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# POLYNOMIAL APPROACH TO POLE PLACEMENT IN MIMO *n-D* SYSTEMS[1]

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Matrix polynomial equations in *n-D* polynomials are employed to assign desired invariant polynomials to general *n-D* multi-input multi-output systems.

## 1. INTRODUCTION

A lot of dynamical properties of a linear system can be naturally expressed in terms of the positions of its poles. That is why the problem of pole placement has become so popular in control. In scalar linear systems, it is sufficient to assign just the characteristic polynomial. In multi-input multi-output systems, however, one must assign separately all the invariant polynomials (and not merely their multiple — the characteristic polynomial). The reason is that the characteristic polynomial itself does not say enough about the internal dynamics of a multi-input multi-output system.

For a scalar 2-*D* system, the problem of pole placement has been solved by several authors recently. We shall follow here the approach of [7] which is based on 2-*D* polynomial equations. The progress described in the present paper is twofold: at first, the multi-input multi-output systems are considered which call for matrix (instead of scalar) polynomial equations. At second, *n-D* systems are involved so that general *n-D* equations ($n \geqq 2$) apply.

## 2. BASIC DEFINITIONS

Throughout the paper, the *n-D* multi-input multi-output systems are described by their transfer matrices which are expressed as matrix polynomial fractions of the type

$$D_L^{-1}N_L \tag{1}$$

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or

$$N_R D_R^{-1} \tag{2}$$

where the matrices $D_L$, $N_L$ and $D_R$, $N_R$ have their entries in $\mathscr{R}[z_1, z_2, \ldots, z_n]$ which is the ring of real polynomials in $n$ indeterminates $z_1, z_2, \ldots, z_n$.

When facing a discrete $n$-$D$ system (such as an $n$-$D$ digital filter [1] or the discrete model of a complex plant described by partial differential equations [2]), the $z_i$'s are usually interpreted as delay operators working in various directions. In the case of a delay-differential system [7], they stand for $\exp(-h_i s)$ where $s$ is a complex variable in the Laplace transform and $h_i$'s are various (possibly noncommesurate) delays durations.

Let us now recall some basic concepts for left fractions such as (1). They are mostly cited from [6]. Similar facts for (2) are dual.

**Definition 1 (Factor Coprimeness).** The matrices $D_L$ and $N_L$ are *left factor coprime* iff they have only unimodular left common factors (i.e. those with nonzero but real determinants).

**Definition 2 (Zero Coprimeness).** The matrices $D_L$ and $N_L$ are *left zero coprime* iff the composite matrix $[D_L N_L]$ has a full row rank for every $n$-tuple $(z_1, z_2, \ldots, z_n)$ of complex numbers.

These two different types of coprimeness project into the following two different concepts of equivalence.

**Definition 3 (Intertwining Equivalence).** The matrices $D_L$ and $D_R$ from $\mathscr{R}[z_1, z_2, \ldots, z_n]$ are *equivalent* (or *intertwined*) iff there are two matrices $N_L$ and $N_R$ from $\mathscr{R}[z_1, z_2, \ldots, z_n]$ such that

$$D_L^{-1} N_L = N_R D_R^{-1} \tag{3}$$

where the both fractions are factor coprime.

**Definition 4 (Strict Equivalence).** The matrices $D_L$ and $D_R$ from $\mathscr{R}[z_1, z_2, \ldots, z_n]$ are *strictly equivalent* iff there are two matrices $N_L$ and $N_R$ from $\mathscr{R}[z_1, z_2, \ldots, z_n]$ such that (3) holds with the both fractions zero coprime.

It can be shown [6] that the matrices $D_L$ and $D_R$ of the same sizes are strictly equivalent iff there are two unimodular matrices $U$ and $V$ in $\mathscr{R}[z_1, z_2, \ldots, z_n]$ such that

$$D_L = U D_R V \tag{4}$$

Needless to say that the strict equivalence implies the intertwining one. The vice versa, however, is not true and two intertwined matrices need not be related by (4) in general.

