

DESIGN OF OBSERVER BASED COMPENSATORS: THE POLYNOMIAL APPROACH

PETER HIPPE

This paper presents the frequency domain design of observer based compensators related to arbitrary observer orders for state reconstruction in direct equivalence to the well known time domain approach. The parameterization of the state feedback and of the state observer problems are possible without recurrence to the time domain representations of the system, however, the equivalent time domain solutions can easily be computed at every design step. An additional result are the doubly coprime factorizations of a system transfer matrix. Thus the results by Nett et al. [13] are generalized to arbitrary observer orders and in addition, the computation of the stable fractional representations becomes possible directly in the frequency domain. A simple example demonstrates the new design.

1. INTRODUCTION

Given a completely controllable and observable system of n th order with p inputs and m outputs, it is well known that by static state feedback $u = -Kx$ the dynamics of the controlled system can be assigned arbitrarily. The usually not completely measurable system state can be reconstructed with the aid of an observer of $(n - \kappa)$ th order with $0 \leq \kappa \leq m$. State feedback plus the observer form a dynamic compensator of order $n - \kappa$. Though this has long been known using the time domain approach, a direct frequency domain design of such compensators was not feasible so far. The existing frequency domain design methods for reduced order compensators (see e.g. [16]) did not give a solution in direct equivalence to the above described time domain approach. In a previous paper Hippe [5] presented the design of the full order compensator directly in the frequency domain. It was shown that the state feedback control for a linear time invariant system of n th order with p inputs and m outputs can either be parameterized by the $p \times n$ state feedback matrix K in the time domain or by the $p \times p$ polynomial matrix $\tilde{D}(s)$ in the frequency domain [5]. Equally the full order observer is parameterized by the $n \times m$ output error feedback matrix L in the time domain or by the $m \times m$ polynomial matrix $\tilde{D}(s)$ in the frequency domain [5].

The design of reduced order observers in the frequency domain remained an open problem, since here any observer design draws on the full order model of the system. Relations with equivalent time domain solutions therefore require a time domain formulation of reduced order observers which also bases on a full order model. This problem was solved by Hippe [8] with the aid of a non-minimal representation of the reduced order observer in the time domain. As a consequence of these results any observer of order $n - \kappa$ with $0 \leq \kappa \leq m$ can be parameterized directly in the frequency domain by an appropriate choice of an $m \times m$ polynomial matrix $\hat{D}(s)$.

The actual choice of $\tilde{D}(s)$ or $\hat{D}(s)$ can either be accomplished by pole placement (with remaining degrees of freedom) or by solving the linear optimal control and the optimal linear estimation schemes. It is well known that both problems can directly be solved in the frequency domain by spectral factorization (cf. [11]). So far, there only existed solutions for the full order case (Kučera [11]) and for the completely reduced order case (Shaked and Sokora [15] and Bekir [2]). The general case of the reduced order linear optimal estimator of order $n - \kappa$ was investigated by Hippe [7]. The polynomial matrices $\tilde{D}(s)$ and $\hat{D}(s)$ resulting from the spectral factorizations parameterize the linear optimal control problem and the optimal linear observer (Kalman filter), respectively, provided they have been adjusted such that their highest degree coefficient matrices $\Gamma_c[\tilde{D}(s)]$ and $\Gamma_r[\hat{D}(s)]$ have appropriate forms.

Once the parameterizing matrices $\tilde{D}(s)$ and $\hat{D}(s)$ for the linear state feedback and for the linear state estimation problems are specified, one can attack the design of the corresponding dynamic compensator.

Anderson and Kučera [1] presented a "relatively simple" procedure to find the compensator matrix fraction descriptions in the full order case starting from the optimal polynomial matrices $\tilde{D}(s)$ and $\hat{D}(s)$. Hippe [5] presented a further simplified solution method for the full order case and in Hippe [6], the compensator matrix fraction descriptions for the reduced (minimal) order case were derived.

Here the general case $n - \kappa$ with $0 \leq \kappa \leq m$ is presented.

After a short summary of the known reduced order observer results in the time domain and of the parameterization of the state feedback control in the frequency domain, the relation between the time and the frequency domain representations of reduced order observers is presented. This makes it possible to define the doubly coprime fractional representations (DCF) of the system transfer matrix as introduced by Nett et al. [13] related to reduced order observers.

Investigating the two dual representations of the closed loop, namely the *controller structure* and the *observer structure* it is shown how the parameterizing matrices for the state feedback control and for the observer are related with these representations.

Finally, the direct computation of the compensator DCFs in the frequency domain is presented. From the DCFs the coprime compensator matrix fraction descriptions

(MFD) can be computed using standard software. These results contain the full order design (Hippe [5]) and the reduced order design (Hippe [6]) as special cases. A simple example is used to demonstrate the design procedure.

2. NOMENCLATURE

| | |
|----------------------|--|
| MFD | matrix fraction description |
| DCF | doubly coprime factorization |
| $\Gamma_c[\cdot]$ | highest column degree coefficient matrix |
| $\Gamma_r[\cdot]$ | highest row degree coefficient matrix |
| $\delta_{cj}[\cdot]$ | j th column degree |
| $\delta_{rj}[\cdot]$ | j th row degree |
| $II[\cdot]$ | polynomial part |
| $SP[\cdot]$ | strictly proper part |

3. PRELIMINARIES

We consider completely controllable and observable linear, time invariant systems of n th order with p inputs and m outputs, described by their state equations

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\tag{3.1}$$

in the time domain. The m -dimensional output vector y with rank $C = m$ is subdivided into

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x\tag{3.2}$$

with y_2 containing the $\kappa \leq m$ outputs which are directly used for the reconstruction of x . The observer of order $n - \kappa$ can be represented by its state equations (Luenberger [12])

$$\dot{z} = Fz + [H_1 \ H_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + TBu\tag{3.3}$$

with

$$z = Tx\tag{3.4}$$

in steady state if

$$TA - FT = [H_1 \ H_2] \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}\tag{3.5}$$

holds. If C_2 and T are linearly independent the observed state vector \hat{x} is given by

$$\hat{x} = \begin{bmatrix} C_2 \\ T \end{bmatrix}^{-1} \begin{bmatrix} y_2 \\ z \end{bmatrix} = [\Psi_2 \ \Theta] \begin{bmatrix} y_2 \\ z \end{bmatrix}.\tag{3.6}$$

For the following it is of importance that

$$\begin{bmatrix} C_2 \\ T \end{bmatrix} [\Psi_2 \ \Theta] = \begin{bmatrix} I_x & 0 \\ 0 & I_{n-x} \end{bmatrix} \quad (3.7)$$

and

$$\Psi_2 C_2 + \Theta T = I_n \quad (3.8)$$

are implied by (3.6). Introducing

$$L_1 = \Theta H_1 \quad (3.9)$$

the observer (3.3) is equally represented by

$$\dot{z} = T(A - L_1 C_1) \Theta z + [TL_1 \mid T(A - L_1 C_1) \Psi_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + TBu \quad (3.10)$$

Because of (3.9) the relation

$$C_2 L_1 = 0 \quad (3.11)$$

results which will also be of importance in the sequel.

Substituting the observed state (3.6) into the state feedback law

$$u = -Kx, \quad (3.12)$$

the compensator is described by

$$u(s) = -F_c(s) y(s) \quad (3.13)$$

with

$$F_c(s) = K\Theta[sI - T(A - L_1 C_1 - BK) \Theta]^{-1} \cdot [TL_1 \mid T(A - L_1 C_1 - BK) \Psi_2] + [0 \mid K\Psi_2]. \quad (3.14)$$

In the sequel we derive time domain equivalent frequency domain results for the closed loop. In order to have uniquely defined relations in every direction, we assume that also the compensator is completely controllable and observable. Therefore it can be represented in a left coprime matrix fraction description (MFD)

$$F_c(s) = D_c^{-1}(s) N_c(s) \quad (3.15)$$

or in a right coprime MFD

$$F_c(s) = \bar{N}_c(s) \bar{D}_c^{-1}(s) \quad (3.16)$$

with $\det D_c(s) = \det \bar{D}_c(s) = \det [sI - T(A - L_1 C_1 - BK) \Theta]$. Likewise the plant transfer matrix

$$F(s) = C(sI - A)^{-1} B \quad (3.17)$$

will be represented in a right coprime MFD

$$F(s) = N(s) D^{-1}(s) \quad (3.18)$$

or in a left coprime MFD

$$F(s) = \bar{D}^{-1}(s) \bar{N}(s) \quad (3.19)$$

in the frequency domain. The denominator matrix $D(s)$ is supposed to be column proper and the denominator matrix $\bar{D}(s)$ to be row proper (Wolovich [17]) unless stated otherwise. The characteristic polynomial of the plant is given by $\det D(s) = \det \bar{D}(s) = \det (sI - A)$.

As was shown e.g. in [5], the dynamics of the state feedback loop without observer are parameterized in the frequency domain by the $p \times p$ polynomial matrix $\bar{D}(s)$ with

$$\Gamma_c[\bar{D}(s)] = \Gamma_c[D(s)] \quad (3.20)$$

and

$$\delta_{c_j}[\bar{D}(s)] = \delta_{c_j}[D(s)], \quad j = 1, 2, \dots, p \quad (3.21)$$

where $\Gamma_c[\cdot]$ is the highest column degree coefficient matrix and $\delta_{c_j}[\cdot]$ is the j th column degree. As a consequence of (3.20) and (3.21) the polynomial matrix $\bar{D}(s)$ contains exactly the same number of free parameters as the state feedback matrix K , namely pn .

