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EQUIVALENCE, INVARIANCE AND DYNAMICAL SYSTEM CANONICAL MODELLING

Part I. Invariant Properties of Observable Models and Associated Transformations

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The first part of this paper considers two different classes of models for linear observable multivariable systems: state-space models and polynomial input-output models. Equivalence relations that do not affect the input-output behavior of the considered models are then introduced, as well as associated sets of canonical forms directly parametrized by the image of all models belonging to the same equivalence class in a complete set of independent invariants for the considered equivalence relations.

1. DYNAMICAL SYSTEM MODELLING

The roots of System Theory lie in the acknowledgment of the comparatively restricted class of mathematical models that can describe the behavior of highly differentiated aspects of reality. The nature of real systems is transparent to System Theory in that they are substituted with an *equivalent* mathematical model i.e. with a set of mathematical relations which, it is assumed, describe the links between input and output evolutions in the real system.

The substitution of a real process with a model is a critical step since it involves an equivalence definition and a certain degree of simplicity in the model may compensate for a possible lack of accuracy. However, once a mathematical model has been selected no further critical decisions should, conceptually, be expected; but this is only partly true since a model does not describe the (unique) reality of a (physical) process but only its behavior so that even models belonging to formally different classes can be considered equivalent as long as they describe the same behavior. Equivalence definitions can also be considered among the elements of a specified class of models, usually corresponding to algebraic manipulations that can be performed on a single model, leading to a new model of the same class which describes the same behavior. In practice the availability of several models in the same class or in different classes to describe the same process can be advantageously exploited by careful selection of the most suitable model for every application.

This work is concerned with linear multivariable systems and their models. The purpose is to show that models which are, formally, quite different but describe the same process are in fact linked by very simple algebra if properly selected within given equivalence classes. The results and algorithms obtained can be used in broad classes of realization problems. This first part of the paper consists of seven sections with the following contents.

Section 2. In this section three different classes of models for linear multivariable systems are considered. The first is given by the usual state–space models; the restriction of this class to observable, reachable and minimal systems are denoted with Σ_o, Σ_c and Σ_m respectively. The second is given by polynomial input–output models. This class can describe only observable and minimal systems; these restrictions are denoted with S_o and S_{oc}. The third is given by polynomial input – partial state – output models, which can describe only reachable and minimal systems; these restrictions are denoted with S_c and S_{co} respectively. For every considered class of models a "natural" equivalence relation is introduced. The term "natural" here means that all the elements belonging to the same equivalence class describe exactly the same dynamical behavior.

Section 3. In this section the classical notion of invariant function with respect to a given equivalence relation is recalled. The associated definitions of complete invariant, set of invariants, complete set of independent invariants, are then given. The second concept that can be associated to equivalence classes i.e. the definition of a set of canonical forms for a given equivalence relation follows. The links between complete sets of independent invariants and canonical forms for the same equivalence relation are then discussed.

Section 4. This section deals with a well-known set of canonical forms for the given equivalence relation on Σ_o and Σ_m and shows how these canonical models are parametrized by the image in a complete set of independent invariants of every element belonging to the same equivalence class.

Section 5. This section defines a set of canonical forms for the considered equivalence relation on S_o and S_{oc} and shows that their parametrization is the image in a complete set of independent invariants of every element belonging to the same equivalence class.

Section 6. The canonical forms that have been independently defined on both Σ_0 and S_o are compared and the elementary algebraic links between these formally different representations deduced.

Section 7. This section contains some short concluding remarks.

References to the contents of parts I and II are made according to the following rules: Definitions, theorems, lemmas, corollaries, properties, remarks, figures and algorithms: (p, n) where p refers to the considered part and n is a progressive number. Equations, relations, formulae and examples: (p, s, n) where p refers to parts, s to sections and n is a progressive number.

2. SETS OF DYNAMICAL MODELS FOR LINEAR MULTIVARIABLE SYSTEMS AND ASSOCIATED EQUIVALENCE RELATIONS

Let $\mathscr{F}$ denote an arbitrary field and r, m and n integers with $n \geqq \max(r, m)$. The linear, finite–dimensional, purely dynamic systems considered in the following will be described by means of the following representations.

1) Sets of State-Space Models Σ, Σ_o, Σ_c and Σ_m

These models consist of the equations

(1.2.1a) $$x(t + 1) = F\,x(t) + G\,u(t)$$

(1.2.1b) $$y(t) = H\,x(t)$$

where $t \in \mathscr{Z}$, $x(t) \in \mathscr{F}^n = \mathscr{X}$ is the state, $u(t) \in \mathscr{F}^r = \mathscr{U}$ is the input, $y(t) \in \mathscr{F}^m = \mathscr{Y}$ is the output, $F \in \mathscr{F}^{(n \times n)}$ is the system dynamical matrix, $G \in \mathscr{F}^{(n \times r)}$ is the input distribution matrix and $H \in \mathscr{F}^{(m \times n)}$ is the output distribution matrix.