Invariant polynomials of $n$-$D$ matrices can be formally defined in the same way as in 1-$D$:

**Definition 5 (Invariant Polynomials).** Let, for an $m \times m$ full rank matrix $D$, the greatest common divisors of its $k \times k$ minors are denoted by $d_k$, $1 \leqq k \leqq m$, and $d_0 = 1$. Then the *invariant polynomials* of $D$ are defined by

$$i_1 = d_1/d_0\,, \quad i_2 = d_2/d_1, \ldots, i_m = d_m/d_{m-1} \tag{5}$$

The *Smith form* of $D$ (in $\mathscr{R}[z_1, z_2, \ldots, z_n]$) is the matrix diag $(i_1, i_2, \ldots, i_m)$.

As expected, two $n$-$D$ polynomial matrices $D_L$ and $D_R$ are intertwined iff they have the same (nonunit) invariant polynomials [6]. Consequently, every $n$-$D$ polynomial matrix is intertwined with its Smith form but, in contrast to 1-$D$, is *not* strictly equivalent to it in general. That is why the unimodular operations (4) are not sufficient to calculate Smith forms in $n$-$D$. Fortunately, just the intertwining equivalence is what we often need to solve control problems.

## 3. FEEDBACK SYSTEMS

Consider an $n$-$D$ linear $l$-input $m$-output *system* given by its transfer matrix

$$A_L^{-1}B_L \tag{6}$$

where $A_L$ and $B_L$ are left factor coprime matrices in $\mathscr{R}[z_1, z_2, \ldots, z_n]$, $A_L$ is $m \times m$ and invertible while $B_L$ is $m \times l$.

Further consider a linear output feedback *controller* described by the transfer matrix

$$Q_R P_R^{-1} \tag{7}$$

where $P_R$, $Q_R$ are, respectively, $m \times m$ and $l \times m$ matrices in $\mathscr{R}[z_1, z_2, \ldots, z_n]$ and $P_R$ is invertible.

As usually, we assume that both the system and controller have their characteristic polynomials equal to the determinants of their transfer matrices which means that they are free of hidden modes.

Let them be connected in the standard feedback structure in Figure 1. Similarly as in [4] for 1-$D$ systems, we can derive that the matrix

$$A_L P_R + B_L Q_R \tag{8}$$

is interwined with the matrix

$$P_L A_R + Q_L B_R \tag{9}$$

where $A_R$, $B_R$ and $P_L$, $Q_L$ are defined via the following factor coprime matrix fractions

$$B_R A_R^{-1} = A_L^{-1} B_L \quad \text{and} \quad P_L^{-1} Q_L = Q_R P_R^{-1} \tag{10}$$

In addition, both the matrices (8) and (9) are intertwined with the denominators of any transfer matrix in the feedback loop in Figure 1 (provided that no cancellation nor extension has been made in these transfer matrices). Consequently, (8) and (9) have the same nonunit invariant polynomials and these polynomials equal (up to real multiples) nonunit invariant polynomials of every realization of the feedback system (provided that no hidden modes are added in this realization).

So to place desirably the poles (invariant polynomials) of the resultant feedback

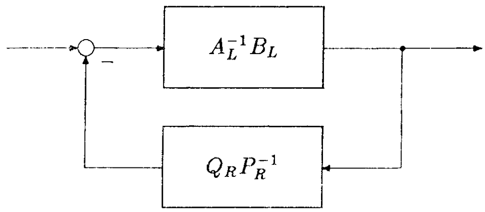


**Fig. 1.** Standard feedback structure.

system, one must first choose an $m \times m$ matrix $C_L$ or an $l \times l$ matrix $C_R$ (both in $\mathscr{R}[z_1, z_2, \ldots, z_n])$ having these desired invariant polynomials. Then the problem reads as follows:

**Formulation 1.** For the $A_L$, $B_L$ and $C_L$ find $P_R$ invertible and $Q_R$ such that

$$A_L P_R + B_L Q_R = C_L \tag{11}$$

or, equivalently, for the $A_R$, $B_R$ and $C_R$ find $P_L$ invertible and $Q_L$ such that

$$P_L A_R + Q_L B_R = C_R \,. \tag{12}$$

## 4. SOLUTION

We have now transformed the problem of pole placement (invariant polynomials assignment) into the solution of one from the $n$-$D$ matrix polynomial equations (11) or (12). The method of solution for such equations can be found in [8] and need not be repeated here. However, it may be interesting to analyze the results of simple examples. We focus our attention on (11) in what follows. The equation (12) is dual.

