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DECENTRALIZED CONTROL OF LINEAR DYNAMICAL SYSTEMS WITH PARTIAL AGGREGATION

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A new method for designing the decentralized control of linear dynamical systems with continuous or numerically controlled stations is presented. The proposed method is intended for the use in designing the suboptimal decentralized control via solving the L-Q problem with output feedback for one subsystem under the assumption that the remaining part of the system is stable. Since the problem can be very complex a special partial aggregation of the mathematical model minimizing the interaction between the considered subsystem and the other part of the system is proposed. The design procedure is demonstrated on an example.

1. INTRODUCTION

One of the problems occurring in connection with large scale systems is the design of a decentralized control station using the output feedback only from one subsystem, supposing that the other part of the system being stable. This problem can be solved by optimizing a given objective function while stability of the overall system has to be ensured. As one of many examples the design of an excitation controller of a synchronous generator integrated into a system of N generators can be mentioned.

Two approaches have been adopted so far:

1. Design of suboptimal output feedback control taking the whole high order mathematical model of the system into the consideration.

2. Design of the control for the isolated subsystem and checking the stability of the global system [1, 6].

In this paper a new approach is developed. It is based on the use of decentralized control design procedures for constructing an optimal output feedback control scheme using a partial aggregated mathematical model of the system [5]. An essential advantage of the proposed method is that it enables to design suboptimal decentralized control for the chosen subsystem with the remaining part of the system being aggregated. Although the properties of the basic system are retained, the aggregation of a part of the system considerably decreases the computation burden. This method is also effective in the special case when the whole system to be controlled is sequentially built by adding subsystems to the first one and the suboptimal behaviour of individual portions of the technological process have to be assured. The proposed solution guarantees the stability and suboptimal control of the whole dynamical system.

At first, the approach is presented for linear discrete-time systems and then the main results are summarized for linear continuous systems in the concluding remarks.

2. PROBLEM STATEMENT

Let us suppose that a large-scale system can be divided into two subsystems. The first is to be controlled, while the second is supposed to be stable. Let us consider a linear time-invariant system $\mathscr{S}$:

$$\begin{aligned} \mathscr{S}:\quad \boldsymbol{x}_1(t+1) &= \boldsymbol{A}_{11}\,\boldsymbol{x}_1(t) + \boldsymbol{A}_{12}\,\boldsymbol{x}_2(t) + \boldsymbol{B}_{11}\,\boldsymbol{u}_1(t)\,, \\ \boldsymbol{x}_2(t+1) &= \boldsymbol{A}_{21}\,\boldsymbol{x}_1(t) + \boldsymbol{A}_{22}\,\boldsymbol{x}_2(t) + \boldsymbol{B}_{21}\,\boldsymbol{u}_1(t)\,, \\ \boldsymbol{y}(t) &= \boldsymbol{C}_1\,\boldsymbol{x}_1(t)\,, \end{aligned} \tag{2.1}$$

where

$\boldsymbol{x}_1(t) \in \mathbb{R}^{n_1}$ and $\boldsymbol{x}_2(t) \in \mathbb{R}^{n_2}$ are the state vectors of the subsystems,
$\boldsymbol{u}_1(t) \in \mathbb{R}^m$ is the control vector of the first subsystem,
$\boldsymbol{y}(t) \in \mathbb{R}^1$ is the output vector of the first subsystem,
$t = 0, 1, \ldots$ is the time sample,
$\boldsymbol{A}_{ij}, \boldsymbol{B}_{i1}, \boldsymbol{C}_1$ are constant matrices of appropriate dimensions.

Further, we assume that $n_1 \ll n_2$, the matrix $\boldsymbol{A}_{22}$ is stable and the triple $(\boldsymbol{A}_{11}, \boldsymbol{B}_{11}, \boldsymbol{C}_1)$ is controllable and observable.

Suppose the objective function in the form

$$J = \sum_{t=t_0}^{\infty} \boldsymbol{x}_1^{\mathrm{T}}(t)\,\boldsymbol{Q}_1\,\boldsymbol{x}_1(t) + \boldsymbol{u}_1^{\mathrm{T}}(t)\,\boldsymbol{R}_1\,\boldsymbol{u}_1(t) \tag{2.2}$$

where $\boldsymbol{Q}_1$ and $\boldsymbol{R}_1$ must be positive definite symmetric matrices.