Example 1. Consider a system of fifth order with two inputs and two outputs with

$$F(s) = \frac{1}{s^5 + 11s^4 + 46s^3 + 93s^2 + 91s + 34} \cdot \begin{bmatrix} s^3 + 5s^2 + 10s + 6 & s^3 + 4s^2 + 3s - 2 \\ s^3 + 5s^2 + 8s + 4 & s^3 + 6s^2 + 13s + 10 \end{bmatrix}$$

or in a right coprime MFD

$$F(s) = N(s) D^{-1}(s) = \begin{bmatrix} 1 & s \\ 1 & s + 2 \end{bmatrix} \begin{bmatrix} s^2 + 5s + 6 & s^2 + 3s + 2 \\ s + 1 & s^3 + 6s^2 + 11s + 6 \end{bmatrix}^{-1}$$

As

$$\Gamma_c[D(s)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \delta_{c_1}[D(s)] = 2, \quad \delta_{c_2}[D(s)] = 3,$$

the polynomial matrix $\bar{D}(s)$ parameterizing the state feedback law (3.12) in the frequency domain is given by

$$\bar{D}(s) = \begin{bmatrix} s^2 + \alpha s + \beta & \varepsilon s^2 + \varphi s + \vartheta \\ \gamma s + \delta & s^3 + \nu s^2 + \psi s + \xi \end{bmatrix}$$

which shows that $pn = 10$ free parameters exist. When choosing e.g. $\alpha = 6$, $\beta = 9$, $\gamma = 0$, $\delta = 0$, $\varepsilon = 1$, $\varphi = 6$, $\vartheta = 9$; $\nu = 9$, $\psi = 27$; $\xi = 27$ all closed loop poles are located at $s = -3$.

Further it is well known that the desired closed loop dynamics assigned in the time domain via $u = -Kx$ will result in the frequency domain if $\bar{D}(s)$ is chosen such that

$$\bar{D}(s) D^{-1}(s) = I + K(sI - A)^{-1} B. \quad (3.22)$$

On the other side the relations

$$D(s) \tilde{D}^{-1}(s) = I - K(sI - A + BK)^{-1} B \quad (3.23)$$

and

$$N(s) \tilde{D}^{-1}(s) = C(sI - A + BK)^{-1} B \quad (3.24)$$

have been shown to hold (see e.g. [13] or [5]).

The relations (3.23) and (3.24) define the system DCFs in the sense of Nett et al. [13]. They allow a computation of the DCFs with known time domain parameters (right hand sides) or a direct computation of these DCFs in the frequency domain (left hand sides), once the parameterizing matrix $\tilde{D}(s)$ for the state feedback is specified.

It is well known that $\tilde{D}(s)$ can be chosen in order to minimize

$$J = \int_0^{\infty} (y^T Q y + u^T R u) dt. \quad (3.25)$$

The optimal $\tilde{D}(s)$ results from spectral factorization (Ježek and Kučera [9]) of

$$\tilde{D}^T(-s) R \tilde{D}(s) = D^T(-s) R D(s) + N^T(-s) Q N(s) \quad (3.26)$$

(Kučera [11]) where the solution $\tilde{D}(s)$ must be aligned to meet the conditions (3.20) and (3.21). If one is interested in the equivalent time domain state feedback matrix K , one easily computes it as

$$Y = K \quad \text{with} \quad X = I \quad (3.27)$$

from the linear diophantine equation

$$X D(s) + Y N_x(s) = \tilde{D}(s) \quad (3.28)$$

where

$$(sI - A)^{-1} B = N_x(s) D^{-1}(s). \quad (3.29)$$

Substitution of (3.27), (3.29) and (3.22) directly shows that (3.27) is a solution to (3.28). So when starting with a time domain design for K , (3.22) can be used to compute the corresponding frequency domain matrix $\tilde{D}(s)$ and from a given $\tilde{D}(s)$, either specified by pole placement as in Example 1 or by spectral factorization of (3.26) (Riccati approach), Eq. (3.28) gives the equivalent time domain matrix K .

4. A NONMINIMAL REPRESENTATION OF THE REDUCED ORDER OBSERVER IN THE TIME AND IN THE FREQUENCY DOMAIN

The frequency domain design of observers uses the left coprime MFD $F(s) = \bar{D}^{-1}(s) \bar{N}(s)$ of the system (Hippe [5]). Figure 1 shows the observer block diagram in the frequency domain. In order to formulate the relations between time and frequency domain representations of observers of $(n - \alpha)$ th order with $0 \leq \alpha \leq m$, one needs a time domain representation of such observers which also uses a full order model.

Figure 2 shows such a block diagram. It is based on a full order plant model, where the α output estimation errors $\varepsilon_2 = y_2 - \hat{y}_2$ are fed back with infinite gain.

This infinite gain reduces the dynamic order of the observer to $n - \kappa$ and it has been shown [8] that the finite dynamics of the system shown in Figure 2 are described by the state equations (3.10) of the reduced order observer.

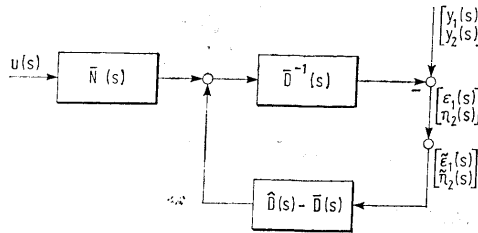


Fig. 1. Frequency domain representation of the observer of $(n - \kappa)$ th order.

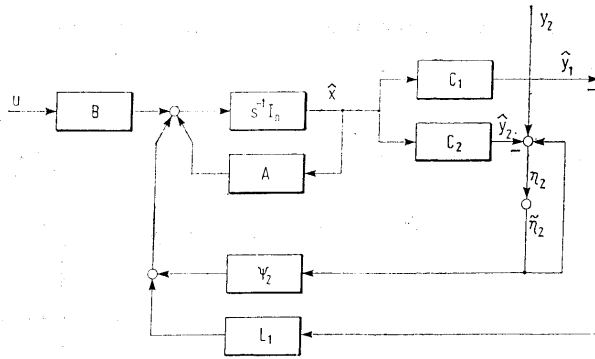


Fig. 2. Nonminimal representation of the observer of $(n - \kappa)$ th order in the time domain.

Thus the two block diagrams in Figures 1 and 2 can be compared.

Cutting open between $\begin{bmatrix} \varepsilon_1 \\ \eta_2 \end{bmatrix}$ and $\begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\eta}_2 \end{bmatrix}$ in Figure 2, the "open loop" gain of the observer is given by

$$\begin{bmatrix} \varepsilon_1(s) \\ \eta_2(s) \end{bmatrix} = \left\{ - \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (sI - A)^{-1} [L_1 \mid \Psi_2] + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right\} \begin{bmatrix} \tilde{\varepsilon}_1(s) \\ \tilde{\eta}_2(s) \end{bmatrix}. \quad (4.1)$$

The "closed loop" behaviour of the observer in Figure 2 from y and u to $\begin{bmatrix} \varepsilon_1 \\ \eta_2 \end{bmatrix}$ is described by

$$\begin{bmatrix} \varepsilon_1(s) \\ \eta_2(s) \end{bmatrix} = \begin{bmatrix} I - C_1 \Theta (sI - F)^{-1} T L_1 & C_1 [-I - \Theta (sI - F)^{-1} T (A - L_1 C_1)] \Psi_2 \\ -C_2 A \Theta (sI - F)^{-1} T L_1 & C_2 [sI - A - A \Theta (sI - F)^{-1} T (A - L_1 C_1)] \Psi_2 \end{bmatrix} \cdot \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} - \begin{bmatrix} C_1 \Theta (sI - F)^{-1} T B \\ C_2 [I + A \Theta (sI - F)^{-1} T] B \end{bmatrix} u(s) \quad (4.2)$$

(Hippe [8]) where the abbreviation $F = T(A - L_1 C_1) \Theta$ has been used.

When introducing the left coprime factorization

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (sI - A)^{-1} \begin{bmatrix} L_1 \\ \Psi_2 \end{bmatrix} + \begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & 0_\kappa \end{bmatrix} = \bar{D}^{-1}(s) \hat{D}(s), \quad (4.3)$$

with $\bar{D}(s)$ as defined in (3.19), simple manipulations are required to show that Figure 1 depicts the frequency domain equivalent of the observer in Figure 2.

The transfer behaviour of this observer configuration is given by

$$\begin{bmatrix} \varepsilon_1(s) \\ \eta_2(s) \end{bmatrix} = \hat{D}^{-1}(s) \bar{D}(s) y(s) - \hat{D}^{-1}(s) \bar{N}(s) u(s) \quad (4.4)$$

which by comparison with (4.2) shows that

$$\det \hat{D}(s) = \det (sI - F) \cong \det [sI - T(A - L_1 C_1) \Theta] \quad (4.5)$$

is the characteristic polynomial of the observer. This comparison further reveals that

$$\begin{aligned} \bar{D}^*(s) &= \hat{D}^{-1}(s) \bar{D}(s) = \\ &= \begin{bmatrix} I - C_1 \Theta (sI - F)^{-1} T L_1 & C_1 [-I - \Theta (sI - F)^{-1} T (A - L_1 C_1)] \Psi_2 \\ -C_2 A \Theta (sI - F)^{-1} T L_1 & C_2 [sI - A - A \Theta (sI - F)^{-1} T (A - L_1 C_1)] \Psi_2 \end{bmatrix} \end{aligned} \quad (4.6)$$

and

$$\bar{N}^*(s) = \hat{D}^{-1}(s) \bar{N}(s) = \begin{bmatrix} C_1 \Theta (sI - F)^{-1} T B \\ C_2 B + C_2 A \Theta (sI - F)^{-1} T B \end{bmatrix} \quad (4.7)$$

constitute the DCFs of the system related to the observer of order $n - \kappa$. The relations (4.6) and (4.7) either allow a computation of the DCFs with given time domain parameters of the observer (right hand sides) or a direct computation of these DCFs in the frequency domain, once the parameterizing matrix $\hat{D}(s)$ of the observer is specified.