**Example 1.** Let be given a 4-$D$ system with

$$A_L = \begin{bmatrix} 1 + z_1 z_2 & 0 \\ z_3 + z_4 & 1 \end{bmatrix}$$

$$B_L = \begin{bmatrix} z_1 \\ z_3 + z_4 \end{bmatrix}$$

and

$$C_L = \begin{bmatrix} 2 + z_1 & 0 \\ 0 & 2 + z_2 \end{bmatrix}$$

When using the algorithm from [8], we result in the controller with

$$P_R = \begin{bmatrix} 2 & 0 \\ 2z_2 z_3 - 3z_3 + 2z_2 z_4 - 3z_4 & 2 + z_2 \end{bmatrix}$$

$$Q_R = [1 - 2z_2 \;\; 0]$$

It is well known that matrix polynomial equations may possess infinitely many solutions. In fact, here any matrices

$$(13) \qquad P_R + B_R T \quad \text{and} \quad Q_R - A_R T$$

where $T$ is an arbitrary compatible matrix with entries in $\mathscr{R}[z_1, z_2, \ldots, z_n]$, where

$$A_R = [1 + z_1 z_2]$$

$$B_R = \begin{bmatrix} z_1 \\ z_3 + z_4 - z_1 z_3 - z_1 z_4 + z_1 z_2 z_3 + z_1 z_2 z_4 \end{bmatrix},$$

give rise to the same invariant polynomials ($i_1 = 1$, $i_2 = (2 + z_1)(2 + z_2)$).

Whenever the $A_L$ and $B_L$ are zero coprime (as in Example 1) then one can assign any invariant polynomials and choose any $C_L$. In particular, all the invariant polynomials can bė taken units by setting $C_L = I$. This choice results in the so called *deadbeat controller* [3, 9] which for the given data reads

$$P_R = \begin{bmatrix} 1 & 0 \\ z_2 z_3 + z_2 z_4 - z_3 - z_4 & 1 \end{bmatrix}$$

$$Q_R = [-z_2 \;\; 0]$$

In fact, this is rather typical situation for $A_L$ and $B_L$ are generically left zero coprime whenever $l \geqq n$ (see [5]).

When, on the contrary, $A_L$ and $B_L$ have some (left) zeros in common then these zeros must occur in every choice of the resulting invariant polynomials with the right multiplicities. This is now well understood in scalar 2-$D$ systems [7] where such common zeros are usually called the fixed poles of the system.

**Example 2.** Consider now

$$(14) \qquad A_L = \begin{bmatrix} 1 & 1 + z_1 \\ -1 - z_1 & 0 \end{bmatrix}, \quad B_L = \begin{bmatrix} 0 \\ 2 + z_2 \end{bmatrix}$$

and, at the moment undeterminate, a right side matrix

$$C_L = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

In such a case, the algorithm from [8] yields directly a parametrization of all acceptable right hand sides. In fact, here the equation (11) is solvable iff $(2 + z_2)$ divides both $c_1(1 + z_1) + c_3$ and $c_2(1 + z_1) + c_4$ at the same time. So for

$$C_1 = \begin{bmatrix} 1 & 0 \\ -1 - z_1 & 2 + z_2 \end{bmatrix}$$

the solution exists being, e.g.,

$$P_R = \begin{bmatrix} 1 & -(1 + z_1)(2 + z_2) \\ 0 & 2 + z_2 \end{bmatrix}$$

and

$$Q_R = [0 \;\; -2z_1 - z_1^2]$$

but

$$C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 + z_2 \end{bmatrix}$$

makes (11) unsolvable.