The problem is that of determining a feedback matrix $\boldsymbol{K}$:

$$\boldsymbol{u}_1(t) = \boldsymbol{K}\boldsymbol{C}_1\,\boldsymbol{x}_1(t) \tag{2.3}$$

with a prescribed information structure and ensuring

i) the minimization of the objective function,

ii) the best possible realization of stability conditions of the global system.

Assuming this conditions, the first subsystems' feedback matrix $\boldsymbol{K}$ must be chosen so that the contribution of the first subsystem to the stability of the whole dynamical system will be optimal. The criterion function taking into account these requirements will be formulated later.

3. MAIN RESULTS FOR THE LINEAR DISCRETE-TIME SYSTEM

Let us define a new state vector $v_1(t)$ as follows

$$v_1(t) = L\, x_2(t)\,. \tag{3.1}$$

where $v_1(t) \in \mathbb{R}^p$.

Matrix L will be referred to as the aggregation matrix [3]. The dimension of matrix L and the magnitude of its elements are determined by the requirements following from the aggregation objective. In our case, this objective is to retain those properties of the aggregated system portion which are essential when considering the stability of the whole system. In addition, the minimal strength of interconnections between the first subsystem and the rest of the system must be achieved. Hence the biggest eigenvalues of the matrix A_{22} have to be retained in the aggregated part of the system. The way of finding L to solve this problem is introduced in [7]. The choice of p depends on the number of eigenvalues, which are kept in the aggregated matrix. Letting $L = A_{12}$, the most simple but not optimal (in the sense of the aggregation objective) results are obtained.

Now, suppose A_{12} in Eq. (2.1) in the form

$$A_{12} = M_{12}L + E_1 \tag{3.2}$$

Minimizing the norm of the matrix E_1 w.r.t. M_{12}

$$\min_{M_{12}} \|E_1\| = \|A_{12} - M_{12}L\|\,, \tag{3.3}$$

we obtain

$$M_{12} = A_{12}L^+\,, \tag{3.4}$$

where

$$L^+ = L^{\mathrm{T}}(LL^{\mathrm{T}})^{-1}$$

Similarly let us suppose in Eq. (2.1)

$$LA_{22} = M_{22}L + E_2\,. \tag{3.5}$$

Minimizing the norm of the matrix E_2 w.r.t. M_{22} we obtain

$$\min_{M_{22}} \|E_2\| = \min_{M_{22}} \|LA_{22} - M_{22}L\|\,.$$

It results in

$$M_{22} = LA_{22}L^+ \tag{3.6}$$

Applying (3.1)–(3.5) to (2.1) the system $\mathscr{S}$ can be rewritten in the following form:

$$\mathscr{S}_1:\quad \begin{aligned} x_1(t+1) &= A_{11}\, x_1(t) + M_{12}\, v_1(t) + E_1\, x_2(t) + B_{11}\, u_1(t) \\ v_1(t+1) &= LA_{21}\, x_1(t) + M_{22}\, v_1(t) + E_2\, x_2(t) + LB_{21}\, u_1(t) \\ x_2(t+1) &= A_{21}\, x_1(t) + A_{22}\, x_2(t) + B_{21}\, u_1(t) \\ y(t) &= C_1\, x_1(t) \end{aligned} \tag{3.7}$$

Let $z_1(t)$ be the state vector of the extended subsystem

$$z_1(t) = \begin{bmatrix} x_1(t) \\ v_1(t) \end{bmatrix} \quad \text{and} \quad z_2(t) = x_2(t)\,,$$

then we obtain for the global system

$$\text{(3.8)} \quad \mathscr{S}_2: \begin{aligned} z_1(t+1) &= \boldsymbol{D}_{11}\, z_1(t) + \boldsymbol{D}_{12}\, z_2(t) + \boldsymbol{B}_1\, \boldsymbol{u}_1(t)\,, \\ z_2(t+1) &= \boldsymbol{D}_{21}\, z_1(t) + \boldsymbol{D}_{22}\, z_2(t) + \boldsymbol{B}_{21}\, \boldsymbol{u}_1(t)\,, \\ y(t) &= \boldsymbol{C}\, z_1(t) \end{aligned}$$

where

$$\boldsymbol{D}_{11} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{L}\boldsymbol{A}_{21} & \boldsymbol{M}_{22} \end{bmatrix}, \quad \boldsymbol{D}_{12} = \begin{bmatrix} \boldsymbol{E}_2 \\ \boldsymbol{E}_2 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} \boldsymbol{B}_{11} \\ \boldsymbol{L}\boldsymbol{B}_{21} \end{bmatrix},$$

$$\boldsymbol{D}_{21} = [\boldsymbol{A}_{21}\ \boldsymbol{0}]\,, \quad \boldsymbol{D}_{22} = \boldsymbol{A}_{22}\,, \quad \boldsymbol{C} = [\boldsymbol{C}_1\ \boldsymbol{0}]\,.$$

