KYBERNETIKA — VOLUME 26 (1990), NUMBER 2

ESTIMATION OF A CENTRALITY PARAMETER AND RANDOM SAMPLING TIME SCHEMES

Part II. Applications

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An A.R.M.A. stationary process is sampled according to a renewal process. While estimating a centrality parameter and using the criterion as in Part I, we investigate the optimal sampling law's support. We prove that in most of the situations this support is finite, and we give numerical results.

0. INTRODUCTION

We present here some particular results for the estimation of centrality parameters of a real discrete time weakly stationary random process by random sampling time schemes.

In [2] we introduced and studied a criterion which is an equivalent to the asymptotic variance of the chosen estimator, i.e. the random mean. We gave several expressions of the criterion value. We proved a necessary condition for a sampling distribution to be optimal.

In the present work we show that in several particular cases:

i) the correlation function of X is *strictly convex*;

ii) the correlation function of X is *positive*;

iii) the process X is an autoregressive or a moving average process.

It is possible to give sufficient conditions to get an optimal sampling distribution with finite support. We use a simple result; the class of the A.R.M.A. processes is stable by random sampling. This result can be found in [3] and [6].

Finally several numerical results make it possible to calculate the obtained *gains* in the studied cases by random sampling: we compare the asymptotic variance of the estimator computed in a random sampling scheme, with the same estimator computed on the original data, without sampling.

1. DEFINITIONS AND NOTATIONS

In what follows $X = (X_n)_{n \in \mathscr{Z}}$ will be a real discrete weakly stationary stochastic random process with mean θ, covariance function C_X and correlation function ϱ_X.

Let us denote by $T = (t_n)_{n \in \mathscr{N}}$ a renewal process on $\mathscr{N}^* = \mathscr{N} - \{0\}$, stochastically independent of X, with a distribution L. (For more details you can see [2].) The potential measure associated with T is Q_L, and the sampled process, which is also weakly stationary, is denoted $\tilde{X} = (\tilde{X}_n)_{n \geqq 0}$, with $\tilde{X}_n = X_{t_n}$, $n \in \mathscr{N}$.

For the estimation of θ we use N observations the instants of which are random sampled and measured during the development of the process.

We choose $\hat{\theta}(N) = N^{-1} \sum_{n=1}^{N} \tilde{X}_n$ as an estimator and consider the asymptotic quadratic criterion

$$a(L) = \lim_{N \to \infty} (N \operatorname{var} \hat{\theta}(N)), \tag{1}$$

where L belongs to $\mathscr{P}_m$, the set of probability distributions on $\mathscr{N}$ with mean smaller than m, $m \geqq 1$. An optimal sampling distribution L_0 is such that

$$L_0 = \arg \operatorname*{Inf}_{L \in \mathscr{P}_m} a(L), \tag{2}$$

2. MAIN RESULTS

2.1 We will begin by the study of two particular and simple cases, which are direct applications of [2]; let us state the hypothesis again.

H$_1$. There exists $\alpha \in \mathscr{R}_+^*$, $\alpha < a$, $a \in]0, 1[$ such that $\varrho_X(k) = O(a^{|k|})$.

Proposition 2.1. (ϱ_X is strictly convex.) Under H$_1$, if the correlation function ϱ_X is *strictly convex*, there exists *unique* optimal sampling distribution L_0, given by

$$\begin{aligned} L_0 &= \beta \delta_{[m]} + (1 - \beta)\, \delta_{[m]+1} \\ m &= \beta[m] + (1 - \beta)([m] + 1) \end{aligned} \tag{3}$$

where $[m]$ is the integer part of $m \in \mathscr{R}$, $m \geqq 1$.

