AR(1) PROCESSES WITH GIVEN MOMENTS OF MARGINAL DISTRIBUTION

JIŘÍ ANDĚL

Let X_t be an AR(1) process given by $X_t = bX_{t-1} + e_t$ where $b \in (-1, 1)$ and e_t is a strict white noise. Sometimes X_t must satisfy also some additional conditions, e.g. $X_t \geqq 0$ or $C \leqq X_t \leqq D$. The problem solved in the paper is how to find a distribution of e_t such that the moments $\mathsf{E}X_t^k$ $(k = 1, \ldots, n)$ have given values.

1. HISTORY OF THE PROBLEM

Simulation procedures are frequently used for demonstrating theoretical results as well as for modelling real situations. In some cases it is necessary to generate pseudorandom numbers not only with a given marginal distribution but also with a given covariance structure. Chamitov [7] writes that the problem was formulated already in 1963. It is particularly important in simulating dynamical systems and in hydrological applications. The first theoretical results concerning an AR (1) process were published by Gaver and Lewis [8]. A review of methods of this kind can be found in [2]. (See also Section 2 of the present paper.) The disadvantage of this method is that the procedure gives explicit results only in a few special cases. Moreover, it cannot be generalized to the AR models of higher order.

Two direct methods were published in 1983. Chamitov [7] publishes some empirical tables. It is not clear, however, which of the tabulated covariance functions should be chosen when one has a given empirical covariance function calculated from a sample or when a covariance function is given analytically. Further, the class of one-dimensional distributions as well as the class of covariance functions is rather limited. The method is based on the following argumentation. If $\xi_1, \ldots, \xi_k$ is a sequence of independent identically distributed random variables, then each of them has, of course, the same distribution. But the members of the ordered sample have distributions different from this one. This point was obviously overlooked.

Sondhi [13] starts with a Gaussian white noise e_t and with a linear filter H. The output Z_t has a normal distribution function G_0 instead of the distribution function F, which is wanted. If we put $X_t = F^{-1}[G_0(Z_t)]$ then it is clear that X_t has the distribution function F. Unfortunately, after the application of the non-linear filter $F^{-1}G_0$ the process X_t has a covariance function which is different from that belonging to the filter H. The procedure suggested by the author is to use Mehler's expansion of the bivariate normal density. It leads to an approximation and the computation is rather complicated. Moreover, there is no possibility how to recognize whether the problem has a solution. Sondhi writes: "Unfortunately, to the best of our knowledge, no tractable procedure is known to decide whether a covariance function is consistent with a given probability distribution function."

For solving the problem a financial support was offered (see [11], p. 193).

In practical situations, the given one-dimensional stationary distribution is usually derived from some of its moments. Anděl [3] proposed a method how to find a distribution of the white noise such that the resulting linear process X_t has given moments $\mathsf{E}X_t^k$ $(k = 1, \ldots, n)$. This approach was generalized to some non-linear processes in [4] and applied to real hydrological data in [6].

2. INTRODUCTION

Consider an AR(1) process X_t given by

$$X_t = bX_{t-1} + e_t \tag{2.1}$$

where $b \in (-1, 1)$ and e_t are independent random variables with the same distribution function G. The problem is to find a G such that all the variables X_t have a given stationary distribution with a distribution function F. The main tool for solving the problem was introduced in [8]. Let

$$\omega(u) = \mathsf{E}\exp(iuX_t) \quad \text{and} \quad \psi(u) = \mathsf{E}\exp(iue_t)$$

be the characteristic function of X_t and e_t, respectively. Since e_t is independent of X_{t-1}, (2.1) yields

$$\omega(u) = \omega(bu)\,\psi(u)\,. \tag{2.2}$$

From here $\psi(u)$ is calculated. This procedure (based on the moment generating function instead of the characteristic function) was applied in [8] for finding G such that X_t has an exponential distribution (see Section 7 of our paper).