Then the interactions between the two subsystems in Eq. (3.8) are much weaker than those between the original subsystems. Minimizing the norms of $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, the norm of $\boldsymbol{D}_{12}$ is minimized too. If its effect could be even neglected, local control scheme could be applied for the first subsystem using any design procedure. In order to check whether $\boldsymbol{D}_{12}$ is sufficiently small, we construct a Lyapunov function and determine the sufficient conditions of stability for the interactions term $\boldsymbol{D}_{12}$. Applying the feedback gain $\boldsymbol{K}$, the closed loop system has the form

$$\text{(3.9)} \qquad \mathscr{S}_3: \begin{aligned} z_1(t+1) &= (\boldsymbol{D}_{11} + \boldsymbol{B}_1\boldsymbol{K}\boldsymbol{C})\, z_1(t) + \boldsymbol{D}_{12}\, z_2(t)\,, \\ z_2(t+1) &= (\boldsymbol{D}_{21} + \boldsymbol{B}_{21}\boldsymbol{K}\boldsymbol{C})\, z_1(t) + \boldsymbol{D}_{22}\, z_2(t) \end{aligned}$$

In order to check the sufficient conditions for the stability of an interconnected system, Lyapunov functions of the subsystems are found. Using the scalar approach, we obtain the Lyapunov function for the global system $V(t)$:

Let

$$\text{(3.10)} \qquad V(t) = a_1\, z_1^{\mathrm{T}}(t)\, \boldsymbol{P}_1\, z_1(t) + a_2\, z_2^{\mathrm{T}}(t)\, \boldsymbol{P}_2\, z_2(t)\,,$$

where

$a_1, a_2 > 0$ are real constants,

$\boldsymbol{P}_1, \boldsymbol{P}_2$ are symmetric positive definite Lyapunov matrices.

Using some properties of the matrix norms for the first difference of (3.10) we obtain an inequality constraint which ensures stability of the global system. We obtain (see Appendix A)

$$\text{(3.11)} \qquad \Delta V(t) \leqq -\boldsymbol{z}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{z} \leqq 0\,,$$

where

$$\boldsymbol{z}^{\mathrm{T}} = [\|z_1(t)\|, \|z_2(t)\|]\,,$$

$$\|z_i(t)\| = (z_i^{\mathrm{T}}(t)\, z_i(t))^{1/2}\,, \quad i = 1, 2\,.$$

The norm of the matrix $\boldsymbol{D}$ is calculated as follows

$$\|\boldsymbol{D}\| = \lambda_{\max}^{1/2}(\boldsymbol{D}^{\mathrm{T}}\boldsymbol{D})$$

and

$$G = \begin{bmatrix} a_1\lambda_{\min}(Q) - a_2\lambda_{\max}(D'^{\mathrm{T}}_{21}P_2D'_{21}) & a_{12} \\ a_{12} & a_2 - a_1\|D_{12}\|^2\lambda_{\max}(P_1) \end{bmatrix}$$

$$a_{12} = -(a_1\|D_{12}\|\,\|P_1D'_{11}\| + a_2\|D^{\mathrm{T}}_{22}P_2D'_{21}\|)$$

$$D'_{11} = D_{11} + B_1KC$$

$$D'_{21} = D_{21} + B_{21}KC$$

$\lambda_{\max}(\cdot)$, $\lambda_{\min}(\cdot)$ are the maximum and minimum eigenvalues of the matrix $(\cdot)$ respectively.