Proof. We recall (cf. [2]) that $g_L(j) = \partial \langle \varrho_X, Q_L \rangle / \partial L_j$, where $\langle\, , \rangle$ is the scalar product of the duality (l^1, l^∞). Employing the strict convexity of ϱ_X on $\mathscr{N}$ we get

$$\varrho_X(n) - 2\varrho_X(n + 1) + \varrho_X(n + 2) > 0 \quad \text{for all} \quad n \in \mathscr{N}. \tag{4}$$

Using H$_1$ and (4), we see that ϱ_X is strictly decreasing and positive on $\mathscr{N}$. Moreover,

i) using Corollary 3.1, of [2], it is easy to prove that g_{L_0} has the same properties;

ii) the strictly decreasing of g_{L_0} is in contradiction with Corollary 3.2, b of [2], consequently $\sum_{j=1}^{\infty} j\, L_0(j) = m$.

If we prove that supp $L_0 = \{s_1, s_2\}$ with $s_2 = s_1 + 1$, the proposition will be proved since L_0 defined in (3) is unique among the distributions satisfying the constraint.

If $m \in \mathcal{N}^*$, then supp $L_0 = \{s_1\}$.

Suppose supp $L_0 = \{s_1, s_2\}$, with $s_2 > s_1 + 1$. Then

$$g_{L_0}(s_1) + s_1 q + k = 0$$
$$g_{L_0}(s_2) + s_2 q + k = 0$$
$$g_{L_0}(n) + nq + k \geqq 0 \qquad \text{for all} \quad n \in \mathcal{N}^*$$

and, consequently,

$$\frac{g_{L_0}(n) - g_{L_0}(s_1)}{n - s_1} \geqq \frac{g_{L_0}(s_2) - g_{L_0}(s_1)}{s_2 - s_1}$$

Now, under the strict convexity of g_{L_0}, this inequality cannot be fulfilled for any n such that $s_1 < n < s_2$; from which the contradiction follows. □

Remarks 2.1. i) The necessary condition characterizing the optimal distribution L_0 in Theorem 3.1 of [2] is here also sufficient.

ii) The set $\mathcal{P}_m$ is a convex subset of $\mathcal{P}$; if ϱ_X is strictly convex, then L_0 is an *extermal point* of $\mathcal{P}_m$.

iii) Finally we notice that supp L_0 is a finite set; at any case an optimal distribution with a non-finite support would not be useful for applications.

Proposition 2.2. (The correlation function is positive.) Under H_1, if $(\varrho_X(k) > 0, \forall k)$, every optimal sampling distribution L_0:

i) satisfies the constraint $\sum_{j=1}^{\infty} j\, L_0(j) = m$;

ii) has a finite support.

Proof. i) By Corollary 3.1 of [2], $g_{L_0} > 0$. If $\sum_{j=1}^{\infty} j\, L_0(j) < m$ and for every $n \in \mathcal{N}^*$ and $s \in \text{supp } L_0$: $g_{L_0}(n) \geqq g_{L_0}(s)$. Now this is consistent with $\lim_{n\to\infty} g_{L_0}(n) = 0$, only if $g_{L_0}(s) \leqq 0$, and the contradiction follows. □

2.2 Autoregressive and moving average processes

The problem is to estimate the mean θ of a process, belonging to the class of A.R.M.A. (p, q) processes ([1]). We know that the class of A.R.M.A. (p, q) processes is stable by random sampling ([3], [6]).

Now we begin with the characterization of the supports of the optimal sampling distributions, in the cases A.R.(p) and M.A.(q).

2.2.1 *AR (p) processes*

Let X be a real autoregressive process of order $p \in \mathcal{N}^*$, the representation of which written, with the innovation white noise process $(\varepsilon_n)_{n\in\mathcal{Z}}$, with variance $\sigma_\varepsilon^2 = 1$, is

$$X_n + a_1 X_{n-1} + \ldots + a_p X_{n-p} = \varepsilon_n \,. \tag{5}$$

To simplify the proofs we assume that:

$\mathbf{H}_2$. The roots $(\alpha_j)_{1 \leqq j \leqq p}$ of the polynomial $A(z) = z^p + a_1 z^{p-1} + \ldots + a_{p-1}z + a_p$ are distinct, different from zero and located inside the unit disk of $\mathbb{C}$.