In the general case, it is difficult to determine G from ψ. It happens also often that ψ calculated from (2.2) is not a characteristic function, which means that for such an F the problem has no solution. The following example is given in [2]. If we look for a G such that X_t have the continuous rectangular distribution on a given interval $[-c, c]$, then using (2.2) we get

$$\psi(u) = b\sin(cu)/\sin(bcu)\,.$$

This is a characteristic function only in the case that $b = 1/n$ for $n = \pm 1, \pm 2, \ldots$

If $b = 1/(2n)$ then ψ corresponds to the distribution concentrated at the points $\pm(2k - 1)\, c/(2n)$ for $k = 1, \ldots, n$ where each point has probability $1/(2n)$. If $b = 1/(2n + 1)$ then ψ is the characteristic function of the distribution concentrated at $\pm 2kc/(2n + 1)$, $k = 0, 1, \ldots, n$, where each point has probability $1/(2n + 1)$. If $b \neq 1/n$ then $|\psi(u)| \to \infty$ as $u \to \pi/(bc)$, and thus ψ cannot be a characteristic function.

In this paper we assume that only some moments of X_t are given. The problem if to find a distribution function G (if it exists) such that (2.1) produces a stationary distribution with the given moments. Our procedure can easily be realized on a computer and it shows automatically when no solution exists.

3. RELATIONS FOR MOMENTS

Define $m_k = \mathsf{E}X_t^k$, $s_k = \mathsf{E}e_t^k$ for $k = 0, 1, \ldots$. It follows from (2.1) that

$$m_k = \sum_{i=0}^{k} \binom{k}{i} b^i m_i s_{k-i}$$

and from here we obtain

$$s_k = (1 - b^k)\, m_k - \sum_{i=1}^{k-1} \binom{k}{i} b^i\, m_i s_{k-i} \tag{3.1}$$

for $k = 1, 2, \ldots$. If the moments m_k are given then (3.1) enables to calculate s_k recurrently. For example, the first four moments are

$$s_1 = (1 - b)\, m_1\,,$$
$$s_2 = (1 - b^2)\, m_2 - 2b(1 - b)\, m_1^2\,,$$
$$s_3 = (1 - b^3)\, m_3 - 3b(1 + 2b)(1 - b)\, m_1 m_2 + 6b^2(1 - b)\, m_1^3\,,$$
$$s_4 = (1 - b^4)\, m_4 - 4b(1 - b)(1 + b + 2b^2)\, m_1 m_3 + 12b^2(1 - b)(1 + 3b)\,.\, m_1^2 m_2 - 6b^2(1 - b^2)\, m_2^2 - 24b^3(1 - b)\, m_1^4\,.$$

Sometimes X_t must fulfil also other conditions. Three following cases are most important.

(i) $X_t \in [C, D]$ for all t, where $C < D$ are given numbers.

(ii) $X_t \geqq 0$ for all t.

(iii) There are no additional restrictions on X_t.

If the numbers $s_1, \ldots, s_n$ are calculated, we must decide if there exists a distribution function G such that s_k are its moments and that X_t fulfil given restrictions.

4. MOMENT PROBLEM ON A FINITE INTERVAL

Theorem 4.1. Let X_t be an AR(1) process defined by (2.1) with $b \in (-1, 1)$. Let $C < D$ be given numbers. Define

$$H = C(1 - b)\,, \quad K = D(1 - b) \quad \text{for} \quad b \geqq 0\,,$$
$$H = C - bD\,, \quad K = D - bC \quad \text{for} \quad b < 0\,.$$

If $H \leqq e_t \leqq K$ for all t, then $C \leqq X_t \leqq D$ for all t. If $\mathsf{P}(e_t \notin [H, K]) > 0$, then $X_t \notin [C, D]$ for infinitely many subscripts t with probability 1.