If the feedback gain matrix K has been chosen to stabilize the first subsystem, then P_i can be found as a solution of the following equations:

$$D'^{\mathrm{T}}_{11}P_1D_{11} - P_1 = -(Q + C^{\mathrm{T}}K^{\mathrm{T}}R_1KC)\,, \tag{3.12}$$

$$D^{\mathrm{T}}_{22}P_2D_{22} - P_2 = -I_{n_2}\,,$$

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad I_{n_2} \text{ is the identity matrix}\,.$$

The matrix Q_2 is a positive definite matrix associated with the extra state vector $v_1(t)$. The state vector $v_1(t)$ is regarded as the interaction between the original first subsystem and the other part of the system. An appropriate choice of Q_2 leads to a minimization of this interaction. The stability conditions for the global system (3.11) yield:

(3.13) (i) $\lambda_{\max}(D'^{\mathrm{T}}_{21}P_2D'_{21}) \to \min$ or $\mathrm{Tr}\,(D'^{\mathrm{T}}_{21}D'_{21}) \to \min$

(ii) $\lambda_{\max}(P_1) \to \min$ or $\mathrm{Tr}\,(P_1) \to \min$.

(iii) the matrix G must be positive semidefinite or definite one.

($\mathrm{Tr}(\cdot)$ denotes trace function of the matrix $(\cdot)$.)

The third condition can be used to determine the range of values for the interaction term $\|D_{12}\|$, which ensures the satisfaction of the sufficient stability conditions. Conditions (3.13i) and (3.13ii) ensure the best possible satisfaction of stability conditions from the viewpoint of the first subsystem. For any positive numbers $a_1, a_2 > 0$ a maximum value of $\|D_{12}\|$ can be found by solving Eq. (3.11). A maximization routine can then be used to find the constants a_1 and a_2 which give the maximum possible range of $\|D^*_{12}\|$. The equation (3.11) yields the requirement on the matrix G

$$G(a_1, a_2, \|D_{12}\|) \geqq 0 \xrightarrow[a_1, a_2]{} \max \|D_{12}\|\,. \tag{3.14}$$

It is possible to calculate $\|D^*_{12}\|$ for instance using a gradient method. If the inequality

$$0 \leqq \|D_{12}\| \leqq \|D^*_{12}\| \tag{3.15}$$

holds, the feedback gain matrix K can be computed taking into account simply the first extended subsystem in Eq. (3.8). The augmented subsystem consists of the original first subsystem and the aggregated mathematical model of the remaining part of the system. Thus, it is possible to design the feedback gain matrix $\boldsymbol{K}$ taking into account all the properties of the original subsystem and the approximation of the rest of the system. The extra part of the original subsystem is determined by the choice of the matrix $\boldsymbol{L}$ in (3.1).

4. CONTROL SYSTEM DESIGN

The aim of this part is to determine the output feedback matrix $\boldsymbol{K}$ which minimizes (3.11) and the extended objective function (2.2) subjected to the constraints (3.12). Let us define the Lagrange function L_1:

$$(4.1)\qquad L_1 = \min_{\boldsymbol{P}_1, \boldsymbol{K}} \max_{\boldsymbol{W}} \operatorname{Tr} \{\boldsymbol{P}_1 + (\boldsymbol{D}_{21} + \boldsymbol{B}_{21}\boldsymbol{KC})^{\mathrm{T}} (\boldsymbol{D}_{21} + \boldsymbol{B}_{21}\boldsymbol{KC}) + \\ + \boldsymbol{W}[(\boldsymbol{D}_{11} + \boldsymbol{B}_1\boldsymbol{KC})^{\mathrm{T}} \boldsymbol{P}_1(\boldsymbol{D}_{11} + \boldsymbol{B}_1\boldsymbol{KC}) - \boldsymbol{P}_1 + \boldsymbol{Q} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{R}_1\boldsymbol{KC})\}$$

where $\boldsymbol{W}$ is a symmetric matrix of Lagrange multipliers. If the last term in Eq. (4.1) is equal to zero, it can be stated using the fact [8] that the minimization of $\operatorname{Tr}(\boldsymbol{P}_1)$ is in the average the same as the minimization of the cost function (2.2). Under the assumption that $\boldsymbol{B}_{21}$ is very "small" it is possible to neglect the second part of the cost function (4.1). The differentiation of (4.1) yields three necessary optimality conditions

