simple definitions of hidden modes, poles and zeros of multivariable constant linear systems are given by means of elementary module theory.

INTRODUCTION

This paper should be read in conjunction with [8], where basic structural properties, such as controllability, observability and state-variable representation, of constant or time-varying generalized linear systems were examined in the context of module theory. Our aim here is to illustrate this approach by giving a simple and transparent definition of hidden modes, poles and zeros of constant multivariable linear systems. The algebraic interpretation of poles and zeros is of course easy via transfer matrices (see, e.g., Kailath [10]). The need has nevertheless been felt by several authors (cf. Francis and Wonham [9], MacFarlane and Karkanias [12], Wonham [15], Wyman and Sain [16], Conte and Perdon [3]) for other types of developments. The first attempt with multivariable hidden modes is due to Rosenbrock [13] (see also Blomberg and Ylinen [1] and Callier and Desoer [2]). The fact that this field of research is still active is demonstrated by a recent and well documented survey due to Schrader and Sain [14].

Elementary module theory permits to define a system without reference to a peculiar choice of equations [8]. It is therefore possible to circumvent some of the difficulties which were previously encountered. Hidden modes, poles and zeros are characterized as the eigenvalues of the derivation d/dt acting on appropriate finitely generated torsion modules.

Preliminary versions were presented in [6, 7].

1. NOTATION AND TERMINOLOGY

1.1. Denote by $k[\mathrm{d}/\mathrm{d}t]$ the set of linear differential operators of the form

$$\sum_{\text{finite}} a_\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$$

with coefficients in a commutative field k of constants. This means that, for any $a \in k$, $\mathrm{d}a/\mathrm{d}t = \dot{a} = 0$.

1.2. Denote by $[w]$ the $k[\mathrm{d}/\mathrm{d}t]$-module spanned by a set $w = \{w_i \mid i \in I\}$.

1.3. A dynamics $\mathscr{D}$ is a finitely generated $k[\mathrm{d}/\mathrm{d}t]$-module, which contains a finite, but possibly empty set $u = (u_1, \ldots, u_m)$ of quantities which play the role of input variables, and such that the quotient module $\mathscr{D}/[u]$ is torsion. The input is *independent*, i.e., $[u]$ is free. Output variables can be chosen as a finite set $y = (y_1, \ldots, y_p)$ of a quantities in $\mathscr{D}$.

2. HIDDEN MODES OR DECOUPLING ZEROS

2.1. Assume that the dynamics $\mathscr{D}$ is uncontrollable, i.e., that $\mathscr{D}$ is not a free $k[\mathrm{d}/\mathrm{d}t]$-module [8]. Then $\mathscr{D}$ contains a unique maximal torsion submodule $\mathscr{T}$, which is finitely generated and therefore finite-dimensional as a k-vector-space. Denote by $\tau\colon \mathscr{T} \to \mathscr{T}$ the k-linear mapping induced by the derivative $\mathrm{d}/\mathrm{d}t$. The *input-decoupling zeros* are the eigenvalues of τ over an algebraic closure $\bar{k}$ of k.

2.2. *Remark.* The dynamics $\mathscr{D}$, with a possible output y, possesses the following state-variable representation [8]:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + Bu$$

$$y = Cx + \sum_{\text{finite}} D_\alpha \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \tag{1}$$

where x, u, y are column vectors with n, m, p components respectively. With respect to the Kalman decomposition into controllable and uncontrollable subspaces, the matrix A can be written

$$\begin{matrix}\text{uncontrollable} \\ \text{controllable}\end{matrix} \left(\begin{array}{c|c} A_1 & 0 \\ \hline * & A_3 \end{array}\right)$$

The input-decoupling zeros are the eigenvalues of A_1.

2.3. Assume that the dynamics $\mathscr{D}$ is unobservable with respect to the output y, i.e., that $[u, y]$ is strictly included in $\mathscr{D}$ [8]. The quotient module $\mathscr{S} = \mathscr{D}/[u, y]$ is torsion and finitely generated. Denote by $\sigma\colon \mathscr{S} \to \mathscr{S}$ the k-linear mapping induced by $\mathrm{d}/\mathrm{d}t$. The *output-decoupling zeros* are the eigenvalues of σ over $\bar{k}$.