Definition 1.1. The set of all triples (F, G, H) with $n \geqq 1$ will be denoted with Σ. Σ_o will denote the subset of Σ of all triples (F, G, H) with rank $(H) = m$ and completely observable, i.e. such that

(1.2.2) $$\operatorname{rank}\left[H^{\mathrm{T}}\ \ F^{\mathrm{T}}H^{\mathrm{T}} \ldots F^{\mathrm{T}(n-m)}H^{\mathrm{T}}\right] = n\,.$$

Σ_c will denote the subset of Σ of all triples (F, G, H) with rank $(G) = r$ and completely reachable, i.e. such that

(1.2.3) $$\operatorname{rank}\left[G\ \ FG \ldots F^{(n-r)}G\right] = n\,.$$

Σ_m will denote the intersection of Σ_o and Σ_c, i.e. the subset of Σ of all the triples (F, G, H) completely reachable and completely observable.

2) Sets of Input-Output Models S_o and S_{oc}

These models consist of the equation

(1.2.4) $$P(z)\,y(t) = Q(z)\,u(t)$$

where $t \in \mathscr{Z}$, $y(t) \in \mathscr{F}^m = \mathscr{Y}$ is the output and $u(t) \in \mathscr{F}^r = \mathscr{U}$ is the input. $P(z)$ and $Q(z)$ are $(m \times m)$ and $(m \times r)$ matrices whose entries are defined over the ring of polynomials in z (unitary advance operator) defined over $\mathscr{F}$ with $n = \deg\det\{P(z)\} \geqq 1$.

Remark 1.1. In the previous definition it has not been assumed that $P(z)$ and $Q(z)$ be left coprime. Their greatest common left divisor can therefore be a non–unimodular polynomial matrix and model (1.2.4) can therefore be *strictly equivalent* (see Definition 1.9) to a non–completely reachable system [1].

Definition 1.2. The set of all pairs $(P(z), Q(z))$ will be denoted by S_o while the subset of S_o of all the pairs $(P(z), Q(z))$ with $P(z)$ and $Q(z)$ left coprime, will be denoted by S_{oc}. The elements of S_{oc} are therefore strictly equivalent to completely observable and completely reachable state space models, i.e. to elements of Σ_m.

Remark 1.2. The considered models (1.2.4) represent, by hypothesis, purely dynamical systems. The entries of the associated transfer matrix $T(z) = P^{-1}(z)\,Q(z)$ ($P(z)$ is nonsingular since $n \geqq 1$) are therefore strictly proper rational functions.

3) Sets of Input — Partial State — Output Models S_c and S_{co}

These models consist of the equations

(1.2.5a) $$R(z)\,w(t) = u(t)$$

(1.2.5b) $$y(t) = S(z)\,w(t)$$

where $t \in \mathscr{T}$, $u(t) \in \mathscr{F}^r = \mathscr{U}$ is the input, $w(t) \in \mathscr{F}^r = \mathscr{W}$ is the partial state and $y(t) \in \mathscr{F}^m = \mathscr{Y}$ is the output. $R(z)$ and $S(z)$ are $(r \times r)$ and $(m \times r)$ matrices whose entries are defined over the ring of polynomials in z (unitary advance operator) defined over $\mathscr{F}$ with $n = \deg\det\{R(z)\} \geqq 1$.

Remark 1.3. In the previous definition it has not been assumed that $R(z)$ and $S(z)$ are right coprime. Their greatest common right divisor can therefore be a non-unimodular polynomial matrix and the model (1.2.5) can be *strictly equivalent* (see Definition 2.2) to a non completely observable system [1].

Definition 1.3. The set of all pairs $(R(z), S(z))$ will be denoted with S_c while the subset of S_c of all pairs $(R(z), S(z))$ with $R(z)$ and $S(z)$ right coprime will be denoted with S_{co}. The elements of S_{co} are therefore strictly equivalent to completely reachable and completely observable state space models, i.e. to elements of Σ_m.

Remark 1.4. The models considered (1.2.5) represent, by hypothesis, purely dynamic systems. The entries of the associated transfer matrix $T(z) = S(z)\,R^{-1}(z)$ ($R(z)$ is nonsingular since $n \geqq 1$) are therefore strictly proper rational functions.