Equation (4.3) allows a derivation of the properties of $\hat{D}(s)$, parameterizing a state observer of order $n - \kappa$ in the frequency domain.

Any parameterizing polynomial matrix $\hat{D}(s)$ has to be row proper in order to contain a minimal number of parameters. If one tried to use a parameterizing polynomial matrix with singular highest degree coefficient matrix, the inherent internal dependencies would result in a non unique parameterization, i.e. two different matrices $\hat{D}(s)$ would yield the same observer. Therefore when designing an observer of n_c th order with $n_c = n - \kappa$, $0 \leq \kappa \leq m$, in the frequency domain, the left coprime MFD (3.19) of the plant should be such that

$$\bar{D}_\kappa(s) = \Pi \left\{ \bar{D}(s) \begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & s^{-1} I_\kappa \end{bmatrix} \right\} \quad (4.8)$$

is row proper, where $\Pi\{\cdot\}$ denotes taking the polynomial part. If the highest row degree coefficient matrix $\Gamma_r[\bar{D}_\kappa(s)]$ should become singular, a regular $\Gamma_r[\bar{D}_\kappa(s)]$ can always be achieved by unimodular left operations (Wolovich [17]). This left unimodular operation must also be applied to the MFD (3.19) before starting the

design of a compensator of order $n_c = n - \kappa$. The rationale for this construction of $\bar{D}_\kappa(s)$ will become apparent in the sequel.

The actual form of $\hat{D}(s)$ can be investigated with the aid of the transfer matrix

$$\begin{aligned}\Phi(s) &= \begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & sI_\kappa \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (sI - A)^{-1} [L_1 \mid \Psi_2] - \begin{bmatrix} 0 & 0 \\ 0 & I_\kappa \end{bmatrix} = \\ &= \begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & sI_\kappa \end{bmatrix} \bar{D}^{-1}(s) \hat{D}(s) - I \end{aligned} \quad (4.9)$$

(see equation (4.3)).

Since $C_2 L_1 = 0$ and $C_2 \Psi_2 = I_\kappa$ the matrix $\Phi(s)$ is strictly proper (Rosenbrock [14], p. 38). In the light of (4.8) this implies that $\hat{D}(s)$ has the following properties:

$$\Gamma_r[\hat{D}(s)] = \Gamma_r[\bar{D}_\kappa(s)], \quad (4.10)$$

and

$$\delta_{rj}[\hat{D}(s)] = \delta_{rj}[\bar{D}_\kappa(s)]; \quad j = 1, 2, \dots, m. \quad (4.11)$$

i.e. the last κ columns of $\hat{D}(s)$ are reduced by s^{-1} in relation to $\bar{D}(s)$.

By construction, $\bar{D}_\kappa(s)$ is row proper and therefore, the following Lemma can be formulated.

Lemma 4.1. Under the restrictions (4.10) and (4.11) the $m \times m$ polynomial matrix $\hat{D}(s)$ characterizing the reduced order observer in the frequency domain contains $m(n - \kappa)$ free parameters.

Proof. Since $\det \bar{D}(s)$ is a polynomial of n th order, $\det \bar{D}_\kappa(s)$ is a polynomial of $(n - \kappa)$ th order. Therefore, as $\bar{D}_\kappa(s)$ is row reduced

$$\sum_{j=1}^m \delta_{rj}[\bar{D}_\kappa(s)] = n - \kappa. \quad (4.12)$$

Because of (4.11) the number of free parameters in each row of $\hat{D}(s)$ is

$$n_{frj} = m\{\delta_{rj}[\bar{D}_\kappa(s)]\} \quad (4.13)$$

and consequently the number n_{fr} of free parameters in $\hat{D}(s)$ is given by

$$n_{fr} = \sum_{j=1}^m n_{frj} = \sum_{j=1}^m m\{\delta_{rj}[\bar{D}_\kappa(s)]\} = m(n - \kappa). \quad (4.14)$$

□

This is exactly the number of free parameters existing in the time domain design of reduced order observers of order $n_c = n - \kappa$, provided one only uses κ plant outputs directly for the reconstruction of x . The case where more than κ outputs y_i together with $n - \kappa$ observer states are used to reconstruct x will be discussed in Section 8.

Example 2. Consider again the system of Example 1 with a left coprime MFD $F(s) = \bar{D}^{-1}(s) \bar{N}(s)$ given by

$$F(s) = \begin{bmatrix} -s^2 - 4s - 4 & -s^3 - 7s^2 - 16s - 11 \\ s^2 + 5s + 6 & s^2 + 7s + 8 \end{bmatrix}^{-1} \begin{bmatrix} -s - 2 & -s - 3 \\ 2 & 2 \end{bmatrix}.$$

Since $n = 5$ and $m = 2$, three observers for state reconstruction can be considered. The full order observer ($\kappa = 0$) is parameterized by $\hat{D}(s)$ meeting the restrictions

$$\Gamma_r[\hat{D}(s)] = \Gamma_r[\bar{D}(s)]$$

and

$$\delta_{rj}[\hat{D}(s)] = \delta_{rj}[\bar{D}(s)]; \quad j = 1, 2, \dots, m.$$

With $\delta_{r1}[\bar{D}(s)] = 3$, $\delta_{r2}[\bar{D}(s)] = 2$ and $\Gamma_r[\bar{D}(s)] = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ we have

$$\hat{D}(s) = \begin{bmatrix} \alpha s^2 + \beta s + \gamma & -s^3 - \delta s^2 - \varepsilon s - \varphi \\ s^2 + \vartheta s + \nu & s^2 + \psi s + \xi \end{bmatrix}.$$

Observer poles at $s = -4$ result e.g. when choosing the $mn = 10$ free parameters such that $\hat{D}(s)$ becomes

$$\hat{D}(s) = \begin{bmatrix} 0 & -s^3 - 12s^2 - 48s - 64 \\ s^2 + 8s + 16 & s^2 + 8s + 16 \end{bmatrix}.$$

Since

$$\bar{D}_1(s) = \begin{bmatrix} -s^2 - 4s - 4 & -s^2 - 7s - 16 \\ s^2 + 5s + 6 & s + 7 \end{bmatrix},$$

the observer of fourth order ($\kappa = 1$) is parameterized by

$$\hat{D}(s) = \begin{bmatrix} -s^2 - \alpha s - \beta & -s^2 - \gamma s - \delta \\ s^2 + \varepsilon s + \varphi & \vartheta s + \nu \end{bmatrix},$$

showing that $m(n - \kappa) = 8$ free parameters exist. With

$$\bar{D}_2(s) = \begin{bmatrix} -s - 4 & -s^2 - 7s - 16 \\ s + 5 & s + 7 \end{bmatrix}$$

the minimal order observer ($\kappa = 2$) is parameterized by

$$\hat{D}(s) = \begin{bmatrix} \alpha s + \beta & -s^2 - \gamma s - \delta \\ s + \varepsilon & s + \varphi \end{bmatrix}.$$

Choosing the $m(n - \kappa) = 6$ free parameters as $\alpha = \beta = 0$; $\gamma = 8$, $\delta = 16$; $\varepsilon = \varphi = 4$ an observer with eigenvalues at $s = -4$ results.

When starting with the time domain approach, Eq. (4.3) directly gives the equivalent polynomial matrix $\hat{D}(s)$, parameterizing the observer in the frequency domain. When starting with a frequency domain matrix $\hat{D}(s)$, the equivalent time domain parameters can be computed as

$$\bar{Y} = [L_1 \quad \Psi_2] \quad \text{with} \quad \bar{X} = \begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & 0_\kappa \end{bmatrix} \quad (4.15)$$

from the linear diophantine equation

$$\bar{N}_x(s) \bar{Y} + \bar{D}(s) \bar{X} = \hat{D}(s) \quad (4.16)$$

where

$$C(sI - A)^{-1} = \bar{D}^{-1}(s) \bar{N}_x(s). \quad (4.17)$$

Substitution of (4.15), (4.17) and (4.3) in (4.16) readily shows that (4.15) is a solution to (4.16). In order to get the complete time domain parameterization of the observer (3.10), T has to be chosen such that

$$T\Psi_2 = 0. \quad (4.18)$$

Choosing $\hat{D}(s)$ as demonstrated in Example 2 corresponds to a "pole placing" observer design. If the system is described by

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw \\ y_1 &= C_1x + v \\ y_2 &= C_2x \end{aligned} \quad (4.19)$$

with $x \in \mathbb{R}^n$, $y_1 \in \mathbb{R}^{m-\kappa}$; $y_2 \in \mathbb{R}^\kappa$, $w \in \mathbb{R}^q$, $v \in \mathbb{R}^{m-\kappa}$ and w and v zero mean white noise with

$$E\{w(t)w^T(\tau)\} = \bar{Q}\delta(t-\tau) \quad (4.20)$$

$$E\{v(t)v^T(\tau)\} = \bar{R}\delta(t-\tau) \quad (4.21)$$

and

$$E\{w(t)v^T(\tau)\} = 0 \quad (4.22)$$

then an optimal reduced order estimator (Kalman filter) can be designed (Gelb [4]). Using the frequency domain description

$$\begin{aligned} y_1(s) &= F_1(s)u(s) + F_{w1}(s)w(s) + v(s) \\ y_2(s) &= F_2(s)u(s) + F_{w2}(s)w(s) \end{aligned} \quad (4.23)$$

of the system (4.19) and defining

$$\begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix} = \bar{D}^{-1}(s)\bar{N}(s) \quad (4.24)$$

and

$$\begin{bmatrix} F_{w1}(s) \\ F_{w2}(s) \end{bmatrix} = \bar{D}^{-1}(s)\bar{N}_w(s) \quad (4.25)$$

and observing that

$$C_2G = II\{sF_{w2}(s)\} \quad (4.26)$$

the optimal reduced order filter of order $n - \kappa$ can directly be designed in the frequency domain by spectral factorization (Ježek and Kučera [9]) of

$$\hat{D}(s) \begin{bmatrix} \bar{R} & 0 \\ 0 & \bar{X} \end{bmatrix} \hat{D}^T(-s) = \bar{D}(s) \begin{bmatrix} \bar{R} & 0 \\ 0 & 0 \end{bmatrix} \bar{D}^T(-s) + \bar{N}_w(s) \bar{Q} \bar{N}_w^T(-s) \quad (4.27)$$

with

$$\bar{X} = C_2G\bar{Q}G^TC_2^T \quad (4.28)$$

(Hippe [7]). A stable filter of order $n - \kappa$ results if $F_{w2}(s)$ does not contain zeros

on the imaginary axis and if

$$\text{rank} \begin{bmatrix} \bar{R} & 0 \\ 0 & \bar{X} \end{bmatrix} = m. \quad (4.29)$$

Again, $\hat{D}(s)$ resulting from the spectral factorization of (4.27) has to be aligned to meet the conditions (4.10) and (4.11). Hence any reduced order observer can directly be designed in the frequency domain and Eq. (4.16) can be used to compute its equivalent time domain representation.