(4.2) a) $\nabla L_{1_{\boldsymbol{P}_1}} = \boldsymbol{I}_{2n_1} + (\boldsymbol{D}_{11} + \boldsymbol{B}_1\boldsymbol{KC})\boldsymbol{W}(\boldsymbol{D}_{11} + \boldsymbol{B}_1\boldsymbol{KC})^{\mathrm{T}} - \boldsymbol{W} = \boldsymbol{0}$

b) $\nabla L_{1_{\boldsymbol{W}}} = (\boldsymbol{D}_{11} + \boldsymbol{B}_1\boldsymbol{KC})^{\mathrm{T}} \boldsymbol{P}_1(\boldsymbol{D}_{11} + \boldsymbol{B}_1\boldsymbol{KC}) - \boldsymbol{P}_1 + \boldsymbol{Q} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{R}_1\boldsymbol{KC} = \boldsymbol{0}$

and $\nabla L_{1_{\boldsymbol{K}}} = \boldsymbol{0}$ implies

c) $\boldsymbol{K} = -(\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{B}_1 + \boldsymbol{R}_1)^{-1} [\boldsymbol{B}_{21}^{\mathrm{T}}\boldsymbol{B}_{21}\boldsymbol{KC} + \boldsymbol{B}_{21}^{\mathrm{T}}\boldsymbol{D}_{21} + \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{D}_{11}\boldsymbol{W}]\, \boldsymbol{C}^{\mathrm{T}} \,. \\ .\, (\boldsymbol{CWC}^{\mathrm{T}})^{-1}$

If $\boldsymbol{C}$ is the identity matrix and $\boldsymbol{B}_{21} = \boldsymbol{0}$ Eq. (4.2c) reduces to the well known equation for the linear quadratic problem

$$\boldsymbol{K} = -(\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{B}_1 + \boldsymbol{R}_1)^{-1}\, \boldsymbol{B}_1\boldsymbol{P}_1\boldsymbol{D}_{11} \,.$$

The three nonlinear matrix equations (4.2) are in the same form used in [2]. Therefore we may suppose the same iterative algorithm. The steps of the solution are as follows:

1. Choose $\boldsymbol{K}^1$ such that $(\boldsymbol{D}_{11} + \boldsymbol{B}_1\boldsymbol{K}^1\boldsymbol{C})$ is stable.
2. Substitute $\boldsymbol{K}^1$ into (4.2a) and (4.2b) and calculate $\boldsymbol{P}_1^1$ and $\boldsymbol{W}^1$.
3. Substitute $\boldsymbol{K}^1$, $\boldsymbol{P}_1^1$ and $\boldsymbol{W}^1$ into (4.2c) to calculate the next approximation $\boldsymbol{K}^2$.
4. If $\|\boldsymbol{K}^2 - \boldsymbol{K}^1\| > \varepsilon > 0$ then $\boldsymbol{K}^1 = \boldsymbol{K}^2$ and go to Step 2.
5. End.

The convergence of the solution represents another interesting problem, and it is not solved in this paper. According to [2] the choice of $\boldsymbol{K}^1$ corresponding to the conditions introduced in the first step represents the necessary convergence condition.

If only maintaining of the stability of the system is required without minimization of the objective function, then the right-hand side of Eq. (3.12a) reduces to the identity matrix, $\lambda_{\min}(Q) = 1$ and $R_1 = 0$.

5. MAIN RESULT FOR LINEAR CONTINUOUS-TIME SYSTEMS

Consider a linear time-invariant dynamical system in the form

$$\text{(5.2)} \qquad \mathscr{S}_4: \quad \begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_{11}u_1 , \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 , \\ y &= C_1 x_1 \end{aligned}$$

with the objective function

$$\text{(5.2)} \qquad J = \int_{t_0}^{\infty} (x_1^{\mathrm{T}} Q_1 x_1 + u_1^{\mathrm{T}} R_1 u_1)\, \mathrm{d}t \to \min .$$

Applying the procedure analogous to Eqs. (3.1)–(3.6) and (3.7), we obtain

$$\mathscr{S}_5: \quad \begin{aligned} \dot{z}_1 &= D_{11}z_1 + D_{12}z_2 + Bu_1 , \\ \dot{z}_2 &= D_{21}z_1 + D_{22}z_2 , \\ y &= Cz_1 \end{aligned}$$

where

$$B = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}$$

Since D_{12} is sufficiently small it is possible to find the condition for disconnecting the two subsystems while maintaining the stability of the global system. Substituting (2.3) into (5.3) the closed loop system can be described as follows

$$\text{(5.4)} \qquad \mathscr{S}_6: \quad \begin{aligned} \dot{z}_1 &= D'_{11}z_1 + D_{12}z_2 , \\ \dot{z}_2 &= D_{21}z_1 + D_{22}z_2 , \end{aligned}$$

where

$$D'_{11} = D_{11} + BKC .$$

In order to check the sufficient stability conditions of the interconnected system, the Lyapunov function for the subsystems are constructed which are followed by the ensuring that some linear combination of them is the Lyapunov function for the global system which is analogous to Eq. (3.10). The main result for the sufficient stability condition is given by the following lemma.