Equivalence Relations on Σ, S_o and S_c

The symbol E will denote the following equivalence relations. On Σ:

(1.2.6) $$(F, G, H)\,E\,(\tilde{F}, \tilde{G}, \tilde{H}) \quad \text{if} \quad \tilde{F} = TFT^{-1}\,, \quad \tilde{G} = TG\,, \quad \tilde{H} = HT^{-1}$$

where $T \in \mathscr{F}^{(n \times n)}$ is a nonsingular matrix. The same equivalence relation will be considered on Σ_o, Σ_c and Σ_m since these subsets are closed with respect to E. On S_o:

(1.2.7) $$(P(z), Q(z))\,E\,(\tilde{P}(z), \tilde{Q}(z)) \quad \text{if} \quad \tilde{P}(z) = M(z)\,P(z)\,, \quad \tilde{Q}(z) = M(z)\,Q(z)$$

where $M(z)$ is an $(m \times m)$ nonsingular unimodular polynomial matrix. Since the inverses and the products of nonsingular unimodular matrices are still nonsingular unimodular matrices it is easy to verify that relation (1.2.7) is an equivalence relation i.e. it is reflexive, symmetric and transitive. The same equivalence relation will be considered on S_{oc} since this subset is closed with respect to E. On S_c:

$$(1.2.8) \quad (R(z), S(z))\, E\, (\tilde{R}(z), \tilde{S}(z)) \quad \text{if} \quad \tilde{R}(z) = R(z)\, M(z), \; \tilde{S}(z) = S(z)\, M(z)$$

where $M(z)$ is an $(r \times r)$ nonsingular unimodular polynomial matrix. Since the set of such matrices is closed with respect to inversion and multiplication, it is easy to verify that relation (1.2.8) is also an equivalence relation i.e. it is reflexive, symmetric and transitive. The same equivalence relation will be considered on S_{co} since this subset is closed with respect to E.

3. COMPLETE SETS OF INDEPENDENT INVARIANTS AND CANONICAL FORMS FOR EQUIVALENCE RELATIONS

Complete Sets of Independent Invariants for Equivalence Relations

Definition 1.4. Denote a set with X and an equivalence relation defined on X with E. Then denote with S a second set and with $f: X \to S$ a function. If x' and x'' are two elements of X, and f is such that $x'Ex''$ implies $f(x') = f(x'')$ then f is called an *invariant* for E. Moreover if $f(x') = f(x'')$ implies $x'Ex''$ then f is called a *complete invariant* for E.

If f is a complete invariant for E then all the elements of X belonging to the same equivalence class have the same image in f; moreover, these classes coincide exactly with the inverse images in f of the elements of the image (or range) of f. There exists, therefore, a bijection between the quotient set X/E and the image of a complete invariant for E.

Definition 1.5. A set of invariants $f_1, \ldots, f_n, f_i: X \to S_i$ for E is called a *complete set of invariants* for E if the function $f = (f_1, \ldots, f_n): X \to S_1 \times \ldots \times S_n$ defined by $c \to (f_1(x), \ldots, f_n(x))$ is a complete invariant for E.

Definition 1.6. A set of invariants for E, $f_1, \ldots, f_n, f_i: X \to S_i$ will be called *independent* if the associated invariant $f = (f_1, \ldots, f_n): X \to S_1 \times \ldots \times S_n$ is surjective.

This condition, which is more restrictive than the one given in [2], implies that no invariant f_i can be expressed as a function of the others. This last condition, however, is much weaker than the given definition of independence.

A complete set of independent invariants for E is also called a *basis* for E on X.

Lemma 1.1 [2] [3]. Let $f: X \to S$ be a complete surjective invariant for E. Then every other invariant for E can be uniquely computed from f.

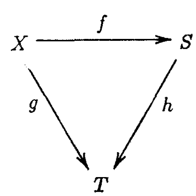

Fig. 1.1.

Proof. Let $f: X \to S$ and $g: X \to T$ be, respectively, a complete surjective invariant and a generic invariant for E. Commutativity in the diagram of Fig. 1.1 can be obtained if and only if for every element s of S the function h is defined as $h(s) = g(x)$ where x is any element of X such that $f(x) = s$. Since f is complete and surjective and g is an invariant, h is well defined for all the elements of S. □

Corollary 1.1. Let f: $X \to S$ be a complete set of independent invariants for E. Then every other invariant for E can be uniquely computed from f.

Property 1.1. Let f: $X \to S$ be a complete set of independent invariants for E. If g: $S \to R$ is a bijection, then $h = g \cdot f: X \to R$ is a complete set of independent invariants for E.