5. TWO DUAL REPRESENTATIONS OF THE CLOSED LOOP

If one considers the closed loop consisting of plant, linear state feedback, and observer, two possible frequency domain representations exist. The *controller structure* is characterized by a right coprime plant and a left coprime compensator MFD and the *observer structure* by a left coprime plant and a right coprime compensator MFD (Hippe [5]).

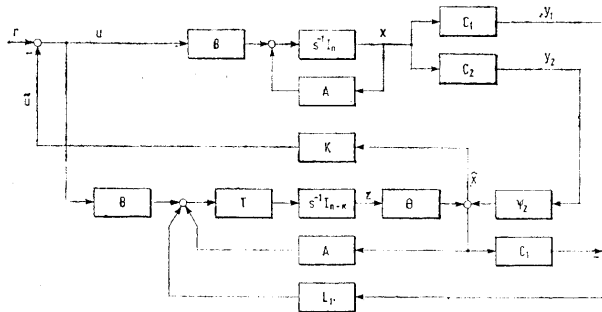


Fig. 3. Closed loop with compensator of $(n - k)$ th order in the time domain.

Figure 3 shows the time domain block diagram of the closed loop with the usual minimal observer representation (3.10). The external input r is the p -dimensional reference vector. The isolated compensator can be described by its transfer behaviour from y and u to the internal quantity \tilde{u} , namely

$$\begin{aligned} \tilde{u}(s) = & \{K\Theta(sI - F)^{-1} [TL_1 \mid T(A - L_1C_1) \Psi_2] + [0 \mid K\Psi_2]\} \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} \\ & + K\Theta(sI - F)^{-1} TB u(s) \end{aligned} \quad (5.1)$$

where again we use the abbreviation $F = T(A - L_1C_1) \Theta$.

If one introduces the left MFDs

$$K\Theta(sI - F)^{-1} [TL_1 \mid T(A - L_1C_1) \Psi_2] + [0 \mid K\Psi_2] = \Delta^{-1}(s) N_C(s) \quad (5.2)$$

and

$$K\Theta(sI - F)^{-1} TB = \Delta^{-1}(s) N_u(s), \quad (5.3)$$

such that $\Delta^{-1}(s)[N_c(s)N_u(s)]$ constitutes a left coprime pair, the closed loop of Figure 3 can equally be represented by the block diagram of Figure 4.

Eliminating the internal feedback loop via $\Delta^{-1}(s)$, the block diagram of Figure 4 assumes the shape shown in Figure 5 with

$$D_c(s) = N_u(s) + \Delta(s). \quad (5.4)$$

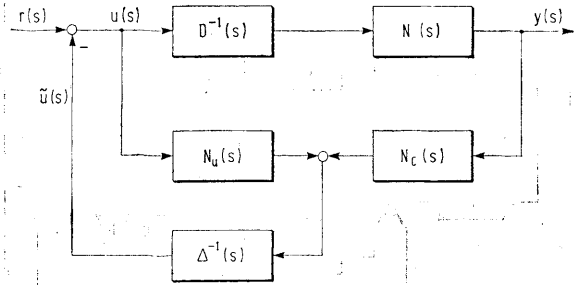


Fig. 4. Closed loop in the frequency domain (controller structure).

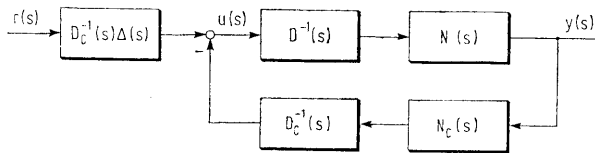


Fig 5. Alternative block diagram of closed loop in controller structure.

It is well known that the basic loop equation for the loop in Figure 5 reads

$$N_c(s)N(s) + D_c(s)D(s) = \Delta(s)\tilde{D}(s). \quad (5.5)$$

The reference behaviour of the closed loop configurations shown in the Figures 3 through 5 is given by

$$y(s) = N(s)\tilde{D}^{-1}(s)r(s). \quad (5.6)$$

This, however, is identical to the reference behaviour when applying the state feedback control (3.12) without observer. Therefore the closed loop configurations shown in the block diagrams of Figures 4 and 5 are said to be in *controller structure*.

An alternative closed loop representation is shown in Figure 6. It is based on the nonminimal observer structure of Figure 2 and this time, the external inputs are the p -dimensional input disturbance vector w and the $(m - \kappa)$ -dimensional output disturbance vector v (see Eq. (4.19) and (4.23)). In analogy to the reduced order optimal estimation scheme, the κ -dimensional output vector y_2 is supposed to be undisturbed.

Considering the isolated compensator, the compensator transfer behaviour from

$\begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\eta}_2 \end{bmatrix}$ to $\begin{bmatrix} \varepsilon_1 \\ \eta_2 \end{bmatrix}$ and u is given by

$$\begin{bmatrix} \varepsilon_1(s) \\ \eta_2(s) \end{bmatrix} = \left\{ - \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (sI - A + BK)^{-1} [L_1 \mid \Psi_2] + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right\} \begin{bmatrix} \tilde{\varepsilon}_1(s) \\ \tilde{\eta}_2(s) \end{bmatrix} \quad (5.7)$$

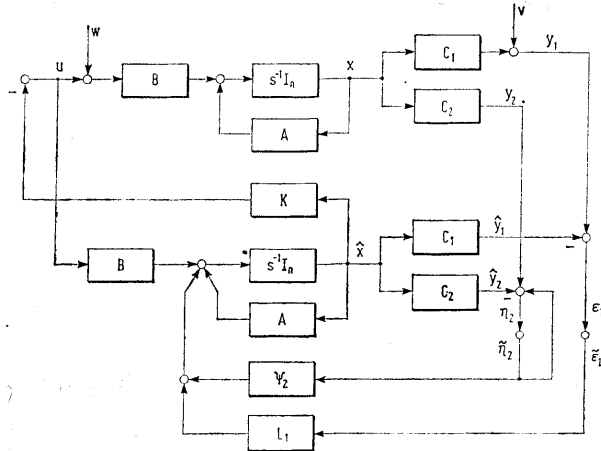


Fig. 6. Closed loop with compensator of order $n - x$ based on the nonminimal observer representation.

and by

$$u(s) = -K(sI - A + BK)^{-1} [L_1 \mid \Psi_2] \begin{bmatrix} \tilde{\varepsilon}_1(s) \\ \tilde{\eta}_2(s) \end{bmatrix}. \quad (5.8)$$

When introducing the right MFDs

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (sI - A + BK)^{-1} [L_1 \mid \Psi_2] - \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \bar{N}_y(s) \bar{A}^{-1}(s) \quad (5.9)$$

and

$$K(sI - A + BK)^{-1} [L_1 \mid \Psi_2] = \bar{N}_c(s) \bar{A}^{-1}(s) \quad (5.10)$$

such that $\begin{bmatrix} \bar{N}_y(s) \\ \bar{N}_c(s) \end{bmatrix} \bar{A}^{-1}(s)$ constitutes a right coprime pair, the closed loop in Figure 6 can equally be represented in the frequency domain by the block diagram shown in Figure 7. Eliminating the internal compensator feedback path via $\bar{A}^{-1}(s)$, the block diagram of Figure 7 assumes the shape shown in Figure 8 with

$$\bar{D}_c(s) = \bar{N}_y(s) + \bar{A}(s). \quad (5.11)$$

It is also well known that the basic loop equation for the loop of Figure 8 reads

$$\bar{N}(s) \bar{N}_c(s) + \bar{D}(s) \bar{D}_c(s) = \hat{D}(s) \bar{A}(s). \quad (5.12)$$

In the closed loop configurations shown in the Figures 6 through 8 the transfer

behaviours from w and $\begin{bmatrix} v \\ 0 \end{bmatrix}$ to the output observation error $\begin{bmatrix} \varepsilon_1 \\ \eta_2 \end{bmatrix}$ namely

$$\begin{bmatrix} \varepsilon_1(s) \\ \eta_2(s) \end{bmatrix} = \hat{D}^{-1}(s) \bar{N}(s) w(s) \quad (5.13)$$

and

$$\begin{bmatrix} \varepsilon_1(s) \\ \eta_2(s) \end{bmatrix} = \hat{D}^{-1}(s) \bar{D}(s) \begin{bmatrix} v(s) \\ 0 \end{bmatrix} \quad (5.14)$$

are the same as if the isolated observer was coupled to the uncontrolled plant (see equation (4.4)). Therefore the closed loop configurations shown in the block diagrams

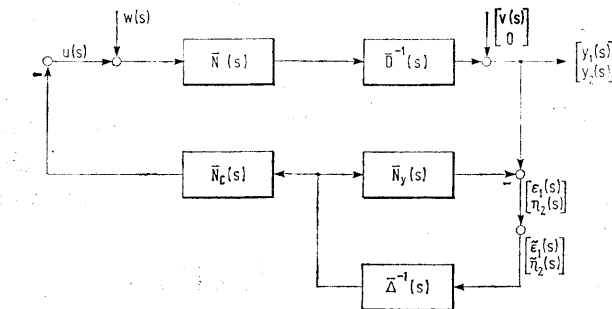


Fig. 7. Closed loop in the frequency domain (observer structure).