Lemma. If D'_{11} is stable in the system (5.4) under the assumption of the stability of D_{22}, then the condition

$$\text{(5.5)} \qquad \|D_{12}\| \leq \frac{\lambda_{\min}(Q)}{4\|P_1\|\,\|D_{22}^{\mathrm{T}}P_2\|}$$

satisfying the Lyapunov matrix equations

$$D_{11}'^{\mathrm{T}} P_1 + P_1 D_{11}' = -[Q + C^{\mathrm{T}} K^{\mathrm{T}} R_1 K C]$$

$$D_{22}^{\mathrm{T}} P_2 + P_2 D_{22} = -I_{n-n_1}$$

ensures that the global system is stable. It makes possible the disconnection of the two subsystems (5.4).

The proof of this Lemma is given in Appendix B.

Now, suppose that for Eq. (5.4) the condition (5.5) holds. Then it is obvious from Eq. (5.5) that P_1 is to be minimized To minimize (5.2) and P_1, we solve the following problem

$$L_2 = \min_{K, P_1} \max_{W} \operatorname{Tr} \{ P_1 + W[(D_{11} + BKC)^{\mathrm{T}} P_1 + P_1 (D_{11} + BKC) + Q + C^{\mathrm{T}} K^{\mathrm{T}} R_1 K C] \} . \tag{5.7}$$

The necessary optimality conditions are:

(5.8) a) $\nabla L_{2_{P_1}} = I_{2n_1} + W(D_{11} + BKC)^{\mathrm{T}} + (D_{11} + BKC) W = \mathbf{0}$.

b) $\nabla L_{2_W} = P_1 (D_{11} + BKC) + (D_{11} + BKC)^{\mathrm{T}} P_1 + Q + C^{\mathrm{T}} K^{\mathrm{T}} R_1 K C = \mathbf{0}$

c) $L_{2_K} = \mathbf{0}$ implies

$$K = -R_1^{-1} B^{\mathrm{T}} P_1 W C^{\mathrm{T}} (C W C)^{-1} \tag{5.8}$$

When C is the identity matrix, Eq. (5.8c) reduces to the well known equation

$$K = -R^{-1} B^{\mathrm{T}} P_1 . \tag{5.9}$$

The nonlinear matrix equations (5.8) can be solved using the two level iterative procedure in the same way as Eqs. (4.2).

6. EXAMPLE FOR THE LINEAR DISCRETE-TIME SYSTEM

Let us consider the linear discrete-time system

$$A_{11} = \begin{bmatrix} 0{\cdot}5 & 0{\cdot}05 & 0{\cdot}04 \\ 0{\cdot}01 & 0 & -0{\cdot}01 \\ -0{\cdot}2 & -0{\cdot}01 & -1{\cdot}1 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0{\cdot}02 & -0{\cdot}01 & 0 & 0 & 0 & 0 & 0{\cdot}1 \\ -0{\cdot}02 & 0 & 0{\cdot}1 & 0 & 0 & 0 & 0{\cdot}02 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0{\cdot}01 \end{bmatrix},$$

$$A_{21}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & 0{\cdot}1 & 0 & 0 & 0{\cdot}05 \\ 0 & 0 & 0 & 0{\cdot}12 & 0 & 0{\cdot}1 & 0 \\ 0 & 0{\cdot}2 & 0 & 0 & 0 & -0{\cdot}1 & 0{\cdot}04 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 0{\cdot}2 & -0{\cdot}01 & 0{\cdot}1 & 0 & 0 & 0{\cdot}15 & 0 \\ 0 & -0{\cdot}15 & 0 & 0{\cdot}05 & -0{\cdot}02 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0{\cdot}05 & 0{\cdot}02 \\ 0{\cdot}1 & 0 & 0 & 0 & 0{\cdot}15 & -0{\cdot}02 & 0 \\ 0{\cdot}02 & 0{\cdot}04 & 0 & 0{\cdot}08 & 0 & 0 & 0{\cdot}12 \\ -0{\cdot}08 & 0 & 0{\cdot}05 & -0{\cdot}02 & 0 & 0 & 0{\cdot}12 \\ -0{\cdot}02 & 0{\cdot}1 & 0 & 0{\cdot}01 & 0{\cdot}2 & 0 & 0 \end{bmatrix},$$