Finally, let $R_1 > R_2 \ldots > R_m$, $1 \leqq m \leqq p$, be the distinct modulus of the roots of A and $(G_j)_{1 \leqq j \leqq m}$ the set of roots of A with R_j as modulus. Remember [5] that in this A.R.(p) case

$$\varrho_X(n) = (C_X(0))^{-1} \left(\sum_{i=1}^{p} \lambda_i \alpha_i^n\right) \tag{6}$$

where

$$\lambda_i = \frac{\alpha_i^p}{\prod_{j=1}^{p} (1 - \alpha_i \alpha_j) \prod_{j \neq i} (\alpha_i - \alpha_j)} .$$

Moreover, denoting by Φ_L and $\hat{C}_{\bar{X}}$ the z-transforms of L and $C_{\bar{X}}$ respectively, it is shown in [3] and [4], that

$$\hat{C}_{\bar{X}}(1) = \sum_{i=1}^{p} \lambda_i \frac{1 + \Phi_L(\alpha_i)}{1 - \Phi_L(\alpha_i)} = F(L) . \tag{7}$$

By [3] (Lemma 2.1) we have

$$\langle \varrho_X, Q_L \rangle = \frac{1}{2} \left(\frac{\hat{C}_{\bar{X}}(1)}{C_X(0)} - 1 \right)$$

so that the minimization of $\langle \varrho_X, Q_L \rangle$ can be replaced by the minimization of $F(L)$ defined in (7), for which all the results of ([2] (3.1)) remain true.

Derivating with respect to $L(n)$ and writing $g_L(n)$ for the nth coordinate of the gradient of $C_{\bar{X}}(1)$ we have

$$g_L(n) = \sum_{i=1}^{p} \lambda_i \frac{\alpha_i^n}{[1 - \Phi_L(\alpha_i)]^2} . \tag{8}$$

We can observe that the $(\lambda_i)_{1 \leqq i \leqq p}$ are all different from zero.

We could not prove that every optimal sampling distribution has a finite support. But, by Theorem 2.3 below, we can show that this result is valid in the different cases we study.

Theorem 2.3. Let H_2 hold. If one of the following conditions is satified, every optimal sampling distribution has a finite support:

i) $R_1 \notin G_1$;

ii) $G_1 = \{R_1\}$;

iii) $\forall j \in \{1, 2, \ldots, m\}$: $R_j \in G_j \Rightarrow G_j \subset \mathscr{R}$.

Proof (see the Appendix).

We can assert that there exists an optimal sampling distribution with finite support if:

– there is a positive real root strictly greater than the other,

– in the set of roots of the largest modulus anyone is a real positive one,

– if there is a real positive root in G_j, $1 \leqq j \leqq m$, then there is no complex one.

2.2.2 *M.A.* (q) *processes*

Let $(\varepsilon_n)_{n\in\mathscr{Z}}$ be a white noise and X a process defined by

$$X_n = \varepsilon_n + b_1\varepsilon_{n-1} + \dots + b_q\varepsilon_{n-q}, \quad b_q \neq 0\,, \tag{9}$$

then $$\varrho_X(j) = 0 \quad \text{for all} \quad |j| > q\,. \tag{10}$$

Theorem 2.4. If X is a M.A.(q) process, then:

i) there exists an optimal sampling distribution $L_0 \in \mathscr{P}_m$, the support of which is included into $[1, q+1] \subset \mathscr{N}$;

ii) the distribution L_0 is an element of the boundary of the set $\mathscr{D}_q \subset \mathscr{R}^q$ defined by

$$\mathscr{D}_q = \{(x_1, \dots, x_q) \in \mathscr{R}^q \mid x_i \geqq 0\,, \quad 1 \leqq i \leqq q\,, \quad \sum_{i=1}^{q} x_i \leqq 1\}\,. \tag{11}$$