Proof. Let $0 \leqq b < 1$. Let $C(1 - b) \leqq e_t \leqq D(1 - b)$ hold for all t. Since

$$X_t = e_t + be_{t-1} + b^2 e_{t-2} + \dots,$$

we obtain

$$X_t \leqq D(1 - b)(1 + b + b^2 + \dots) = D\,,$$
$$X_t \geqq C(1 - b)(1 + b + b^2 + \dots) = C\,.$$

Now, consider the case $-1 < b < 0$. We can write

$$X_t = e_t + b^2 e_{t-2} + b^4 e_{t-4} + \dots + b(e_{t-1} + b^2 e_{t-3} + b^4 e_{t-5} + \dots)\,.$$

Thus

$$X_t \leqq (D - bC)(1 + b^2 + b^4 + \dots) + b(C - aD)(1 + b^2 + b^4 + \dots) = D\,,$$
$$X_t \geqq (C - bD)(1 + b^2 + b^4 + \dots) + b(D - bC)(1 + b^2 + b^4 + \dots) = C\,.$$

A proof of the last assertion of Theorem 4.1 is similar to that of Lemma 10.2. □

Theorem 4.2. A sequence of numbers $\{s_k\}_0^{2r}$ is a system of moments on an interval $[H, K]$ if and only if both the matrices

$$\mathbf{A} = (s_{i+j})_{i,j=0}^{r}\,, \quad \mathbf{B} = ((H + K)s_{i+j+1} - HKs_{i+j} - s_{i+j+2})_{i,j=0}^{r-1}$$

are positive semidefinite.

Proof. See [9], p. 90. □

Theorem 4.3. A sequence of numbers $\{s_k\}_0^{2r+1}$ is a system of moments on an interval $[H, K]$ if and only if both the matrices

$$\mathbf{A} = (s_{i+j+1} - Hs_{i+j})_{i,j=0}^{r}\,, \quad \mathbf{B} = (Ks_{i+j} - s_{i+j+1})_{i,j=0}^{r}$$

are positive semidefinite.

Proof. See [9], p. 91. □

Let us remark that an infinite sequence $\{s_k\}_0^\infty$ is a system of moments on an interval $[H, K]$ if and only if the infinite quadratic forms

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} s_{i+j} x_i x_j\,, \quad \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [(H + K)\, s_{i+j+1} - HKs_{i+j} - s_{i+j+2}]\, x_i x_j$$

are positive semidefinite. This is equivalent to the condition that the forms

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (s_{i+j+1} - Hs_{i+j})\, x_i x_j\,, \quad \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (Ks_{i+j} - s_{i+j+1})\, x_i x_j$$

are positive semidefinite (see [9], p. 92).

If we have numbers $s_0 = 1, s_1, \dots, s_n$ then it can be decided using Theorems 4.2 and 4.3 if they are moments or not. If they are, it is important to know at least one distribution having just moments $s_1, \dots, s_n$. It is known that there exists a discrete distribution with this property and the points at which the probability is concentrated are given as the roots of some polynomials. Such points are shortly called "points of concentration". Since this method is very general and not known among the statisticians, we describe it in this paper in detail. Before doing it, let us make the following remark. In the most situations only four moments are investigated. The author believes that then the use of the generalized Tukey's λ-system (see [12]) can be recommended. The procedure gives smooth densities and the sample can be quickly generated on a computer.

Now, we return to the problem of general n.

Theorem 4.4. Let $n = 2r - 1$. Let the matrices $\mathbf{A}$ and $\mathbf{B}$ from Theorem 4.3 for a sequence of numbers $\{s_k\}_0^{2r-1}$ be positive definite. Then the points of concentration of so called lower main representation are the roots of

$$\begin{vmatrix} s_0 & s_1 & \dots & s_{r-1} & 1 \\ s_1 & s_2 & \dots & s_r & z \\ \dots & \dots & \dots & \dots & \dots \\ s_r & s_{r+1} & \dots & s_{2r-1} & z^r \end{vmatrix} = 0 .$$

The points of concentration of so called upper main representation are the roots of

$$(K - z)(z - H)\begin{vmatrix} s'_0 & s'_1 & \dots & s'_{r-2} & 1 \\ s'_1 & s'_2 & \dots & s'_{r-1} & z \\ \dots & \dots & \dots & \dots & \dots \\ s'_{r-1} & s'_r & \dots & s'_{2r-3} & z^{r-1} \end{vmatrix} = 0$$

where $s'_k = (H + K)\, s_{k+1} - HKs_k - s_{k+2}$.