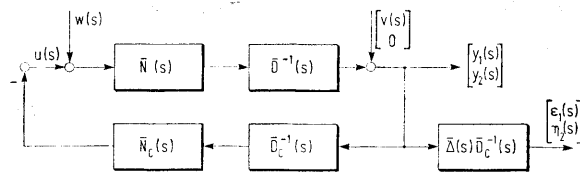


Fig. 8. Alternative block diagram of closed loop in observer structure.

of Figures 7 and 8 are said to be in *observer structure*. The observer structure is mainly of theoretical interest since it can only be realized in the full order case. For $\kappa > 0$ the quantity $\bar{\Delta}(s)\bar{D}_c^{-1}(s)$ (see Fig. 8) is no longer proper.

The state feedback control is parameterized by the $p \times p$ polynomial matrix $\bar{D}(s)$ and the observer by the $m \times m$ polynomial matrix $\hat{D}(s)$. Both matrices, however, appear in two different diophantine equations which are related with the *controller structure* (5.5) and with the *observer structure* (5.12) of the closed loop. Neither equation can be solved for the desired compensator transfer matrix $F_C(s)$ since in (5.5) the parameterizing matrix $\hat{D}(s)$ for the observer and in (5.12) the parameterizing matrix $\bar{D}(s)$ for the state feedback control does not appear explicitly. A solution to this problem is presented in the next two sections.

6. THE COMPUTATION OF THE LEFT COPRIME COMPENSATOR MFD

Consider any polynomial solutions $Y(s)$ and $X(s)$ of the linear diophantine equation

$$Y(s)N(s) + X(s)D(s) = \tilde{D}(s) \quad (6.1)$$

where $\tilde{D}(s)$ characterizes the controlled plant dynamics (see Section 3 and Example 1). Since $D(s)$ and $N(s)$ are relatively coprime, such polynomial matrices exist (Kučera [10]).

With $\Pi[\cdot]$ denoting the polynomial part and $SP[\cdot]$ denoting the strictly proper part of a rational matrix it is obvious that for a given transfer matrix $P(s)$

$$P(s) = \Pi[P(s)] + SP[P(s)] \quad (6.2)$$

holds.

Lemma 6.1. Consider the plant transfer matrix $F(s) = \bar{D}^{-1}(s)\bar{N}(s)$, a solution $Y(s)$ of (6.1) and the $m \times m$ polynomial matrix $\hat{D}(s)$ characterizing the observer dynamics (see Section 4 and Example 2). Then the strictly proper part of $Y(s)\bar{D}^{-1}(s)\hat{D}(s)$ is given by

$$SP\{Y(s)\bar{D}^{-1}(s)\hat{D}(s)\} = K(sI - A)^{-1} [L_1 \mid \Psi_2]. \quad (6.3)$$

Proof. Using the basic relation (4.3) we can write

$$Y(s)\bar{D}^{-1}(s)\hat{D}(s) = Y(s)C(sI - A)^{-1} [L_1 \mid \Psi_2] + Y(s) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $Y(s)$ is a polynomial matrix it remains to be shown that

$$SP\{Y(s)C(sI - A)^{-1} [L_1 \mid \Psi_2]\} = K(sI - A)^{-1} [L_1 \mid \Psi_2] \quad (6.4)$$

or equivalently that

$$W(s) = [Y(s)C - K](sI - A)^{-1} [L_1 \mid \Psi_2]$$

is a polynomial matrix. Right multiplication of (6.1) by $D^{-1}(s)$ yields

$$Y(s)N(s)D^{-1}(s) + X(s) = \tilde{D}(s)D^{-1}(s),$$

or with (3.17), (3.18) and (3.22)

$$[Y(s)C - K](sI - A)^{-1}B = I - X(s). \quad (6.5)$$

The right hand side of (6.5) is a polynomial matrix. As we have assumed a completely controllable plant, $(sI - A)^{-1}B$ is a coprime pair and consequently, $[Y(s)C - K]$ has the form $\hat{N}(s)(sI - A)$ with $\hat{N}(s)$ being a polynomial matrix. Therefore $[Y(s)C - K](sI - A)^{-1}$ constitutes a polynomial matrix which completes the proof. \square

With this preliminary result we can formulate the solution procedure for the left coprime compensator factorization.

Theorem 6.1. With the polynomial matrix

$$V(s) = \Pi[Y(s) \bar{D}^{-1}(s) \hat{D}(s)] \quad (6.6)$$

the doubly coprime left factorization of the compensator is given by

$$N_C^*(s) = \Delta^{-1}(s) N_C(s) = Y(s) - V(s) \hat{D}^{-1}(s) \bar{D}(s) \quad (6.7)$$

and

$$D_C^*(s) = \Delta^{-1}(s) D_C(s) = X(s) + V(s) \hat{D}^{-1}(s) \bar{N}(s). \quad (6.8)$$

The left coprime compensator MFD and the observer matrix $\Delta(s)$ can easily be computed from (6.7) and (6.8) by prime factorization of $[N_C^*(s) D_C^*(s)] = \Delta^{-1}(s) [N_C(s) D_C(s)]$ (Kučera [10]).

Proof. We first observe that

$$\begin{aligned} V(s) &= \Pi[Y(s) \bar{D}^{-1}(s) \hat{D}(s)] \\ &= \Pi \left\{ Y(s) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + Y(s) C(sI - A)^{-1} [L_1 \mid \Psi_2] \right\} \\ &= Y(s) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + Y(s) C(sI - A)^{-1} [L_1 \mid \Psi_2] - SP\{Y(s) C(sI - A)^{-1} \\ &\quad \cdot [L_1 \mid \Psi_2]\} \\ V(s) &= Y(s) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + [Y(s) C - K](sI - A)^{-1} [L_1 \mid \Psi_2]. \end{aligned} \quad (6.9)$$

Using the relations (6.9), (4.6), (3.7), (3.8), (3.11) and $F = T(A - L_1 C_1) \Theta$ the numerator matrix (6.7) is given by

$$\begin{aligned} N_C^*(s) &= Y(s) - V(s) \hat{D}^{-1}(s) \bar{D}(s) = \\ &= Y(s) - \left[Y(s) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + [Y(s) C - K](sI - A)^{-1} [L_1 \mid \Psi_2] \right] \\ &\quad \cdot \begin{bmatrix} I - C_1 \Theta (sI - F)^{-1} T L_1 & C_1 [-I - \Theta (sI - F)^{-1} T(A - L_1 C_1)] \Psi_2 \\ -C_2 A \Theta (sI - F)^{-1} T L_1 & C_2 [sI - A - A \Theta (sI - F)^{-1} T(A - L_1 C_1)] \Psi_2 \end{bmatrix} \\ &= Y(s) - Y(s) \begin{bmatrix} I - C_1 \Theta (sI - F)^{-1} T L_1 & -C_1 [I + \Theta (sI - F)^{-1} T(A - L_1 C_1)] \Psi_2 \\ 0 & 0 \end{bmatrix} \\ &\quad - [Y(s) C - K](sI - A)^{-1} \{L_1 - (L_1 C_1 + \Psi_2 C_2 A) \Theta (sI - F)^{-1} T L_1 \mid \\ &\quad \mid [sI - \Psi_2 C_2 A - L_1 C_1 - (L_1 C_1 + \Psi_2 C_2 A) \Theta (sI - F)^{-1} T(A - L_1 C_1)] \Psi_2\} \\ &= Y(s) - Y(s) \begin{bmatrix} I - C_1 \Theta (sI - F)^{-1} T L_1 & -C_1 [I + \Theta (sI - F)^{-1} T(A - L_1 C_1)] \Psi_2 \\ 0 & 0 \end{bmatrix} \\ &\quad - [Y(s) C - K](sI - A)^{-1} \{\Theta T L_1 - (\Theta T L_1 C_1 + A - \Theta T A) \Theta (sI - F)^{-1} T L_1 \mid \\ &\quad \mid [sI - A + \Theta T A - \Theta T L_1 C_1 - (\Theta T L_1 C_1 + A - \Theta T A) \Theta (sI - F)^{-1} T(A - L_1 C_1)] \\ &\quad \cdot \Psi_2\} = Y(s) - \end{aligned}$$

$$\begin{aligned}
& - Y(s) \left[\begin{array}{c|c} I - C_1 \Theta (sI - F)^{-1} T L_1 & -C_1 [I + \Theta (sI - F)^{-1} T (A - L_1 C_1)] \Psi_2 \\ \hline 0 & 0 \end{array} \right] \\
& - [Y(s) C - K] (sI - A)^{-1} \{ (sI - \Theta T A + \Theta T L_1 C_1 - \Theta T L_1 C_1 - A + \Theta T A) \cdot \\
& \cdot \Theta (sI - F)^{-1} T L_1 \mid [sI - A + (sI - \Theta T A + \Theta T L_1 C_1 - \Theta T L_1 C_1 - A + \Theta T A) \cdot \\
& \cdot \Theta (sI - F)^{-1} T (A - L_1 C_1)] \Psi_2 \} = \\
& = Y(s) - Y(s) \left[\begin{array}{c|c} I - C_1 \Theta (sI - F)^{-1} T L_1 & -C_1 [I + \Theta (sI - F)^{-1} T (A - L_1 C_1)] \Psi_2 \\ \hline 0 & 0 \end{array} \right] \\
& - [Y(s) C - K] [\Theta (sI - F)^{-1} T L_1 \mid \Psi_2 + \Theta (sI - F)^{-1} T (A - L_1 C_1) \Psi_2] = \\
& = Y(s) - Y(s) \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array} \right] + K [\Theta (sI - F)^{-1} T L_1 \mid \Theta (sI - F)^{-1} T (A - L_1 C_1) \Psi_2 + \Psi_2] \\
& = K \Theta (sI - F)^{-1} [T L_1 \mid T (A - L_1 C_1) \Psi_2] + [0 \mid K \Psi_2] = \Delta^{-1}(s) N_c(s),
\end{aligned}$$

because of (5.2)