2.4. *Remark.* With respect to the Kalman decomposition of (1) into observable and unobservable subspaces, A can be written

$$\begin{array}{r} \text{observable} \\ \text{unobservable} \end{array} \left(\begin{array}{c|c} * & 0 \\ \hline * & A_2 \end{array}\right)$$

The input-decoupling zeros are the eigenvalues of A_2.

2.5. The set of *hidden modes* or *decoupling zeros* is the union of the sets of input-decoupling and output-decoupling zeros.

2.6. *Example.* The system $\dot{y} = \dot{u}$, where $m = p = 1$, has the minimal realization

$$\dot{x} = 0$$
$$y = x + u \tag{2}$$

which is uncontrollable. There is only one hidden mode, zero, which is an input-decoupling zero.

3. POLES AND ZEROS

3.1. We use the notations of Section 2.1. The quotient module $\mathscr{F} = \mathscr{D}/\mathscr{T}$ is free. Since the intersection of $[u]$, which is assumed to be free, and $\mathscr{T}$ is trivial, i.e., $[u] \cap T = 0$, there is a canonical injection of $[u]$ into $\mathscr{F}$. We therefore might consider with a slight abuse of notations the components of u as elements of $\mathscr{F}$. The quotient $\Delta = \mathscr{F}/[u]$ is a finitely generated torsion module. Denote by $\delta: \Delta \to \Delta$ the k-linear mapping induced by $\mathrm{d}/\mathrm{d}t$. The *poles* of the dynamics $\mathscr{D}$ are the eigenvalues of δ over $\bar{k}$.

3.2. *Remark.* The poles are the eigenvalues of the matrix A_3 defined in 2.2.

3.3. When working with a dynamics $\mathscr{D}$ with output y, we must "kill" the effects of uncontrollability and unobservability, which are taken into account by the hidden modes. Let $\mathscr{T}_1$ be the maximal torsion submodule of $[u, y]$. The quotient module $\mathscr{F}' = [u, y]/\mathscr{T}_1$ is free. As in 3.1, we might consider the components of u as elements of $\mathscr{F}'$. The quotient $\Delta' = \mathscr{F}'/[u]$ is a finitely generated torsion module, on which $\mathrm{d}/\mathrm{d}t$ induces a k-linear mapping $\delta': \Delta' \to \Delta'$. The poles of the dynamics $\mathscr{D}$ with output y are the eigenvalues of δ' over $\bar{k}$.

3.4. *Remark.* With respect to the Kalman decomposition, take in (1) the maximal controllable and observable subspace. The matrix A thus reads

$$\begin{array}{r} \text{controllable and observable } \{ \\ \text{uncontrollable and/or unobservable } \{ \end{array} \left(\begin{array}{c|c} A_4 & * \\ \hline & * \end{array}\right)$$

The poles are the eigenvalues of A_4.

3.5. *Example* (continued). From (2) we see that the system $\dot{y} = \dot{u}$ has no poles. This is confirmed by the fact that its transfer function is 1.

3.6. Denote by $\mathscr{S}$, $[y, \mathscr{T}_1] \subseteq \mathscr{S} \subseteq [u, y]$, the maximal module such that the quotient $\mathfrak{Z} = \mathscr{S}/[y, \mathscr{T}_1]$ is torsion (and, of course, finitely generated). The system with input u and output y is *left-invertible* [4, 5] if, and only, if, the two modules $\mathscr{I}$ and $[u, y]$ coincide[1]. The *inverse dynamics* is, by definition, $\mathfrak{I}$. Denote by ε: $\mathfrak{I} \to \mathfrak{I}$ the k-linear mapping induced by $\mathrm{d}/\mathrm{d}t$. The *zeros* are the eigenvalues of ε over $\bar{k}$.

3.7. *Remark.* When the system is left-invertible, the zeros are the poles of the left-inverse system.

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[1] In [4,5] we employed the more or less equivalent language of *differential k-vector spaces*. It should be clear to the reader that the *differential dimension* of such a vector space corresponds to the *rank* of the module [11]. The differential dimension is, therefore, zero if, and only if, the module is torsion.