Proof. See [9], p. 122. □

Theorem 4.5. Let $n = 2r$. Let the matrices $\mathbf{A}$ and $\mathbf{B}$ from Theorem 4.2 for a sequence of numbers $\{s_k\}_0^{2r}$ be positive definite. Then the points of concentration of the lower main representation are the roots of

$$(z - H)\begin{vmatrix} s_1 - Hs_0 & s_2 - Hs_1 & \dots & s_r - Hs_{r-1} & 1 \\ s_2 - Hs_1 & s_3 - Hs_2 & \dots & s_{r+1} - Hs_r & z \\ \dots & \dots & \dots & \dots & \dots \\ s_{r+1} - Hs_r & s_{r+2} - Hs_{r+1} & \dots & s_{2r} - Hs_{2r-1} & z^r \end{vmatrix} = 0 .$$

The points of concentration of the upper main representation are the roots of

$$(K - z)\begin{vmatrix} Ks_0 - s_1 & Ks_1 - s_2 & \dots & Ks_{r-1} - s_r & 1 \\ Ks_1 - s_2 & Ks_2 - s_3 & \dots & Ks_r - s_{r+1} & z \\ \dots & \dots & \dots & \dots & \dots \\ Ks_r - s_{r+1} & Ks_{r+1} - s_{r+2} & \dots & Ks_{2r-1} - s_{2r} & z^r \end{vmatrix} = 0 .$$

Proof. See [9], p. 122. □

It is recommended to prefer the lower main representation. One of the reasons for it may be that in the case $n = 2r - 1$ it needs a smaller number of the points of concentration.

If $n = 2r$, we find the points $z_1, \ldots, z_{r+1}$ of the lower main representation. Then we calculate probabilities $p_1, \ldots, p_{r+1}$ from

$$z_1^k p_1 + \ldots + z_{r+1}^k p_{r+1} = s_k \quad (k = 0, 1, \ldots, r).$$

The solution has the property that $p_1 \geqq 0, \ldots, p_{r+1} \geqq 0$ and that

$$z_1^k p_1 + \ldots + z_{r+1}^k p_{r+1} = s_k$$

holds also for $k = r + 1, \ldots, 2r$. If $n = 2r - 1$, the procedure is analogous.

5. SPECIAL DISTRIBUTIONS ON $[-1, 1]$

If X_t has the continuous rectangular distribution on $[-1, 1]$, then its moments are

$$m_k = 1/(k + 1) \quad \text{for } k \text{ even}, \quad m_k = 0 \quad \text{for } k \text{ odd}.$$

Because $C = -1$, $D = 1$, we get $H = -1 + |b|$, $K = 1 - |b|$. For simplicity, we consider only the case $b \in (0, 1)$. Using the results of Section 3 we get

$$s_1 = 0, \quad s_2 = (1 - b^2)/3, \quad s_3 = 0, \quad s_4 = (1 - b^2)(3 - 7b^2)/15.$$

We restrict ourselves to $n = 4$. Then

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & (1 - b^2)/3 \\ 0 & (1 - b^2)/3 & 0 \\ (1 - b^2)/3 & 0 & (1 - b^2)(3 - 7b^2)/15 \end{pmatrix},$$

$$\boldsymbol{B} = (2/15)(1 - b)(1 - 2b)\begin{pmatrix} 5 & 0 \\ 0 & (1 + b)(1 - 3b) \end{pmatrix}.$$