Successively using (6.9), (4.7), (3.7), (3.8), (3.11), $F = T(A - L_1 C_1) \Theta$, (6.1), (3.22) (3.17) and (3.18), the denominator matrix (6.8) is given by

$$\begin{aligned}
D_c^*(s) & = X(s) + V(s) \hat{D}^{-1}(s) \bar{N}(s) = X(s) + \left\{ Y(s) \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array} \right] + \right. \\
& + [Y(s) C - K] (sI - A)^{-1} [L_1 \mid \Psi_2] \left. \right\} \left[\begin{array}{c} C_1 \Theta (sI - F)^{-1} T B \\ \hline C_2 B + C_2 A \Theta (sI - F)^{-1} T B \end{array} \right] \\
& = X(s) + Y(s) \left[\begin{array}{c|c} C_1 \Theta (sI - F)^{-1} T B \\ \hline 0 \end{array} \right] + \\
& + [Y(s) C - K] (sI - A)^{-1} [(L_1 C_1 + \Psi_2 C_2 A) \Theta (sI - F)^{-1} T B + \Psi_2 C_2 B] \\
& = X(s) + Y(s) \left[\begin{array}{c|c} C_1 \Theta (sI - F)^{-1} T B \\ \hline 0 \end{array} \right] + [Y(s) C - K] (sI - A)^{-1} \cdot \\
& \cdot [B - (sI - \Theta T A + \Theta T L_1 C_1 - \Theta T L_1 C_1 - A + \Theta T A) \Theta (sI - F)^{-1} T B] \\
& = I + K (sI - A)^{-1} B - Y(s) C (sI - A)^{-1} B + Y(s) C (sI - A)^{-1} B - \\
& - K (sI - A)^{-1} B + Y(s) \left[\begin{array}{c|c} C_1 \Theta (sI - F)^{-1} T B & -C_1 \Theta (sI - F)^{-1} T B \\ \hline 0 & 0 \end{array} \right] + \\
& + K \Theta (sI - F)^{-1} T B = I + K \Theta (sI - F)^{-1} T B = \Delta^{-1}(s) D_c(s)
\end{aligned}$$

where the latter becomes obvious by inspection of (5.3) and (5.4).

The right hand side of (5.5) constitutes the generalized characteristic polynomial of the MIMO control loop. Substituting the compensator matrices $N_c(s)$ and $D_c(s)$ in (5.5) by the right hand sides of (6.7) and (6.8) one obtains

$$\begin{aligned}
N_c^*(s) N(s) + D_c^*(s) D(s) & = \\
& = [Y(s) - V(s) \hat{D}^{-1}(s) \bar{D}(s)] N(s) + [X(s) + V(s) \hat{D}^{-1}(s) \bar{N}(s)] D(s) \\
& = Y(s) N(s) + X(s) D(s) + V(s) \hat{D}^{-1}(s) [\bar{N}(s) D(s) - \bar{D}(s) N(s)] = \bar{D}(s).
\end{aligned}$$

Since $N_C^*(s) = \Delta^{-1}(s) N_C(s)$ and $D_C^*(s) = \Delta^{-1}(s) D_C(s)$ this becomes

$$\Delta^{-1}(s) N_C(s) N(s) + \Delta^{-1}(s) D_C(s) D(s) = \bar{D}(s),$$

and by left multiplication with $\Delta(s)$ the basic loop equation (5.5) results. \square

7. THE COMPUTATION OF THE RIGHT COPRIME COMPENSATOR MFD

Consider any polynomial solutions $\bar{Y}(s)$ and $\bar{X}(s)$ of the linear diophantine equation

$$\bar{N}(s) \bar{Y}(s) + \bar{D}(s) \bar{X}(s) = \hat{D}(s) \quad (7.1)$$

where $\hat{D}(s)$ characterizes the observer dynamics (see Section 4 and Example 2). Since $\bar{D}(s)$ and $\bar{N}(s)$ are relatively coprime such polynomial matrices exist (Kučera, [10]).

Lemma 7.1. Consider the plant transfer matrix $F(s) = N(s) D^{-1}(s)$, a solution $\bar{Y}(s)$ of (7.1) and the $p \times p$ polynomial matrix $\bar{D}(s)$ parameterizing the linear state feedback control in the frequency domain (see Section 3). Then the strictly proper part of $\bar{D}(s) D^{-1}(s) \bar{Y}(s)$ is given by

$$SP[\bar{D}(s) D^{-1}(s) \bar{Y}(s)] = K(sI - A)^{-1} [L_1 \mid \Psi_2]. \quad (7.2)$$

Proof. Using the basic relation (3.22) one obtains

$$\bar{D}(s) D^{-1}(s) \bar{Y}(s) = \bar{Y}(s) + K(sI - A)^{-1} B \bar{Y}(s). \quad (7.3)$$

Since $\bar{Y}(s)$ is a polynomial matrix, it remains to be shown that

$$SP[K(sI - A)^{-1} B \bar{Y}(s)] = K(sI - A)^{-1} [L_1 \mid \Psi_2] \quad (7.4)$$

or equivalently that

$$\bar{W}(s) = K(sI - A)^{-1} \{B \bar{Y}(s) - [L_1 \mid \Psi_2]\}$$

is a polynomial matrix. Left multiplication of (7.1) by $\bar{D}^{-1}(s)$ yields

$$\bar{D}^{-1}(s) \bar{N}(s) \bar{Y}(s) + \bar{X}(s) = \bar{D}^{-1}(s) \hat{D}(s)$$

or using (3.19), (3.17) and (4.3)

$$C(sI - A)^{-1} \{B \bar{Y}(s) - [L_1 \mid \Psi_2]\} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - \bar{X}(s). \quad (7.5)$$

The right hand side of (7.5) is a polynomial matrix. As we have assumed complete observability, $C(sI - A)^{-1}$ is a coprime pair and consequently $\{B \bar{Y}(s) - [L_1 \mid \Psi_2]\}$ has the form $(sI - A) \hat{N}(s)$ with $\hat{N}(s)$ being a polynomial matrix. Therefore $(sI - A)^{-1} \{B \bar{Y}(s) - [L_1 \mid \Psi_2]\}$ constitutes a polynomial matrix which completes the proof. \square

Now we can formulate the solution procedure for the right coprime compensator factorization.

Theorem 7.1. With the polynomial matrix

$$\bar{V}(s) = \Pi[\bar{D}(s) D^{-1}(s) \bar{Y}(s)] \quad (7.6)$$

the doubly coprime right factorization of the compensator is given by

$$\bar{N}_c^*(s) = \bar{N}_c(s) \bar{A}^{-1}(s) = \bar{Y}(s) - D(s) \bar{D}^{-1}(s) \bar{V}(s) \quad (7.7)$$

and

$$\bar{D}_c^*(s) = \bar{D}_c(s) \bar{A}^{-1}(s) = \bar{X}(s) + N(s) \bar{D}^{-1}(s) \bar{V}(s). \quad (7.8)$$

The right coprime compensator MFD and the matrix $\bar{A}(s)$, containing the controlled plant dynamics, can easily be computed from (7.7) and (7.8) by prime factorization of

$$\begin{bmatrix} \bar{N}_c^*(s) \\ \bar{D}_c^*(s) \end{bmatrix} = \begin{bmatrix} \bar{N}_c(s) \\ \bar{D}_c(s) \end{bmatrix} \bar{A}^{-1}(s) \text{ (Kučera [10]).}$$

Proof. The polynomial matrix $\bar{V}(s)$ is given by

$$\begin{aligned} \bar{V}(s) &= \Pi\{\bar{D}(s) D^{-1}(s) \bar{Y}(s)\} = \Pi\{\bar{Y}(s) + K(sI - A)^{-1} B \bar{Y}(s)\} \\ &= \bar{Y}(s) + K(sI - A)^{-1} B \bar{Y}(s) - SP[K(sI - A)^{-1} B \bar{Y}(s)] \\ &= \bar{Y}(s) + K(sI - A)^{-1} \{B \bar{Y}(s) - [L_1 \mid \Psi_2]\}. \end{aligned} \quad (7.9)$$

Using the relations (7.9) and (3.23) the numerator matrix (7.7) of the compensator has the form

$$\begin{aligned} \bar{N}_c^*(s) &= \bar{Y}(s) - D(s) \bar{D}^{-1}(s) \bar{V}(s) \\ &= \bar{Y}(s) - [I - K(sI - A + BK)^{-1} B] [\bar{Y}(s) + K(sI - A)^{-1} \{B \bar{Y}(s) - \\ &\quad - [L_1 \mid \Psi_2]\}] \\ &= K(sI - A + BK)^{-1} B \bar{Y}(s) + K(sI - A + BK)^{-1} BK(sI - A)^{-1} \cdot \\ &\quad \cdot \{B \bar{Y}(s) - [L_1 \mid \Psi_2]\} - K(sI - A)^{-1} \{B \bar{Y}(s) - [L_1 \mid \Psi_2]\} \\ &= K(sI - A + BK)^{-1} [BK - sI + A - BK] (sI - A)^{-1} \{B \bar{Y}(s) - \\ &\quad - [L_1 \mid \Psi_2]\} + K(sI - A + BK)^{-1} B \bar{Y}(s) \\ &= K(sI - A + BK)^{-1} [L_1 \mid \Psi_2] = \bar{N}_c(s) \bar{A}^{-1}(s) \end{aligned}$$

because of (5.10).