$$B_{11}^{T} = [0\ 0{\cdot}1\ 1],$$

$$B_{21}^{T} = [0{\cdot}\ 0\ 0.001\ 0{\cdot}0005\ 0\ 0\ 0{\cdot}001],$$

$$C_1 = [0\ 1\ 1].$$

Let $L = A_{12}$.

The performance index is given

$$J_1 = \sum_{t=t_0}^{\infty} z_1^T Q z_1 + u_1^2 r_1,$$

where

$$Q = \operatorname{diag}\{0{\cdot}5\ 1\ 1\ 5\ 5\ 5\},$$

$$r_1 = 1.$$

Choosing $K^1 = 0{\cdot}3$ after the fourth iteration it is obtained from (4.2)

$$\|K^2 - K^1\| = \|0{\cdot}3416 - 0{\cdot}36037\| \leqq 0{\cdot}02.$$

$$K = 0{\cdot}3416$$

The matrix G is given for $\|D_{12}\| = 0{\cdot}025$ as follows:

$$G = \begin{bmatrix} a_1 \,.\, 0{\cdot}5 - a_2 \,.\, 0{\cdot}05739 & -(a_1 \,.\, 0{\cdot}47 + a_2 \,.\, 0{\cdot}034959) \\ -(a_1 \,.\, 0{\cdot}47 + a_2 \,.\, 0{\cdot}034959) & a_2 - a_1 \,.\, 0{\cdot}03501 \end{bmatrix}$$

Without minimizing the relation (3.13) it is obvious that for $a_1 = a_2 = 1$ the matrix G is a positive definite one and therefore, the system is stable and the output feedback gain matrix K minimizes the given objective function.

7. CONCLUSIONS

A method of decentralized control design based on an aggregated model is presented. This design method is particularly applicable for a global system consisting of many recognizable subsystems. In such case, the design method is used to calculate local control schemes for each subsystem in turn incorporating the effects of the previous local control schemes at each stage. This simplifies the control calculation when compared to the use of a centralized control design with a full system model. The method yields a robust design which optimizes the objective function of the subsystems while ensuring the best stability conditions of the global system.

APPENDIX A

We get from Eqs. (3.8) and (3.9)

$$\Delta V(t) = V(t+1) - V(t) = z_1^{\mathrm{T}}(t)\left[\left(-\boldsymbol{Q} - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{R}_1\boldsymbol{K}\boldsymbol{C}\right)a_1 + \right.$$

$$\left. + a_2(\boldsymbol{D}_{21} + \boldsymbol{B}_{21}\boldsymbol{K}\boldsymbol{C})^{\mathrm{T}}\boldsymbol{P}_2(\boldsymbol{D}_{21} + \boldsymbol{B}_{21}\boldsymbol{K}\boldsymbol{C})\right] z_1(t) + z_2^{\mathrm{T}}(t) \,.$$

$$.\left[-a_2\boldsymbol{I} + a_1\boldsymbol{D}_{12}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{D}_{12}\right] z_2(t) + 2z_1^{\mathrm{T}}(t)\ .$$

$$.\left[a_1(\boldsymbol{D}_{11} + \boldsymbol{B}_1\boldsymbol{K}\boldsymbol{C})^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{D}_{12} + a_2(\boldsymbol{D}_{21} + \boldsymbol{B}_{21}\boldsymbol{K}\boldsymbol{C})^{\mathrm{T}}\boldsymbol{P}_2\boldsymbol{D}_{22}\right] z_2(t)$$

