THE ORDER OF NODE REMOVALS IN INFLUENCE DIAGRAM

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Shachter [1] has shown that in order to solve the inference problem in influence diagram, some nodes are to be removed in any order. The problem of efficient order of this set was left open. This paper shows one possible way to order the removal of these nodes. The nature of this order is to obtain the minimal arc addition after removing a node. The number of arc reversals in the next steps will be lowered and one can obtain the solution with fewer arc reversals. Further, the topology order of influence diagram seems to be useful from the implementation point of view.

1. INTRODUCTION: BACKGROUND

Recent research indicates that influence diagrams are effective for representing decision knowledge in expert systems. They are a powerful communication tool for the participants in the decision process. The advantages of using influence diagram as a probabilistic model for uncertainty handling are shown in [1] and [2].

An influence diagram is a network consisting of an acyclic directed graph, associated node sample spaces and conditional probability distributions.

The random variables and their corresponding sample spaces are denoted as $x_1, \ldots, x_n$ respectively $\Omega_i, \ldots, \Omega_n$. Let N be the set $\{1, \ldots, n\}$ and $C(i) \subset N$ be the indices of the random variables which condition x_i. The nodes in $C(i)$ are called the direct predecessors of the node i in influence diagram.

For any subset $J \subset N$ we define:

$$C(J) = \bigcup_{j \in J} C(j)$$

$$C^{-1}(J) = \{i \in N \mid J \cap C(i) \neq \emptyset\}$$

These nodes are known as the direct predecessors and direct successors of the nodes in J.

The general inference problem is to find $P\{f(x_J) \mid x_K\}$ where J and K are arbitrary subsets of N and f is an arbitrary measurable function on Ω_J.

In [1] it was shown that in order to solve the general inference problem, it was necessary to create a new variable $x_0 = f(x_J)$ with conditional variables x_J and remove all variables except x_0 and x_K in any order. The desired expression is the resulting conditional probability distribution for x_0.

The node removal theorem states that any node in an influence diagram may be removed from the diagram. First, order its successors, and reverse the arcs from the node to each successor in the order.

The arc reversal theorem says that in an influence diagram containing an arc from i to j but no other directed path from i to j, it is possible to transform the diagram to one with an arc from j to i instead, and in the new diagram both i and j inherit each other's direct predecessors. The new conditional probability distributions for x_j and x_i are given as:

$$P\{x_j \mid x_{C(i) \cup C(j) - \{i\}}\} = \sum_{\Omega_i} \pi(x_j \mid x_{C(j)})\, \pi(x_i \mid x_{C(i)})$$

$$P\{x_i \mid x_{\{j\} \cup C(i) \cup C(j) - \{i\}}\} = \frac{\pi(x_j \mid x_{C(j)})\, \pi(x_i \mid x_{C(i)})}{P\{x_j \mid x_{C(i) \cup C(j) - \{i\}}\}}\,.$$

Shachter [1] showed an algorithm to calculate the set $R(J, K)$ of nodes that will have to be removed to solve the inference problem $P\{f(x_J) \mid x_K\}$. And the nodes to be ignored are in the set $N - R(J, K) - K$.

Although it is possible to remove the nodes $R(J, K)$ in any order, some orders may be more efficient than others.

In Section 2 an advantage of topology order is shown and in Section 3 the problem of the order of node removals will be discussed.

2. TOPOLOGY ORDER OF NODES

It is well known from graph theory that a directed graph is acyclic if and only if there is an order of the nodes such that any successor of a node in the graph follows it in the order as well. Such orders are called topology orders of the graph.

Kučera [3] describes an algorithm for topology order of a graph with complexity $O(n + m)$, where n is the number of nodes and m is the number of arcs in the graph.

It is not difficult to prove the following proposition.

Proposition 1. In order to remove any node in an influence diagram with topology order, it is possible to reverse all the arcs from the node gradually in the topology order, and after removing the node, the topology order to the new diagram remains.

Proof. The nodes of the influence diagram can be ordered in such a way that all of the arcs are oriented to one direction (say from left to right).

Let i be an arbitrary node in the diagram and $j_1, \ldots, j_k$ its direct successors in topology order. After reversing the arcs $(i, j_1), \ldots, (i, j_{r-1})$ there is no directed path from i to j_r except the arc (i, j_r), because the set of direct successors of i in the new diagram is $\{j_r, \ldots, j_k\}$, which is in topology order.

After reversing the last arc (i, j_k), all the arcs orientating from right to left have the node i as their direct successor. Thus after removing the node i, all the arcs in the new diagram are orientated from left to right, i.e. the topology order remains. $\square$

3. THE ORDER OF NODE REMOVALS

First we consider the problem of arc addition after removing a node. Let i be an arbitrary node in the diagram. Our goal is to compute the number of added arcs after removing i.

Suppose the direct successors of i, denoted by $j_1, \ldots, j_k$, are in topology order, so that we can gradually reverse the arcs $(i, j_1), \ldots, (i, j_k)$.

Let $C'(j_r)$, $C_r(i)$ be the new sets of direct predecessors of the node j_r (respectively i) after reversing the arcs $(i, j_1), \ldots, (i, j_r)$, $1 \leqq r \leqq k$.