With the aid of (3.24), (7.9), (7.1), (4.3), (3.19) and (3.17) one can show that the denominator matrix (7.8) is given by

$$\begin{aligned} \bar{D}_c^*(s) &= \bar{X}(s) + N(s) \bar{D}^{-1}(s) \bar{V}(s) \\ &= \bar{X}(s) + C(sI - A + BK)^{-1} B [\bar{Y}(s) + K(sI - A)^{-1} \{B \bar{Y}(s) - [L_1 \mid \Psi_2]\}] \\ &= C(sI - A)^{-1} [L_1 \mid \Psi_2] + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - C(sI - A)^{-1} B \bar{Y}(s) + \\ &\quad + C(sI - A + BK)^{-1} B \bar{Y}(s) + C(sI - A + BK)^{-1} BK(sI - A)^{-1} \\ &\quad \cdot \{B \bar{Y}(s) - [L_1 \mid \Psi_2]\} \\ &= C(sI - A + BK)^{-1} [BK - sI + A - BK] (sI - A)^{-1} B \bar{Y}(s) - \end{aligned}$$

$$\begin{aligned}
& - [L_1 | \Psi_2] + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + C(sI - A + BK)^{-1} B Y(s) \\
& = C(sI - A + BK)^{-1} [L_1 | \Psi_2] + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \bar{D}_c(s) \bar{A}^{-1}(s)
\end{aligned}$$

which becomes obvious by inspection of (5.9) and (5.11).

The right hand side of (5.12) constitutes the generalized characteristic polynomial of the MIMO control loop.

Substituting the compensator matrices $\bar{N}_c(s)$ and $\bar{D}_c(s)$ in (5.12) by the right hand sides of (7.7) and (7.8) one obtains

$$\begin{aligned}
& \bar{N}(s) \bar{N}_c^*(s) + \bar{D}(s) \bar{D}_c^*(s) \\
& = \bar{N}(s) [\bar{Y}(s) - D(s) \bar{D}^{-1}(s) \bar{V}(s)] + \bar{D}(s) [\bar{X}(s) + N(s) \bar{D}^{-1}(s) \bar{V}(s)] \\
& = \bar{N}(s) \bar{Y}(s) + \bar{D}(s) \bar{X}(s) + [\bar{D}(s) N(s) - \bar{N}(s) D(s)] \bar{D}^{-1}(s) \bar{V}(s) = \hat{D}(s).
\end{aligned}$$

Since $\bar{N}_c^*(s) = \bar{N}_c(s) \bar{A}^{-1}(s)$ and $\bar{D}_c^*(s) = \bar{D}_c(s) \bar{A}^{-1}(s)$ this becomes

$$\bar{N}(s) \bar{N}_c(s) \bar{A}^{-1}(s) + \bar{D}(s) \bar{D}_c(s) \bar{A}^{-1}(s) = \hat{D}(s)$$

and by right multiplication with $\bar{A}(s)$ the basic loop equation (5.12) results. \square

8. GENERAL REMARKS

The above presented results are valid for any observer/compensator order n_c within the limits

$$n \geq n_c \geq n - m \quad (8.1)$$

or equivalently

$$0 \leq \kappa \leq m \quad (8.2)$$

and therefore, they contain known results as special cases. The full order case (Hippe [5]) is characterized by

$$\kappa = 0, \quad C_1 = C, \quad T = \Theta = I,$$

and vanishing C_2 and Ψ_2 , and the minimal order case (Hippe [6]) by

$$\kappa = m, \quad C_2 = C, \quad \Psi_2 = \Psi$$

and vanishing C_1 and L_1 . The state feedback control $u = -Kx$ is independent of the observer/compensator order (as long as this order stays within the limits defined by (8.1)) and it can be parameterized in the frequency domain by the $p \times p$ polynomial matrix $\bar{D}(s)$ which has the following properties

$$\Gamma_c[\bar{D}(s)] = \Gamma_c[D(s)] \quad (8.3)$$

$$\delta_{c_j}[\bar{D}(s)] = \delta_{c_j}[D(s)], \quad j = 1, 2, \dots, p \quad (8.4)$$

where $\Gamma_c[\cdot]$ is the highest column degree coefficient matrix and $\delta_{cj}[\cdot]$ is the j th column degree.

These two restrictions reduce the amount of free parameters in $\bar{D}(s)$ to the minimum value pn , which corresponds to the number of free parameters in the state feedback matrix K . Of course $\Gamma_c[D(s)]$ is supposed to have full rank, i.e. the denominator matrix $D(s)$ in the MFD (3.18) must be column proper.

The parameterization of the observer of order $n - \varkappa$ bases on the matrix $\bar{D}_\varkappa(s)$, which is related with the denominator matrix $\bar{D}(s)$ in the MFD (3.19) by

$$\bar{D}_\varkappa(s) = \Pi \left\{ \bar{D}(s) \begin{bmatrix} I_{m-\varkappa} & 0 \\ 0 & s^{-1}I_\varkappa \end{bmatrix} \right\} \quad (8.5)$$

where $\Pi[\cdot]$ denotes taking the polynomial part.

If $\bar{D}_\varkappa(s)$ is not row proper, i.e. the highest row degree coefficient matrix $\Gamma_r[\bar{D}_\varkappa(s)]$ is singular, it can be made proper by unimodular left operations. These left operations then have to be applied to the plant MFD (3.19) also. The observer of order $n - \varkappa$ is parameterized by $\hat{D}(s)$ which has the following properties

$$\Gamma_r[\hat{D}(s)] = \Gamma_r[\bar{D}_\varkappa(s)] \quad (8.6)$$

$$\delta_{rj}[\hat{D}(s)] = \delta_{rj}[\bar{D}_\varkappa(s)], \quad j = 1, 2, \dots, m \quad (8.7)$$

where $\Gamma_r[\cdot]$ is the highest row degree coefficient matrix and $\delta_{rj}[\cdot]$ is the j th row degree. The two restriction (8.6) and (8.7) reduce the amount of free parameters in $\hat{D}(s)$ to the minimum number $m(n - \varkappa)$, which is also the amount of free parameters in the time domain design of reduced order observers, provided one only uses the \varkappa outputs y_2 for the reconstruction (3.6) of the plant state x . This, on the other hand, implies that $p(m - \varkappa)$ compensator transfer functions are strictly proper.

The polynomial matrices $\bar{D}(s)$ and $\hat{D}(s)$ therefore parameterize the observer based compensator of order $n_c = n - \varkappa$ and they contain

$$n_f = np + m(n - \varkappa) \quad (8.8)$$

free parameters. In a general compensator of $(n - \varkappa)$ th order, a total amount of

$$n_t = mp + (m + p)(n - \varkappa) \quad (8.9)$$

free parameters exist. The difference $n_t - n_f = p(m - \varkappa)$ results from the fact, that generally the compensator transfer functions only need to be proper. In the time domain approach, this corresponds to the case when all m plant outputs y are used for the reconstruction of the state x . In the frequency domain approach presented herein, these additional degrees of freedom can be exploited by modifying the appropriate $p(m - \varkappa)$ constant elements in the first $(m - \varkappa)$ columns of $V(s)$ or $\bar{V}(s)$ (see Example 3).

The actual computation of the compensator transfer matrices goes along the lines defined in Theorem 6.1 for the left coprime MFD and in Theorem 7.1 for the right coprime MFD. Since the left coprime compensator MFD is the one usually

used (controller structure) its design shall be shortly summarized.

- i) solve $Y(s)N(s) + X(s)D(s) = \bar{D}(s)$
- ii) compute $V(s) = H[Y(s)\bar{D}^{-1}(s)\hat{D}(s)]$
- iii) compute $N_c^*(s) = Y(s) - V(s)\hat{D}^{-1}(s)\bar{D}(s)$
 $D_c^*(s) = X(s) + V(s)\hat{D}^{-1}(s)\bar{N}(s)$
- iv) find $D_c(s), N_c(s)$ and $\Delta(s)$ by prime factorization of
 $[N_c^*(s)D_c^*(s)] = \Delta^{-1}(s)[N_c(s)D_c(s)]$.

The parameterizing matrices $\bar{D}(s)$ or $\hat{D}(s)$ can either be specified arbitrarily (see Examples) or they may be computed from the optimal linear control or the optimal linear estimation problems (see (3.26) and (4.27)).

Thus the DCFs related to arbitrary observer orders can directly be computed in the frequency domain. From a (stable) choice of the parameterizing matrices the system DCFs (3.23), (3.24), (4.6), (4.7) can directly be computed. The corresponding DCFs of the compensator follow from the Theorems 6.1 and 7.1. Thus the DCFs in the sense of Nett et al [13] can

- directly be computed without recurrence to time domain representations
- be generalized to arbitrary observers for state reconstruction, namely observers of order $n - \kappa$ with $0 \leq \kappa \leq m$.

Given the DCFs of the compensator the left and right MFDs of the compensator are easily computed using standard software (Chang [3]).