We have $|\boldsymbol{A}| = 4(1 - b^2)^2 (1 - 4b^2)/135$. The matrix $\boldsymbol{A}$ is positive definite for $0 < b < 1/2$ and positive semidefinite for $b = \frac{1}{2}$. The matrix $\boldsymbol{B}$ is positive definite for $0 < b < \frac{1}{3}$ and positive semidefinite for $b = \frac{1}{2}$, $b = \frac{1}{3}$. Let $0 < b < \frac{1}{3}$. Inserting into the first part of Theorem 4.5 we obtain

$$(z - H)\begin{vmatrix} -H & s_2 & 1 \\ s_2 & -Hs_2 & z \\ -Hs_2 & s_4 & z^2 \end{vmatrix} =$$

$$= (2/45)(1 - b)(1 - 2b)(1 - b^2)(z + 1 - b) \,.$$
$$. \left[5z^2 - 2(1 + 2b)z - (1 - 3b)(1 + b)\right].$$

The roots are

$$z_1 = b - 1, \quad z_{23} = (1/5)\left[1 + 2b \mp (6 - 6b - 11b^2)^{1/2}\right].$$

If we insert $b = \frac{1}{2}$ (which is the case not covered by Theorem 4.5), we have $z_1 = -0{\cdot}5$, $z_2 = 0{\cdot}3$, $z_3 = 0{\cdot}5$. Solving the system of linear equations we get $p_1 = 0{\cdot}5$, $p_2 = 0$, $p_3 = 0{\cdot}5$, which is the exact solution (cf. Section 2). Similarly for $b = \frac{1}{3}$ we obtain $z_1 = -\frac{2}{3}$, $z_2 = 0$, $z_3 = \frac{2}{3}$ and $p_1 = p_2 = p_3 = \frac{1}{3}$ which also agrees with the exact solution.

Let X_t have the density

$$f(x) = 2^{-p-q+1}\,\mathrm{B}^{-1}(p, q)\,(x + 1)^{p-1}(x - 1)^{q-1}\,, \quad -1 < x < 1\,,$$

where $p > 0$ and $q > 0$ are given parameters and B is the beta function. In the special case when $p = q = 1$ we get the rectangular distribution mentioned above. The moments corresponding to f are

$$m_k = (-1)^k \sum_{i=0}^{k} \binom{k}{i} (-1)^i\, 2^i \frac{p(p + 1) \dots (p + i - 1)}{(p + q)(p + q + 1) \dots (p + q + i - 1)}\,.$$

For example, $p = q = 0{\cdot}5$ gives a U-distribution of X_t. Consider its moments $m_1, \dots, m_6$. For $b = 0{\cdot}15$ the problem has no solution and for $b = 0{\cdot}1$ we obtain the distribution of e_t in the form

$$\begin{array}{llll} z_1 = -0{\cdot}9\,, & z_2 = -0{\cdot}372\,, & z_3 = 0{\cdot}333\,, & z_4 = 0{\cdot}895\,, \\ p_1 = 0{\cdot}268\,, & p_2 = 0{\cdot}225\,, & p_3 = 0{\cdot}231\,, & p_4 = 0{\cdot}276\,. \end{array}$$

For $p = 0{\cdot}5$ and $q = 3$ using $m_1, \dots, m_6$ we come to the conclusion that for $b = 0{\cdot}35$ there is no solution and for $b = 0{\cdot}3$ the distribution of e_t is

$$\begin{array}{llll} z_1 = -0{\cdot}7\,, & z_2 = -0{\cdot}434\,, & z_3 = 0{\cdot}060\,, & z_4 = 0{\cdot}658\,, \\ p_1 = 0{\cdot}599\,, & p_2 = 0{\cdot}239\,, & p_3 = 0{\cdot}139\,, & p_4 = 0{\cdot}023\,. \end{array}$$

6. MOMENT PROBLEM ON $[0, \infty)$

Theorem 6.1. Let X_t be an AR(1) process defined by (2.1) with $0 \leqq b < 1$. If $e_t \geqq 0$ for all t, then $X_t \geqq 0$ for all t. If $\mathsf{P}(e_t < 0) > 0$, then $X_t < 0$ for infinitely many subscripts t with probability 1.