Proof. i) Let $L \in \mathscr{P}_m$; define $L' \in \mathscr{P}_m$ by:

$$* \; L'_j = L_j\,, \quad \forall j \leqq q\,,$$

$$* \; L'_{q+1} = \sum_{j=q+1}^{\infty} L_j\,,$$

$$* \; L_k = 0\,, \quad \forall k > q+1\,.$$

The values of $Q_L(\tau)$, $\tau \in \mathscr{N}$, where Q_L is the potential measure of L, are a polynomial function of $L_1, L_2, \dots, L_\tau$, consequently

$$Q_L(j) = Q_{L'}(j) \quad \text{for all} \quad j \leqq q$$

and by (10)

$$F(L) = \sum_{j=1}^{q} \varrho_X(j)\, Q_L(j)\,,$$

consequently: $F(L) = F(L')$, and if L is an optimal sampling distribution, and L' the support of which is included into $[1, q+1] \subset \mathscr{N}$ is another one.

ii) Let x be in the interior of $\mathscr{D}_q$. Then

$$\sum_{i=1}^{q} x_i < 1\,, \quad x_i > 0\,, \quad 1 \leqq i \leqq q\,. \tag{12}$$

For such an element to be optimal it is necessary (cf. [2] (15)) that the gradient vector of $F(g_x(k))_{k\in\mathscr{N}^*}$ equals zero. But, here that gradient can be written:

$$\left(\sum_{h=0}^{q-1} \varrho_X(h+1)\, a(h), \sum_{h=0}^{q-2} \varrho_X(h+2)\, a(h), \dots, \varrho_X(q-1) + \varrho_X(q)\, a_1, \varrho_X(q)\right) \in \mathscr{R}^q$$

with $a(h)$ defined as in [2], Corollary 3.1. Consequently by (12), if the gradient of F equals zero then the same is true for all $\varrho_X(j)$'s; there is a contradiction with (9) and (10) and the theorem is proved. □

3. COMPLEMENTARY STUDIES FOR PARTICULAR CASE

We will now give some details about optimal sampling schemes for A.R.(p) and M.A.(q) processes with p and q equal to 1 or 2. In this context, we also give numerical results in order to evaluate the gains procured by the random sampling schemes.

3.1 A.R.(1) and A.R.(2) processes

Proposition 3.1. Let X be a real A.R.(1) process $X_{n+1} = \varrho X_n + \varepsilon_n$, $|\varrho| < 1$, $n \in \mathscr{Z}$.
i) If $\varrho > 0$, $L_0 = [1 - (m - [m])] \delta_{[m]} + (m - [m]) \delta_{[m]+1}$;
ii) If $\varrho < 0$, $L_0 = \delta_1$ with δ_a the Dirac distribution on $a \in \mathscr{R}$.
Proof. i) is a direct outcome of [2], Theorem 3.2;
ii) means that we do not use random sampling scheme, and is easy to establish. □

Remark 3.1. Observe that there is a difference of results according to the algebraical sign of ϱ. The part ii) gives the same results as [8], by random sampling of the observation instants, for continuous time similar processes, where obviously the only case $\varrho < 0$ was studied.

Proposition 3.2. Let X be an A.R.(2) process, the roots of $A(z)$ being distinct and satisfying H_2. Then Card (supp L_0) $\leqq 3$ and if the roots are positive, card . . (supp L_0) $= 2$.
Proof. See [4]. □

3.2 M.A.(1) and M.A.(2) processes

Proposition 3.3. a) Let us assume that X is a M.A.(1) process:
– if $\varrho_X(1) < 0$ then $L_0 = \delta_1$,
– if $\varrho_X(1) > 0$ then $L_0 = \delta_2$.
b) Let us assume that X is a M.A.(2) process:
– if $\varrho_X(k) \geqq 0$ $k = 1, 2$, then $L_0 = \delta_3$,
– if $\varrho_X(1) \leqq 0$, $\varrho_X(2) \geqq 0$
  if $2\varrho_X(2) + \varrho_X(1) < 0$, then $L_0 = \delta_1$,
  if $2\varrho_X(2) + \varrho_X(1) > 0$, then $L_0(1) = \alpha$, $L_0(3) = 1 - \alpha$,
  $\alpha \in]0, 1[$, $L_0(j) = 0$, $j \notin \{1, 3\}$,
– if $\varrho_X(k) \leqq 0$, $k = 1, 2$, then $L_0 = \delta_1$,
– if $\varrho_X(1) \geqq 0$; $\varrho_X(2) \leqq 0$, then $L_0 = \delta_2$.
Proof. See [4]; different other proofs are presented for other particular cases. □