Using some properties of the matrix norms we obtain

$$\Delta V(t) \leqq$$

$$\leqq z^{\mathrm{T}}\begin{bmatrix} a_1\lambda_{\min}(\boldsymbol{Q} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{R}_1\boldsymbol{K}\boldsymbol{C}) - a_2\lambda_{\max}(\boldsymbol{D}_{21}'^{\mathrm{T}}\boldsymbol{P}_2\boldsymbol{D}_{21}') & -g_{12} \\ -g_{12} & a_2 - a_1\lambda_{\max}(\boldsymbol{D}_{12}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{D}_{12}) \end{bmatrix} z$$

$$g_{12} = a_1\lambda_{\max}^{1/2}\left[(\boldsymbol{D}_{11}'\boldsymbol{P}_1\boldsymbol{D}_{12})^{\mathrm{T}}(\boldsymbol{D}_{11}'\boldsymbol{P}_1\boldsymbol{D}_{12})\right] +$$

$$+ a_2\lambda_{\max}^{1/2}\left[(\boldsymbol{D}_{21}'^{\mathrm{T}}\boldsymbol{P}_2\boldsymbol{D}_{22})^{\mathrm{T}}(\boldsymbol{D}_{21}'\boldsymbol{P}_2\boldsymbol{D}_{22})\right]$$

or more strongly

$$\Delta V(t) \leqq z^{\mathrm{T}}\begin{bmatrix} a_1\lambda_{\min}(\boldsymbol{Q}) - a_2\lambda_{\max}(\boldsymbol{D}_{21}'^{\mathrm{T}}\boldsymbol{P}_2\boldsymbol{D}_{21}') & a_{12} \\ \boldsymbol{a_{12}} & a_2 - a_1(\|\boldsymbol{D}_{12}\|^2\,\lambda_{\max}(\boldsymbol{D}_1)) \end{bmatrix} z$$

APPENDIX B

Proof of Lemma.

In order to check the sufficient stability conditions of the interconnected system (5.4) Lyapunov functions of the subsystems are found.

Let

(B.1) $$V = a_1V_1 + a_2V_2\,, \quad a_1 > 0\,, \quad a_2 > 0$$

and

(B.2) $$\frac{\mathrm{d}V}{\mathrm{d}t} = a_1z_1^{\mathrm{T}}(\boldsymbol{D}_{11}'^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{P}_1\boldsymbol{D}_{11}')\,z_1 + a_2z_2^{\mathrm{T}}(\boldsymbol{D}_{22}^{\mathrm{T}}\boldsymbol{P}_2 + \boldsymbol{P}_2\boldsymbol{D}_{22})\,z_2 +$$

$$+ 2z_1^{\mathrm{T}}\left[a_1\boldsymbol{P}_1\boldsymbol{D}_{12} + a_2\boldsymbol{D}_{21}^{\mathrm{T}}\boldsymbol{P}_2\right] z_2$$

If $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are given by Eqs. (5.6) using some properties of the matrix norms for the first derivative of (B.2) we get

(B.3) $$\frac{\mathrm{d}V}{\mathrm{d}t} \leqq -z^{\mathrm{T}}\begin{bmatrix} a_1\lambda_{\min}(\boldsymbol{Q}) & -\xi_{12} \\ -\xi_{12} & a_2 \end{bmatrix} z \leqq 0$$

where

$$\xi_{12} = a_1\|\boldsymbol{P}_1\|\,\|\boldsymbol{D}_{12}\| + a_2\|\boldsymbol{D}_{21}^{\mathrm{T}}\boldsymbol{P}_2\|$$

We get from the stability conditions of the overall system for $\|D_{12}\|$

(B.4) $$\|D_{12}\| \leqq \frac{-a_2\|D_{21}^T P_2\| + \sqrt{(a_1 a_2 \lambda_{\min}(Q))}}{a_1\|P_1\|}$$

Let us find $a_1 > 0$ and $a_2 > 0$ such that $\|D_{12}\| \to \max$. We get from Eq. (B.4)

(B.5)

$$\left(\frac{a_1}{a_2}\right)^2 \|D_{12}\|^2 \|P_1\|^2 + \left(\frac{a_1}{a_2}\right)(2\|D_{12}\| \|P_1\| \|D_{21}^T P_2\| - \lambda_{\min}(Q)) + \|D_{21}^T P_2\|^2 \leqq 0$$

Since we require a_1 and a_2 to be real constants, Eq. (B.5) yields the following solution

$$\|D_{12}\| = \frac{\lambda_{\min}(Q)}{4\|P_1\| \|D_{21}^T P_2\|}$$

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