Proof. The first assertion is clear. If $\mathsf{P}(e_t < 0) > 0$, then there exist numbers $c > 0$ and $q \in (0, 1)$ such that $\mathsf{P}(e_t < -c) \geqq q$. Now, we apply Lemma 10.2. □

Theorem 6.2. A sequence of numbers $\{s_k\}_0^n$ is a system of moments on $[0, \infty)$ if and only if the matrices

$$\boldsymbol{A} = (s_{i+j})_{i,j=0}^{[n/2]}\,, \quad \boldsymbol{B} = (s_{i+j+1})_{i,j=0}^{[(n-1)/2]}$$

are positive semidefinite; [ ] denotes the integer part.

Proof. See [9], p. 237. □

An infinite sequence $\{s_k\}_0^\infty$ is a system of moments on $[0, \infty)$ if and only if the infinite quadratic forms

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} s_{i+j} x_i x_j\,, \quad \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} s_{i+j+1} x_i x_j$$

are positive semidefinite (see [1], p. 100).

Theorem 6.3. Let the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ from Theorem 6.2 be positive definite for a given sequence of numbers $\{s_k\}_0^{2r}$. Define

$$Q_{2r}(z) = \begin{vmatrix} 1 & s_0 & s_1 & \dots & s_{r-1} \\ z & s_1 & s_2 & \dots & s_r \\ \dots & \dots & \dots & \dots & \dots \\ z^r & s_r & s_{r+1} & \dots & s_{2r-1} \end{vmatrix},$$

$$Q_{2r+1}(z) = \begin{vmatrix} z & s_1 & s_2 & \dots & s_r \\ z^2 & s_2 & s_3 & \dots & s_{r+1} \\ \dots & \dots & \dots & \dots & \dots \\ z^{r+1} & s_{r+1} & s_{r+2} & \dots & s_{2r} \end{vmatrix}.$$

Then the roots of $Q_{2r}(z)$ are the points of concentration of the upper main representation and the roots of $Q_{2r+1}(z)$ are the points of concentration of the lower main representation.

Proof. See [9], p. 260. □

7. EXPONENTIAL DISTRIBUTION

Let $b \in (0, 1)$. If X_t has the exponential distribution $Ex(\lambda)$ with the density

$$f(x) = \lambda^{-1} \exp(-x/\lambda), \quad x > 0,$$

then in the model (2.1) the e_t must be a variable which equals to zero with probability b and which has $Ex(\lambda)$ with probability $1 - b$ (see [8]). The moments of e_t are $s_k = k!\,(1 - b)\,\lambda^k$. Let us consider the case of four moments. Both $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive definite for $b \in (0, 1)$, the roots of $Q_5(z)$ are

$$z_1 = 0, \quad z_2 = (3 - 3^{1/2})\,\lambda, \quad z_3 = (3 + 3^{1/2})\,\lambda$$

and the corresponding probabilities are

$$p_1 = (1 + 2b)/3, \quad p_2 = (1 - b)(2 + 3^{1/2})/6, \quad p_3 = (1 - b)(2 - 3^{1/2})/6.$$

Numerically, if $\lambda = 1$ and $b = 0{\cdot}5$, we have

$$z_1 = 0, \quad z_2 = 1{\cdot}268, \quad z_3 = 4{\cdot}732,$$
$$p_1 = 0{\cdot}667, \quad p_2 = 0{\cdot}311, \quad p_3 = 0{\cdot}022.$$

8. MOMENT PROBLEM ON $(-\infty, \infty)$

This case is described in [3] and so we introduce briefly only main results.

Theorem 8.1. A sequence of numbers $\{s_k\}_0^{2r}$ is a system of moments on $(-\infty, \infty)$ if and only if the matrix

$$\boldsymbol{A} = (s_{i+j})_{i,j=0}^{r}$$

is positive semidefinite.

Proof. See [9], p. 246. □

A sequence $\{s_k\}_0^\infty$ is a system of moments on $(-\infty, \infty)$ if and only if the infinite quadratic form

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} s_{i+j} x_i x_j$$

is positive semidefinite (see [9], p. 246).