4. NUMERICAL RESULTS ANALYSIS

The analysis of numerical results allows us to draw some conclusions in addition to the previous theoretical results. If we call δ_1 the Dirac distribution at the point $n = 1$ the sampling distribution $L = \delta_1$ is such that $\tilde{X} = X$.

We measure the gain of a random sampling scheme by reference to $F(\delta_1)$, which is the equivalent to the variance of $\hat{\theta}_n$ without sampling. The gain for the optimal distribution L_0 is computed for several A.R.M.A.(p, q); it is called $G(L_0)$ and expressed in percentage by

$$G(L_0) = \frac{F(\delta_1) - F(L_0)}{F(\delta_1)} \times 100 . \tag{13}$$

4.1 A.R.(1) models

Let $(\varepsilon_n)_{n\in\mathcal{N}}$ be a white noise with variance $\sigma_\varepsilon^2 = 1$ and X a process defined by $X_{n+1} = \alpha X_n + \varepsilon_{n+1}$, $n \in \mathcal{Z}$, $\alpha \in [0, 1[$.

i) By the results of the paragraph 3.2 and (8), if we suppose that $m = 5, 3$ (i.e. a sampling rate higher than 18%) we have

$$L_0 = 0{\cdot}7\delta_5 + 0{\cdot}3\delta_6 \quad \text{(Prop. 3.1)}$$

$$G(L_0) = 100 \times \left[1 - \frac{1+\Phi}{1-\Phi} \times \frac{1-\alpha}{1+\alpha}\right], \quad \Phi = \Phi_{L_0}(\alpha) = 0{\cdot}7\alpha^5 + 0{\cdot}3\alpha^6 .$$

α	0·00	0·15	0·25	0·40	0·60	0·70	0·80	0·90	0·95	0·99
$G(L_0)$ %	9·52	26·08	39·91	56·41	71·32	75·98	79·00	80·62	81·00	81·12

$m = 5{\cdot}3$

ii) Now let $\alpha = 0{\cdot}5$ and look at results obtained for several values of m:

m	1·3	2·3	3·3	5·3	7·3	9·3
$G(L_0)$	17·4	48·6	58·7	64·8	66·2	66·5

$\alpha = 0{\cdot}5$

$$L_0 = (1 - (m - [m]))\,\delta_{[m]} + (m - [m])\,\delta_{[m]+1} .$$

A conclusion is: as soon as the algebraic sign is known we can take the decision to sample or not to sample; in addition it seems absurd not to take advantage of a sampling scheme if we suspect α to be higher than 0·5.

4.2 M.A.(1) models

Let the process X be defined by

$$X_n = \varepsilon_n + a\varepsilon_{n-1}, \quad n \in \mathcal{Z} .$$

We recall that, if $\varrho_X(1) < 0$, then $L_0 = \delta_1$ and if $\varrho(1) > 0$ then $L_0 = \delta_2$.

a	0·01	0·05	0·1	0·2	0·3	0·5
$G(L_0)$	1·96	9·07	16·52	27·77	35·50	44·4

$m = 5{\cdot}3$

The gains remain important, but they seem to be lower than in the A.R.(1) cases.

4.3 More general models

For higher order A.R.M.A. models, the optimal sampling distributions are often impossible to predict by the investigation of the roots or poles of the models. The correlation may in some singular cases help us, for example when ϱ_X is positive. Here are the gains always important, as it can be seen in Table 1, where optimal sampling distributions and gains are given in terms of the poles of the models.