Canonical Forms for Equivalence Relations [3]

Definition 1.7. Let E be an equivalence relation on X. A subset C of X is called a set of *canonical forms* for E if every $x \in X$ is equivalent under E to one and only one element of C; this element is *the* canonical form of x. The function $g: X \to C$ thus defined is therefore a complete invariant for E. Obviously g can be assumed surjective without loss of generality.

Complete Sets of Independent Invariants and Canonical Forms

Let f: $X \to S$ be a complete set of independent invariants and C a set of canonical forms for E. Then (Corollary 1.1) there exists a unique function h: $S \to C$ such that $g = h \cdot f$. Since g is complete, h is a bijection. Moreover if i: $C \to X$ is the injection $i(c) = c$ then it follows that $h^{-1} = f \cdot i$. The following theorem has thus been proved.

Theorem 1.1 [2]. Let C be a set of canonical forms for an equivalence relation E on X and f a complete set of independent invariants for E. Then there exists a unique bijection between C and the image of f.

4. CANONICAL FORMS ON Σ_o AND Σ_m

Let (F, G, H) be an element of Σ_o with dim $(F) = n$ and

$$(1.4.1) \qquad H = \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} \qquad G = [g_1 \dots g_r] .$$

Consider then the m sequences of vectors given by

$$(1.4.2) \qquad \begin{matrix} h_1^T \; F^T h_1^T \dots F^{T(n-m+1)} h_1^T \\ \vdots \\ h_m^T \; T^T h_m^T \dots F^{T(n-m+1)} h_m^T \end{matrix}$$

Now order vectors (1.4.2) as follows

$$h_1^T \dots h_m^T\, F^T h_1^T \dots F^T h_m^T \dots F^{T(n-m+1)} h_m^T \tag{1.4.3}$$

and select, in sequence (1.4.3), the vectors linearly independent of preceding ones. Let $F^{T\nu_i^o} h_i^T$ be the first vector, belonging to the ith row of (1.4.2), linearly dependent on preceding ones in (1.4.3), i.e. such that

$$F^{T\nu_i^o} h_i^T = \sum_{j=1}^{m} \sum_{k=1}^{\nu_{ij}^o} \alpha_{ijk}^o F^{T(k-1)} h_j^T \tag{1.4.4}$$

where, because of the order of the vectors in sequence (1.4.3), the integers ν_{ij}^o are given by

$$\begin{aligned} \nu_{ij}^o &= \nu_i^o && \text{for} \quad i = j \\ \nu_{ij}^o &= \min(\nu_i^o + 1, \nu_j^o) && \text{for} \quad i > j \\ \nu_{ij}^o &= \min(\nu_i^o, \nu_j^o) && \text{for} \quad i < j \end{aligned} \tag{1.4.5}$$

The *total* number of scalars α_{ijk}^o thus defined is therefore given by

$$l = \sum_{i=1}^{m} \sum_{j=1}^{m} \nu_{ij}^o . \tag{1.4.6}$$

As is well known, dependence relation (1.4.4) implies also the linear dependence of all subsequent vectors belonging to the ith row of (1.4.2) (i.e. of the type $F^{T(\nu_i^o+k)} h_i^T$, $k \geq 1$) from their antecedents in sequence (1.4.3).

The linearly independent vectors selected in sequence (1.4.3) are called *regular vectors* [4].

Remark 1.5. Since rank $(H) = m$, all the integers ν_i^o are greater than zero.

Remark 1.6. Because of the complete observability of all the elements of Σ_o, the regular vectors constitute a basis for $\mathscr{X}$, i.e. $\nu_1^o + \dots + \nu_m^o = n$.

Denote now with

$$b_{ijk}^o = \langle g_j, F^{T(k-1)} h_i^T \rangle \tag{1.4.7}$$

the scalar products of the columns of G, g_j with the regular vectors.

Definition 1.8. The integers ν_i^o $(i = 1, \dots, m)$ obtained by means of the outlined procedure are called in the literature *Kronecker invariants* of the pair (F^T, H^T) since they are coincident (modulo ordering) with Kronecker's minimal column indices for the singular matrix pencil $[zI - F^T \mid H^T]$ [5], [6]. These indices will, in the following, be called *structural invariants* of (F, H) or *observation invariants* of (F, G, H). The scalars α_{ijk}^o which appear in (1.4.4) will be called *characteristic parameters* of the pair (F, H), and the scalars b_{ijk}^o which appear in (1.4.7) will be called *input parameters* of (F, G, H).