Theorem 8.2. If $s_1, \ldots, s_{2r}$ are given numbers such that the matrix $\boldsymbol{A}$ given in Theorem 8.1 is positive definite, then the points of concentration of so called canonical representation which includes a given point w are the roots of

$$Q(z) = \begin{vmatrix} s_0 & s_1 & \ldots & s_{r-1} & 1 & 1 \\ s_1 & s_2 & \ldots & s_r & w & z \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ s_{r+1} & s_{r+2} & \ldots & s_{2r} & w^{r+1} & z^{r+1} \end{vmatrix} .$$

Proof. See [9], p. 247. □

9. A SPECIAL CASE ON $(-\infty, \infty)$

We proved in Section 5 that for $b \in (\frac{1}{3}, \frac{1}{2})$ there exists no distribution of e_t such that X_t belongs to $[-1, 1]$ and has the moments $m_1 = 0$, $m_2 = \frac{1}{3}$, $m_3 = 0$, $m_4 = \frac{1}{5}$. However, if we do not insist on the condition $X_t \in [-1, 1]$, then $\boldsymbol{A}$ being positive definite for $b \in (0, \frac{1}{2})$ allows to find a solution on the real line. The polynomial $Q(z)$ with $w = 0$ has the roots

$$z_{13} = \mp[(3 - 7b^2)/5]^{1/2}, \quad z_2 = 0,$$

and the corresponding probabilities are

$$p_1 = p_3 = \tfrac{5}{6}(1 - b^2)/(3 - 7b^2), \quad p_2 = \tfrac{4}{3}(1 - 4b^2)/(3 - 7b^2).$$

For example, if $b = 0{\cdot}4$ then

$$\begin{aligned} z_1 &= -0{\cdot}613, & z_2 &= 0, & z_3 &= 0{\cdot}613, \\ p_1 &= 0{\cdot}372, & p_2 &= 0{\cdot}256, & p_3 &= 0{\cdot}372. \end{aligned}$$

In this special case we obtain $-1{\cdot}532 \leqq X_t \leqq 1{\cdot}532$.

10. APPENDIX

Lemma 10.1. Let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be two sequences of events satisfying the following conditions.

(i) The events $A_1, A_2, \ldots$ are independent.
(ii) For each n, the events A_n and B_n are independent.
(iii) $\sum \mathsf{P}(A_n) = \infty$.
(iv) $\mathsf{P}(B_n) \to 1$.

Then infinitely many events $A_n \cap B_n$ occur with probability one.

Proof. See [5]. □

Lemma 10.2. Let X_t be an AR(1) process defined by (2.1) with $0 \leqq b < 1$. If there exist numbers $c > 0$ and $q \in (0, 1)$ such that $\mathsf{P}(e_t < -c) \geqq q$, then $X_t < 0$ for infinitely many subscripts with probability one.

Proof. Let j_k be the smallest integer such that

$$j_k q^k \geqq 1 \quad \text{for} \quad k = 1, 2, \dots$$

Arrange the positive integers into the subsets $S_1, S_2, \dots$ in the following way. Let $S_1 = \{1, \dots, j_1\}$. Let S_2 contain the elements of j_2 couples

$$(j_1 + 1, j_1 + 2), \quad (j_1 + 3, j_1 + 4), \dots, (j_1 + 2j_2 - 1, j_1 + 2j_2),$$

let S_3 contain elements of j_3 triples starting with $(j_1 + 2j_2 + 1, j_1 + 2j_2 + 2, j_1 + 2j_2 + 3)$ and so on. The numbers $1, \dots, j_1$, the last numbers of couples, the last numbers of triples etc. denote $n_1, n_2, \dots$. If $n_i \in S_k$ then we use the decomposition

$$X_{n_i} = U_{n_i} + Z_{n_i}$$

where

$$U_{n_i} = e_{n_i} + be_{n_i-1} + \dots + b^{k-1} e_{n_i-k+1},$$

$$Z_{n_i} = b^k e_{n_i-k} + b^{k+1} e_{n_i-k-1} + \dots$$

Define $\mu = \mathsf{E} e_t$, $\sigma^2 = \operatorname{var} e_t$. Then

$$\mathsf{E} Z_{n_i} = \mu b^k/(1 - b), \quad \operatorname{var} Z_{n_i} = \sigma^2 b^{2k}/(1 - b^2)$$

and thus

$$\mathsf{E} Z_{n_i}^2 = \operatorname{var} Z_{n_i} + (\mathsf{E} Z_{n_i})^2 \to 0 \quad \text{as} \quad i \to \infty.$$