Optimal sampling distribution and gains for A.R.(2) models

$$x_n + a_1 x_{n-1} + a_2 x_{n-2} = \varepsilon_n$$

$$m = 5{\cdot}3 \quad \lambda_1 = \frac{\alpha_1(1 - \alpha_2^2)}{(\alpha_1 - \alpha_2)(1 - \alpha_1\alpha_2)} \quad \lambda_2 = -\frac{\alpha_2(1 - \alpha_1^2)}{(\alpha_1 - \alpha_2)(1 - \alpha_1\alpha_2)};$$

α_1 and α_2 roots of $A(z) = z^2 + a_1 z + a_2$.

α_1^{-1} and α_2^{-1}		$F(\delta_1)$	L_0	$F(L_0)$ $L_0 \neq \delta_1$	$100\,\frac{F(\delta_1) - F(L_0)}{F(\delta_1)}$
1·5	2	**1·16**	$0{\cdot}85\delta_5 + 0{\cdot}15\delta_7$	0·078	66%
−1·5	2	0·1	δ_3	−0·096	32%
−2	2	0·333	$0{\cdot}85\delta_5 + 0{\cdot}15\delta_7$	0·0008	39%
2	10	0·856	$0{\cdot}7\ \delta_5 + 0{\cdot}3\ \delta_6$	0·0237	61%
−2	10	−0·275	δ_6		0%
−3 ±	4i	−0·211	δ_1		0%
3 ±	4i	0·238	δ_3	−0·005	33%

The computer program gives a local optimum, sometimes function of initial condition of the algorithm. We have chosen, for each example, nine different initial conditions.

APPENDIX

Proof of Theorem 2.3

By (8): $g_L(n) = \sum_{j=1}^{m} R_j^m\, \gamma_j(n)$; where, if we denote by $\operatorname{Arg}\alpha_k$ the argument of $\alpha_k \in \mathbb{C}$,

$$\gamma_j(n) = \sum_{\{k \mid \alpha_k \in G_j\}} \frac{\lambda_k\, e^{-in \operatorname{Arg}\alpha_k}}{(1 - \Phi_L(\alpha_k))^2}. \tag{A1}$$

Moreover, by [2], Corollary 3.2, if L_0 minimizes $\langle \varrho, Q_{L_0} \rangle$ and if the support of L_0, denoted by $\{n_k;\ k \in \mathcal{N}^*\}$, is not finite: $g_{L_0}(n) \geqq 0$ for all $n \geqq 1$ and $g_{L_0}(n_k) = 0$ for all $k \in \mathcal{N}^*$. Consequently, if we set:

$$\gamma(n) = \frac{g_{L_0}(n)}{R_1^n} \quad \text{for all} \quad n \in \mathcal{N}^* \tag{A2}$$

we have

$$\gamma(n) = \gamma_1(n) + \sum_{j=2}^{m} \left(\frac{R_j}{R_1}\right)^n \gamma_j(n) \geqq 0 \quad \text{for all } n \in \mathcal{N}^*, \tag{A3}$$

and

$$\gamma(n_k) = 0 \quad \text{for all} \quad k \in \mathcal{N}^* . \tag{A4}$$

The $\gamma_j(n)$ being bounded, we deduce from A3 that:

$$\lim_{k \to \infty} \gamma_1(n_k) = 0 . \tag{A5}$$

We easily prove that the entire series the general term of which is $\gamma(n)\, z^n$, converges inside the unit disk of $\mathbb{C}$; moreover $\gamma(n) \geqq 0$ for all n, consequently $z = 1$ is a singular point of $f(z) = \sum_{j=0}^{\infty} \gamma(n)\, z^n$ ([7]).

i) Suppose $R_1 \notin G_1$, then we can write:

$$f(z) = \sum_{\{k \mid \alpha_k \in G_1\}} \left\{ \frac{\lambda_k}{1 - \Phi_{L_0}(\alpha_k)} \frac{1}{1 - z\, \mathrm{e}^{-\mathrm{i}\,\mathrm{Arg}\alpha_k}} \right\} + \\ + \sum_{j=2}^{n} \left\{ \sum_{\{k \mid \alpha_k \in G_j\}} \frac{\lambda_k}{1 - \Phi_L(\alpha_k)} \frac{1}{1 - z(R_j/R_1)\, \mathrm{e}^{\mathrm{i}\,\mathrm{Arg}\alpha_k}} \right\}$$

and $\mathrm{Arg}\, \alpha_k \neq 0$ for $\alpha_k \in G_1$, $z = 1$ is not a singular point of f, consequently there is a contradiction and L_0 has a finite support.

