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ON THE DIRECTABILITY OF AUTOMATA

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We present some partial results on the hypothesis due to Černý [1] and a necessary and sufficient condition for the directability of an automaton.

1. THE DIRECTABILITY OF AUTOMATA

Let $\mathscr{A} = (A, \Sigma, \delta)$ be a finite automaton, where A is the finite set of states, Σ is the finite set of input signals and $\delta\colon A \times \Sigma \to A$ is the transition function. This function can be extended to the set $A \times \Sigma^*$, where Σ^* is the set of all words over Σ, and it defines for every $s \in \Sigma^*$ a mapping

$$s^{\mathscr{A}}\colon A \to A\,, \quad a \mapsto as^{\mathscr{A}} = \delta(a, s)\,.$$

For every $B \subseteq A$, $|B|$ will designate the number of elements in B and $B\Sigma^*$ is the set $\{bw^{\mathscr{A}} \mid b \in B,\ w \in \Sigma^*\}$. An automaton $\mathscr{A} = (A, \Sigma, \delta)$ is strongly connected if for every state $a \in A$, $a\Sigma^* = A$.

An automaton $\mathscr{A}$ is directable if there exists a word $s \in \Sigma^*$, called a directing word, and a state $c \in A$ such that $As^{\mathscr{A}} = \{c\}$. Then $\cap\,(a\Sigma^* \mid a \in A) \neq \emptyset$. This is the smallest subautomaton of $\mathscr{A}$ and also the unique strongly connected subautomaton of $\mathscr{A}$. $C(\mathscr{A})$ or $(C(A), \Sigma, \delta)$ will designate this subautomaton and we shall call $C(\mathscr{A})$, and also $C(A)$, the centre of $\mathscr{A}$. If there exists a word $t \in \Sigma^*$ such that $At^{\mathscr{A}} \subseteq C(A)$, we shall call $\mathscr{A}$ semidirectable and t a semidirecting word of $\mathscr{A}$. Let $\mathscr{S}$ be the class of all semidirectable automata and $\mathscr{D}$ the class of all directable automata.

Theorem 1.1. $\mathscr{A} \in \mathscr{D} \Leftrightarrow \mathscr{A} \in \mathscr{S}$ and $C(\mathscr{A}) \in \mathscr{D}$.

Proof. If $As^{\mathscr{A}} = \{c\}$ for some $s \in \Sigma^*$, $c \in A$, then $c \in \cap(a\Sigma^* \mid a \in A) = C(A)$ and $\mathscr{A} \in \mathscr{S}$. Naturally $C(\mathscr{A}) \in \mathscr{D}$.

If $At^{\mathscr{A}} \subseteq C(A)$ and $C(A)\,s^{\mathscr{A}} = \{c\}$ for some $t, s \in \Sigma^*$, then $Ats^{\mathscr{A}} = \{c\}$ and $\mathscr{A} \in \mathscr{D}$. □

Remark 1.1. When $\mathscr{A} \in \mathscr{D}$ then for every state $c \in C(A)$ there exists a word $s_c \in \Sigma^*$ such that $As_c^{\mathscr{A}} = \{c\}$ and for every directing word s of $\mathscr{A}$, $As^{\mathscr{A}} \in C(A)$.

Let $\mathscr{A} = (A, \Sigma, \delta)$ be a semidirectable automaton with n states. Then $C(A) = \cap (a\Sigma^* \mid a \in A)$ and for every $a \in A$ there exists $s_a \in \Sigma^*$ such that $as_a^{\mathscr{A}} \in C(A)$. From these words s_a we construct a semidirecting word of $\mathscr{A}$.

If $C(A) = A$, then every word is semidirecting.

Let $C(A) \neq A$, $a \in A \setminus C(A)$ and $s_a \in \Sigma^*$ such that $as_a^{\mathscr{A}} \in C(A)$. Then

$$|As_a^{\mathscr{A}} \cap (A \setminus C(A))| < |A \setminus C(A)| .$$

If $As_a^{\mathscr{A}} \cap (A \setminus C(A)) \neq \emptyset$, we repeat this procedure until we get such words $s_a, s_b, \dots \dots, s_w$, $s \in \Sigma^*$ that $s = s_a s_b \dots s_w$ and $As^{\mathscr{A}} \subseteq C(A)$.

Therefore Theorem 1.1 has

Corollary 1.1. A finite automaton $\mathscr{A}$ is directable iff it has the smallest subautomaton, the centre, which is directable.

2. THE HYPOTHESIS OF ČERNÝ

Let $l(\mathscr{A})$ be the length of the shortest directing word of $\mathscr{A} = (A, \Sigma, \delta) \in \mathscr{D}$ and $D(n, m)$ the class

$$\{\mathscr{A} \in \mathscr{D} \mid |A| = n , \quad |C(A)| = m\} .$$

Let

$$l(n) = \max (l(\mathscr{A}) \mid \mathscr{A} \in D(n, m) , \quad 1 \leqq m \leqq n) .$$

In [1] Černý has presented the following hypothesis.

Černý's hypothesis. $l(n) = (n - 1)^2$, $n \in \mathbb{N}$.

In [2] Černý, Pirická and Rosenauerová have proved the hypothesis for $n \leqq 5$. By Corollary 1.1. we sharpen this hypothesis.

If $m = n$, then $l(\mathscr{A}) = l(C(\mathscr{A}))$.

Let $m < n$, s be the semidirecting word that we can get by repeating the procedure presented in the proof of Corollary 1.1. by choosing every state c and every word s_c such that the word s_c is so short than possible, and $lg(s)$ be the length of the word s. Since $|A \setminus C(A)| = n - m$, we find that $lg(s) \leqq \sum_{i=0}^{n-m} i$.

Theorem 2.1. Let $\mathscr{A} \in D(n, m)$, $m, n \in \mathbb{N}$. Then

$$l(\mathscr{A}) \leqq \sum_{i=0}^{n-m} i + l(C(\mathscr{A})) .$$

The sum $\sum_{i=0}^{n-m} i$ is better upper bound than $(n - m)^2$ that one can get from the conjecture presented by Pin [5].

Corollary 2.1. If an automaton $\mathscr{A} \in D(n, m)$, $m, n \in \mathbb{N}$, fulfils the condition

$$l(C(\mathscr{A})) \leqq (m - 1)^2 ,$$

then

$$l(\mathscr{A}) \leqq (n-1)^2 .$$

Especially

$$l(\mathscr{A}) < (n-1)^2 \quad \text{for all} \quad n > 2 , \quad n \neq m .$$

Proof. When $m \neq n$, $n > 2$, then $l(\mathscr{A}) \leqq \sum_{i=0}^{n-m} i + (m-1)^2 < (n-m)(n-1) + (m-1)(n-1) = (n-1)^2$. □

Now also the first claim is obvious.

Since Černý's hypothesis was proved in [2] for automata $\mathscr{A} \in D(n, m)$, $n \leqq 5$, we get

Corollary 2.2. For all automata $\mathscr{A} \in D(n, m)$, $m \leqq 5$,

$$l(\mathscr{A}) \leqq (n-1)^2 .$$

Remark 2.1. If Černý's hypothesis is valid for strongly connected directable automata, then the upper bound $(n-1)^2$ presented by Černý can be sharpened for all directable automata with centre $C(A) \neq A$, where $|A| > 2$.

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