Example 3. We consider again the system used in the Examples 1 and 2. The parameterizing matrix for the state feedback was (Example 1)

$$\bar{D}(s) = \begin{bmatrix} s^2 + 6s + 9 & s^2 + 6s + 9 \\ 0 & s^3 + 9s^2 + 27s + 27 \end{bmatrix}.$$

Using an observer of fourth order ($\kappa = 1$), the parameterizing matrix for the observer may be chosen as

$$\hat{D}(s) = \begin{bmatrix} -s^2 - 8s - 16 & -s^2 - 8s - 16 \\ s^2 + 8s + 16 & 0 \end{bmatrix}.$$

To determine the left coprime compensator MFD we follow Theorem 6.1. The linear diophantine equation (6.1), namely

$$\begin{aligned} Y(s) \begin{bmatrix} 1 & s \\ 1 & s + 2 \end{bmatrix} + X(s) \begin{bmatrix} s^2 + 5s + 6 & s^2 + 3s + 2 \\ s + 1 & s^3 + 6s^2 + 11s + 6 \end{bmatrix} = \\ = \begin{bmatrix} s^2 + 6s + 9 & s^2 + 6s + 9 \\ 0 & s^3 + 9s^2 + 27s + 27 \end{bmatrix} \end{aligned}$$

is e.g. solved by [3]

$$Y(s) = \begin{bmatrix} s^2 + 3.5s + 2.5 & 3.5s + 7.5 \\ -4s^2 - 19.5s - 23.5 & -5.5s - 5.5 \end{bmatrix}; \quad X(s) = \begin{bmatrix} 0 & -1 \\ 4 & 5 \end{bmatrix}.$$

Now compute

$$V(s) = II[Y(s) \bar{D}^{-1}(s) \hat{D}(s)] = \begin{bmatrix} s^2 + 6.5s + 7 & -s - 4 \\ -4s^2 - 31.5s - 58 & 4s + 30 \end{bmatrix}$$

and use (6.7) and (6.8) to get

$$\begin{aligned} N_C^*(s) &= Y(s) - V(s) \hat{D}^{-1}(s) \bar{D}(s) = \\ &= \begin{bmatrix} -4s^3 - 42s^2 - 144s - 160 & 8s^4 + 118s^3 + 636s^2 + 1472s + 1216 \\ 17s^3 + 168s^2 + 528s + 512 & 45s^4 + 604s^3 + 2958s^2 + 6192s + 4576 \end{bmatrix} \frac{1}{(s+4)^4} \\ D_C^*(s) &= X(s) + V(s) \hat{D}^{-1}(s) \bar{N}(s) = \\ &= \begin{bmatrix} s^4 + 17s^3 + 102s^2 + 256s + 224 & -6s^2 - 48s - 96 \\ -s^3 - 60s^2 - 432s - 832 & s^4 + 19s^3 + 98s^2 + 128s - 96 \end{bmatrix} \frac{1}{(s+4)^4}. \end{aligned}$$

By prime factorization one finally obtains

$$\begin{aligned} N_C(s) &= \begin{bmatrix} -4s - 10 & 8s^2 + 54s + 76 \\ 17s + 32 & 45s^2 + 244s + 286 \end{bmatrix} \\ D_C(s) &= \begin{bmatrix} s^2 + 9s + 14 & -6 \\ -s - 52 & s^2 + 11s - 6 \end{bmatrix} \\ \Delta(s) &= \begin{bmatrix} s^2 + 8s + 16 & 0 \\ 0 & s^2 + 8s + 16 \end{bmatrix} \end{aligned}$$

Obviously the compensator transfer functions related to y_1 are strictly proper so that $p(m - \kappa) = 2$ additional degrees of freedom result, if these transfer functions are allowed to be simply proper.

Modifying

$$V(s) = \begin{bmatrix} s^2 + 6.5s + 7 + \alpha & -s - 4 \\ -4s^2 - 31.5s - 58 + \beta & 4s + 30 \end{bmatrix}$$

one obtains

$$\begin{aligned} N_C^*(s) &= \\ &= \begin{bmatrix} -\alpha s^4 - (4 + 13\alpha) s^3 - (42 + 62\alpha) s^2 - (144 + 128\alpha) s - (160 + 96\alpha) \\ -\beta s^4 + (17 - 13\beta) s^3 + (168 - 62\beta) s^2 + (528 - 128\beta) s + (512 - 96\beta) \end{bmatrix} \\ &\quad \cdot \frac{1}{(s+4)^4} \\ D_C^*(s) &= \begin{bmatrix} s^4 + 17s^3 + (102 + 2\alpha) s^2 + (256 + 16\alpha) s + (224 + 32\alpha) \\ -s^3 - (60 - 2\beta) s^2 - (432 - 16\beta) s - (832 - 32\beta) \\ - (6 - 2\alpha) s^2 - (48 - 16\alpha) s - (96 - 32\alpha) \\ s^4 + 19s^3 + (98 + 2\beta) s^2 + (128 + 16\beta) s + (-96 + 32\beta) \end{bmatrix} \frac{1}{(s+4)^4}. \end{aligned}$$

Choosing $\alpha = 8$ and $\beta = 45$ a compensator with strictly proper transfer functions in the y_2 channel results:

$$N_C(s) = \begin{bmatrix} -8s^2 - 44s - 58 & -2s + 12 \\ -45s^2 - 208s - 238 & -71s - 74 \end{bmatrix}$$

$$D_C(s) = \begin{bmatrix} s^2 + 9s + 30 & 10 \\ -s + 38 & s^2 + 11s + 84 \end{bmatrix}$$

$$A(s) = \begin{bmatrix} s^2 + 8s + 16 & 0 \\ 0 & s^2 + 8s + 16 \end{bmatrix}$$

Obviously the parameters α and β parameterize the negative feedthrough in the y_1 -channel of the compensator and they constitute two additional degrees of freedom to influence closed loop properties. When using Theorem 7.1 instead, an introduction of α and β in the first column of $\bar{V}(s)$ has the same consequences.

9. CONCLUSIONS

Using a new nonminimal representation of reduced order observers in the time domain the direct parameterization of such observers in the frequency domain has been developed. Consequently, the state feedback as well as the observer problems can be formulated in the frequency domain without recurrence to time domain parameters. If the equivalent time domain quantities are of interest, they can be computed from corresponding linear diophantine equations.

Also the doubly coprime factorizations of a transfer matrix related to reduced order observers can now be computed using either the time or the frequency domain representations.

Starting from the parameterizing polynomial matrices $\tilde{D}(s)$ for the controlled plant and $\hat{D}(s)$ for the observer the computation of the resulting observer based compensator has been presented. There are two possible frequency domain descriptions for the closed loop, namely the controller/observer structures. They are based on a right/left coprime plant and a left/right coprime compensator MFD. Both compensator MFDs can be computed by algorithms which simply require the solution of one linear diophantine equation.

When using κ plant outputs together with $n_c = n - \kappa$ observer states for the reconstruction of the plant state x , $p(m - \kappa)$ compensator transfer functions are strictly proper. The transfer functions of a general compensator of $(n - \kappa)$ th order may only be proper and therefore, $p(m - \kappa)$ additional degrees of freedom exist. These additional degrees of freedom can be exploited in the presented design by a modification of the appropriate $p(m - \kappa)$ constant elements in the first $(m - \kappa)$ columns of the polynomial matrices $V(s)$ or $\bar{V}(s)$, respectively.

A simple example was used to demonstrate the proposed design method.

(Received November 2, 1989.)

REFERENCES

- [1] B. D. O. Anderson and V. Kučera: Matrix fraction construction of linear compensators. *IEEE Trans. Automat. Control* *AC-30* (1985), 1112—1114.
- [2] E. Bekir: A unified solution to the singular and nonsingular linear minimum-variance estimation problem. *IEEE Trans. Automat. Control* *AC-33* (1988), 590—591.
- [3] B.-C. Chang: Programs for Solving Polynomial Equations. Technical Report No. 8209, Electrical Engineering Dept., Rice University, Houston, Texas 1982.
- [4] A. Gelb: Applied Optimal Estimation. MIT Press, Cambridge, Mass. 1974.
- [5] P. Hippe: Parameterization of the full-order compensator in the frequency domain. *Internat. J. Control* *48* (1988), 1583—1603.
- [6] P. Hippe: Parameterization of reduced-order compensator in the frequency domain. *Internat. J. Control* *49* (1989), 1143—1162.
- [7] P. Hippe: Design of reduced-order optimal estimators directly in the frequency domain. *Internat. J. Control* *50* (1989), 2599—2614.
- [8] P. Hippe: A nonminimal representation of reduced-order observers. *Automatica* *26* (1990), 405—409.
- [9] J. Ježek and V. Kučera: Efficient algorithm for matrix spectral factorization. *Automatica* *21* (1985), 663—669.
- [10] V. Kučera: Discrete Linear Control: The Polynomial Approach. Wiley-Interscience, New York 1979.
- [11] V. Kučera: New results in state estimation and regulation. *Automatica* *17* (1981), 745—748.
- [12] D. G. Luenberger: An introduction to observers. *IEEE Trans. Automat. Control* *AC-16* (1971), 596—602.
- [13] C. N. Nett, C. A. Jacobson and M. J. Balas: A connection between state-space and doubly coprime fractional representations. *IEEE Trans. Automat. Control* *AC-29* (1984), 831—832.
- [14] H. H. Rosenbrock: State Space and Multivariable Theory. Th. Nelson and Sons Ltd., London 1970.
- [15] U. Shaked and E. Soroka: A simple solution to the singular linear minimum-variance estimation problem. *IEEE Trans. Automat. Control* *AC-32* (1987), 81—84.
- [16] W. A. Wolovich: Frequency domain state feedback and estimation. *Internat. J. Control* *17* (1973), 417—428.
- [17] W. A. Wolovich: Linear Multivariable Systems. Springer-Verlag, New York—Berlin—Heidelberg 1974.

Dr.-Ing. Peter Hippe, Institut für Regelungstechnik, Universität Erlangen-Nürnberg, Cauerstraße 7, W-8520 Erlangen. Federal Republic of Germany.