i) Suppose $G_1 = \{R_1\}$. then

$$\gamma_1(n) = \frac{\lambda_1}{1 - \Phi_{L_0}(R_1)}$$

and $\gamma_1(n)$ is a non-zero constant, for all $n \geqq 1$ and there is a contradiction with (A5) consequently L_0 has a finite support.

iii) We have to prove that: if $R \in G$, then if the support of L_0 is finite, $-R_j$ is the only other possible element of G_j. Let us study the case where $G_1 = \{R_1, -R_1\}$, then

$$\gamma_1(n) = \frac{\lambda_1}{(1 - \Phi_{L_0}(R_1))^2} + \frac{(-1)^n \lambda_2}{(1 - \Phi_{L_0}(-R_1))^2} . \tag{A6}$$

It is impossible for the support of L_0 to contain a non-finite subset of natural numbers of the two evenness, because we would have $\lambda_1 = \lambda_2 = 0$.

Let us suppose, for example, that the support of L_0 contains only even natural numbers larger than some $k_0 \in \mathcal{N}$. By (A5) we deduce that:

$$\gamma_1(2p) = \frac{\lambda_1}{[1 - \Phi_{L_0}(R_1)]^2} + \frac{\lambda_2}{[1 - \Phi_{L_0}(-R_1)]^2} = 0 \quad \text{for all} \quad p \in \mathcal{N}^* .$$

By (A3), the conclusion is:

$$\sum_{j=2}^{n} \lambda_j(2p) \left[\frac{R_j}{R_1}\right]^{2p} \geqq 0 \quad \text{for all} \quad p \in \mathcal{N}^* .$$

By the same argument as that used in i) applied to the series of general term $\gamma(n)\, z^n$

with

$$\gamma(n) = \begin{cases} 0 & \text{if} \quad n = 2p + 1 \\ \gamma_2(2p) + \sum_{j=3}^{m} \gamma_j(2p) \left[\dfrac{R_j}{R_2}\right]^{2p} & \text{if} \quad n = 2p \end{cases}$$

and if $R_2 \notin G_2$, we conclude that L_0 has a finite support, then, as in i) we prove that if $G_2 = \{R_2\}$, L_0 also has a finite support. By the previous argument on even natural numbers, we prove that the only case for which the support of L_0 may be infinite is that in which

$$G_j = \{R_j, -R_j\} \quad \text{for all} \quad j \in [1, m] \subset \mathcal{N}. \tag{A7}$$

So it is necessary to employ (A6) to verify that the support of L_0 is finite.

When (A7) is verified, we easily compute λ_i; we know that $G_1 = \{R_1, -R_1\}$ and deduce that $\lambda_1 = \lambda_2 \neq 0$, in contradiction with (A6). Consequently, the support of L_0 does not contain infinitely many even natural numbers.

Then, suppose that the support of L_0 contains only odd natural numbers after $k_0 \in \mathcal{N}$. The convergence towards zero of $\gamma_1(n_k)$ implies that

$$\gamma_1(2p + 1) = \frac{\lambda_1}{(1 - \Phi_{L_0}(R_1))^2} - \frac{\lambda_2}{(1 - \Phi_{L_0}(-R_1))^2}. \tag{A8}$$

By the same argument as in 1), we deduce that the only case where L_0 may be optimal is the case in which (A7) is satisfied. Then for $\lambda_1 = \lambda_2 \neq 0$, (A8) is equivalent to $\Phi_{L_0}(R_1) = \Phi_{L_0}(-R_1)$ that is possible, if $R_1 \neq 0$, only if all the odd natural numbers do not belong to the support of L_0. The theorem is proved. □

(Received April 6, 1988.)

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