This is rather surprising for $C_1$ and $C_2$ have not only the same invariant polynomials but they are even strictly equivalent:

$$C_1 = UC_2$$

where

$$U = \begin{bmatrix} 1 & 0 \\ -1 - z_1 & 1 \end{bmatrix}$$

Further studies will be sure useful to explain (system theoretically) why (11) possesses no solution for $C_L = C_2$ even if its invariant polynomials ($i_1 = 1, i_2 = 2 + z_2$) can be easily assigned to the given system (6), either by employing $C_L = C_1$ or by using, from the very beginning, another matrix fraction representation for the plant, say $\bar{A}_L = U^{-1}A_L$ and $\bar{B}_L = U^{-1}B_L$ (here clearly $\bar{A}_L^{-1}\bar{B}_L = A_L^{-1}B_L$).

Let us note that the same situation may happen in 1-$D$ if $A_L$ and $B_L$ are not coprime and also there it is not yet understood. However, in 1-$D$ one can always pre-cancel the fraction (6) to result in coprime left side of (11) before a $C_L$ is chosen which is not the case in $n$-$D$ (for $n \geqq 2$).

Fortunately, when using the method [8] to solve (11), one can calculate a parametrization of all acceptable right sides before choosing $C_L$ (as in the Example 2). Such a way, all the problems above are overcome.

**Example 3.** As another example, consider an unstable delay-differential plant with the transfer matrix

$$A_L^{-1}B_L = \begin{bmatrix} s & 1 + s \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} \exp(-hs) \\ 1 \end{bmatrix} \tag{15}$$

Using again the method of [7], we first solve (11) for $C_L = I$ to get (substituting for brevity $d = \exp(-hs)$)

$$\begin{bmatrix} s & 1 + s \\ 0 & s \end{bmatrix} \begin{bmatrix} d - 1 & d - d^2 \\ 1 & -d \end{bmatrix} + \begin{bmatrix} d \\ 1 \end{bmatrix} [-s \;\; 1 + ds] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{16}$$

and the right side fraction for (15) as a by-product

$$B_R A_R^{-1} = \begin{bmatrix} -1 - s + sd \\ s \end{bmatrix} [s^2]^{-1} \tag{17}$$

It is easy to see from (16) that the fraction (15) is zero coprime. As a consequence, any stable invariant polynomials can be assigned to stabilize the plant (15).

However, the solution appearing in (16) can not be used directly as $\det P = 0$. In such a case, one must apply (13) to get a suitable solution.

In practical cases one usually wishes to design a proper controller. Clearly, the controller made directly from (16) is an improper one but this is just a preparatory step. It is well known that, to get a proper controller, the degrees of $C_L$ must be sufficiently high. Applying the result of [10], one must take the first-row-degree of $C_L$ to be the first-row-degree of the matrix $[A_L\ B_L]$ plus the maximum degree of $\begin{bmatrix} A_R \\ B_R \end{bmatrix}$ minus one which here equals 2. Similarly, the second-row-degree of $C_L$ must be, at least, 2 as well. For example,

$$C_L = \begin{bmatrix} (1+s)^2 & 0 \\ 0 & (1+s)^2 \end{bmatrix} \tag{18}$$

will do the job. Using now the algorithm [7], we obtain the controller (7) with

$$P_R = \begin{bmatrix} 1 + s + \exp(-hs) & 1 + s - \exp(-hs) - \exp(-2hs) \\ 1 & \exp(-hs) - s \end{bmatrix}$$

$$Q_R = [-s \quad 1 + s(2 + \exp(-hs))]$$

This proper retarded controller stabilizes the plant (15) by assigning (18), i.e. the finite number of poles characterized by the invariant polynomials $i_1 = i_2 = (1 + s^2)$.

Finally, recall that if the system (6) is strictly causal (i.e., $A_L(0)$ is invertible and $B_L(0) = 0$) then every solution of (11) is causal ($P_R(0)$ is invertible) for a causal $C_L$. Otherwise, noncausal solutions also exist and if a causal one is desired then (13) can be used if necessary.

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