Introduce events

$$A_i = \{U_{n_i} < -c\}, \quad B_i = \{Z_{n_i} < c\}.$$

Since

$$\mathsf{P}(Z_{n_i} \geqq c) \leqq c^{-2} \, \mathsf{E} Z_{n_i}^2 \to 0,$$

we have $\mathsf{P}(Z_{n_i} < c) \to 1$. It is clear that

$$\sum_{i=1}^{\infty} \mathsf{P}(A_i) = \underbrace{q + \dots + q}_{j_1} + \underbrace{q^2 + \dots + q^2}_{j_2} + \dots = \infty.$$

The events $A_1, A_2, \dots$ are independent and for each i the events A_i and B_i are also independent. Thus the assertion follows from Lemma 10.1. □

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REFERENCES

[1] N. I. Achiezer: Klassičeskaja problema momentov i nekotoryje voprosy analiza svjazannyje s neju. Gos. izd. fiz.-mat. lit., Moskva 1961.

[2] J. Anděl: Marginal distributions of autoregressive processes. In: Trans. Ninth Prague Conf. on Inform. Theory, Statist. Dec. Functions, Random Processes, Academia, Prague 1983, pp. 127—135.

[3] J. Anděl: On linear processes with given moments. J. Time Ser. Anal. *8* (1987), 373—378.
[4] J. Anděl and M. Garrido: On stationary distributions of some time series models. In: Trans. Tenth Prague Conf. on Inform. Theory, Statist. Dec. Functions, Random Processes, Academia, Prague 1988, pp. 193—202.
[5] J. Anděl and V. Dupač: An extension of the Borel lemma. Comment. Math. Univ. Carolin. *32* (1989), 405—407.
[6] J. Anděl and K. Zvára: Simulation methods in time series. In: Proc. 2nd Internat. Symp. on Numerical Analysis, Prague 1987 (I. Marek, ed.). Teubner-Texte zur Mathematik 107, Teubner, Leipzig 1988, pp. 99—113.
[7] G. P. Chamitov: Imitacija slučajnych processov. Izd. Irkutskogo instituta narodnogo chozjajstva, Irkutsk 1983.
[8] D. P. Gaver and P. A. W. Lewis: First-order autoregressive gamma sequences and point processes. Adv. Appl. Probab. *12* (1980), 727—745.
[9] M. T. Krejn and A. A. Nudelman: Problema momentov Markova i ekstremalnyje zadači. Izd. Nauka, Moskva 1973.
[10] M. T. Krein and A. A. Nudelman: The Markov Moment Problem and Extremal Problems. (Transl. of Math. Monographs, Vol. 50.) American Mathematical Society, Providence 1977.
[11] A. J. Lawrance and P. A. W. Lewis: Modelling and residual analysis of nonlinear autoregressive time series in exponential variables. J. Roy. Statist. Soc. Ser. B *47* (1985), 165—202.
[12] J. S. Ramberg, E. J. Dudewicz, P. R. Tadikamalla and E. F. Mykytka: A probability distribution and its uses in fitting data. Technometrics *21* (1979), 201—214.
[13] M. M. Sondhi: Random processes with specified spectral density and first-order probability density. Bell System Technical J. *62* (1983), 679—701.

Prof. RNDr. Jiří Anděl, DrSc., Univerzita Karlova, matematicko-fyzikální fakulta (Charles University — Faculty of Mathematics and Physics), Sokolovská 83, 186 00 Praha 8. Czechoslovakia.