Theorem 1. The sets $C'(j_r)$, $C_r(i)$ are given as:

$$C'(j_r) = \{j_1, \ldots, j_{r-1}\} \cup C(i) \cup C^-(\{j_1, \ldots, j_r\}) \tag{1}$$

$$C_r(i) = \{j_1, \ldots, j_r\} \cup C(i) \cup C^-(\{j_1, \ldots, j_r\}) \tag{2}$$

(where $C^-(X) = C(X) - \{i\}$ for any subset $X \subset N$).

Proof. (By mathematical induction on r.)

For $r = 1$, using the arc reversal theorem:

$$C'(j_1) = C(i) \cup C^-(j_1)$$

$$C_1(i) = \{j_1\} \cup C(i) \cup C^-(j_1)$$

Suppose we have reversed the arcs $(i, j_1), \ldots, (i, j_r)$, the new sets of direct predecessors of j_r and i are given by (1), (2). After reversing the arc (i, j_{r+1}), by the arc reversal theorem, we obtain:

$$C'(j_{r+1}) = C_r(i) \cup C^-(j_{r+1}) = \{j_1, \ldots, j_r) \cup C(i) \cup C^-(\{j_1, \ldots, j_r\}) \cup C^-(j_{r+1}) =$$
$$= \{j_1, \ldots, j_r\} \cup C(i) \cup C^-(\{j_1, \ldots, j_r, j_{r+1}\})$$
$$C_{r+1}(i) = \{j_{r+1}\} \cup C_r(i) \cup C^-(j_{r+1}) =$$
$$= \{j_1, \ldots, j_{r+1}\} \cup C(i) \cup C^-(\{j_1, \ldots, j_{r+1}\}) \qquad \square$$

Theorem 2. After removing the node i, the number of added arcs is given as:

$$A(i) = \sum_{r=1}^{k} (|C_r(i)| - |C(j_r)|) - |C(i)| - k\,, \tag{3}$$

which satisfies the relation:

$$\frac{k(k-3)}{2} - |C(i)| \leqq A(i) \leqq \frac{k(k-3)}{2} + (k-1)\,(|C(i)| + |C(C^{-1}(i))|) \tag{4}$$

Proof. Before reversing the first arc, the number of arcs is

$$F(i) = \sum_{r=1}^{k} |C(j_r)| + |C(i)|$$

After reversing the last arc and removing the node i we have:

$$L(i) = \sum_{r=1}^{k} |C'(j_r)| = \sum_{r=1}^{k} |C_r(i)| - k$$

Thus the number of added arcs is:

$$A(i) = L(i) - F(i) = \sum_{r=1}^{k} (|C_r(i)| - |C(j_r)|) - |C(i)| - k$$

From (2) and the definition of $C^-(j_r)$ we have:

$$(5) \qquad |C_r(i)| - |C(j_r)| = |\{j_1, \ldots, j_r\} \cup C(i) \cup C^-(\{j_1, \ldots, j_r\})| - |C^-(j_r)| - 1$$
$$= r + |C(i) \cup C^-(\{j_1, \ldots, j_r\})| - |C^-(j_r)| - 1$$

Since

$$|C^-(j_r)| \leqq |C(i) \cup C^-(\{j_1, \ldots, j_r\})| \leqq |C(i)| + |C^-(j_1, \ldots, j_r)|$$

from (5) we get

$$r - 1 \leqq |C_r(i)| - |C(j_r)| \leqq r - 1 + |C(i)| + |C^-(j_1, \ldots, j_r)| - C^-(j_r)|$$

Thus

$$A(i) \geqq \frac{k(k-1)}{2} - |C(i)| - k = \frac{k(k-3)}{2} - |C(i)|$$

$$(6) \quad A(i) \leqq \frac{k(k-3)}{2} + (k-1)\,|C(i)| + \sum_{r=1}^{k} (|C^-)(j_1, \ldots, j_r)| - |C^-(j_r)|) \leqq$$
$$\leqq \frac{k(k-3)}{2} + (k-1)\,(|C(i)| + |C(C^{-1}(i))|)$$

(In inequality (6) have used the relation:

$$|C^-(\{j_1, \ldots, j_r\})| - |C^-(j_r)| \begin{cases} = 0 & \text{for } r = 1 \\ \leqq |C(C^{-1}(i))| & \text{for } r > 1) \end{cases}$$ □

Now consider a special case, when the node i has only one direct successor, denoted by j. The previous theorem gives:

$$A(i) = |C_1(i)| - |C(j)| - |C(i)| - 1 = |C(i) - C(j)| - |C(i)| - 1$$

and

$$-1 - |C(i)| \leqq A(i) \leqq -1$$

Thus after removing i, the number of arcs in the new diagram decreases. Such nodes will be the first ones to be removed in solving the inference problem.

The set $R(J, K)$ can now be ordered according to the increasing values of $A(i)$, and the process of node removal may be realized in the given sequence.

Another procedure for the order of node removal is following: find some node $i \in R(J, K)$ with property $A(i) = \min A(j)$, $j \in R(J, K)$; get the node i from $R(J, K)$ and remove the node i from the diagram; repeat finding with the current values of $A(j)$ until $R(J, K) = \emptyset$.

4. CONCLUSION

We have investigated the problem of arc addition in influence diagram as a consequence of arc reversal and node removal. The results may be used to derive an effective order of node removals in solving the general inference problem.

It is necessary to remark that the effectivity of the order of node removals can also depend on the size of node sample spaces and their conditional distributions. The case of the same space dimension (e.g. all nodes are binary) was considered in this paper.

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