A set of scalars $(\nu_i^o, \alpha_{ijk}^o, b_{ijk}^o)$ has been associated to every element (F, G, H)

of Σ_o. A function

$$f_0 = (f_i^{ov}, f_{ijk}^{o\alpha}, f_{ijk}^{ob}) : \Sigma_o \rightarrow N^m \times \mathscr{F}^l \times \mathscr{F}^{(n\times r)}$$

has thus been implicitly defined. Here, and in the following, N denotes the set of natural numbers. It is now possible to prove the following theorem.

Theorem 1.2. The function $f_o = (f_i^{ov}, f_{ijk}^{o\alpha}, f_{ijk}^{ob})$ constitutes a complete set of independent invariants for equivalence relation (1.2.6) on Σ_o.

Proof.

Invariance of f_o

Let (F, G, H) and (F', G', H') be two elements of Σ_o with $(F, G, H)\, E\, (F', G', H')$. It will be proved that $f_o(F, G, H) = f_0(F', G', H')$. Because of the given definition for E there exists a nonsingular matrix $T \in \mathscr{F}^{(n\times n)}$ such that $F' = TFT^{-1}$, $G' = TG$ and $H' = HT^{-1}$. Sequence (1.4.3) for (F', G', H') is given by

$$(1.4.8) \qquad (T^{-1})^{\mathrm{T}} h_1^{\mathrm{T}} \ldots (T^{-1})^{\mathrm{T}} h_m^{\mathrm{T}} \ldots (T^{-1})^{\mathrm{T}} F^{\mathrm{T}(n-m+1)} h_m^{\mathrm{T}} .$$

Because of the nonsingularity of T^{-1} the linear dependence relationships among vectors (1.4.8) are obviously the same as among vectors (1.4.3). It follows, therefore, that $f_i^{ov}(F, G, H) = f_i^{ov}(F', G', H')$ and $f_{ijk}^{o\alpha}(F, G, H) = f_{ijk}^{o\alpha}(F', G', H')$. Now denote with R the basis of $\mathscr{X}$ given by the regular vectors ordered as follows

$$(1.4.9) \qquad R = [h_1^{\mathrm{T}} \ldots F^{\mathrm{T}(v_1{}^o-1)} h_1^{\mathrm{T}} \mid \ldots \mid h_m^{\mathrm{T}} \ldots F^{\mathrm{T}(v_m{}^o-1)} h_m^{\mathrm{T}}] .$$

Because of the given definition (1.4.7) the scalars b_{ijk}^o are the entries of the matrix $R^{\mathrm{T}}G$. When the triple (F', G', H') is considered, because of (1.4.8) it immediately follows that the scalars $b_{ijk}'^{o}$ are the entries of the matrix $R'^{\mathrm{T}}G' = ((T^{-1})^{\mathrm{T}} R)^{\mathrm{T}}\, TG = R^{\mathrm{T}}G$. Therefore $f_{ijk}^{ob}(F, G, H) = f_{ijk}^{ob}(F', G', H')$ and, consequently, $f_o(F, G, H) = f_o(F', G', H')$.

Completeness of f_o

Let (F, G, H) and (F', G', H') be two elements of Σ_o such that $f_o(F, G, H) = f_o(F', G', H') = (v_i^o, \alpha_{ijk}^o, b_{ijk}^o)$. It will be proved that $(F, G, H)\, E\, (F', G', H')$. Since $v_i^o = v_i'^o$ it follows that the regular vectors associated to (F', G', H') are generated exactly in the same way as vectors (1.4.9), i.e.

$$(1.4.10) \qquad R' = [h_1'^{\mathrm{T}} \ldots F'^{\mathrm{T}(v_1{}^o-1)} h_1'^{\mathrm{T}} \mid \ldots \mid h_m'^{\mathrm{T}} \ldots F'^{\mathrm{T}(v_m{}^o-1)} h_m'^{\mathrm{T}}] .$$

Now define the nonsingular matrix

$$(1.4.11) \qquad T^{\mathrm{T}} = R'R^{-1}$$

so that

$$(1.4.12) \qquad R' = T^{\mathrm{T}}R$$

$$(1.4.13) \qquad F'^{\mathrm{T}(k-1)} h_i'^{\mathrm{T}} = T^{\mathrm{T}} F^{\mathrm{T}(k-1)} h_i^{\mathrm{T}} \quad (i = 1, \ldots, m;\ k = 1, \ldots, v_i^o) .$$

Relation (1.4.13) for $i = 1, \ldots, m$ and $k = 1$ implies $H' = HT$. Moreover, since

$\alpha^o_{ijk} \approx \alpha'^o_{ijk}$ it also holds that

(1.4.14) $$F'^{\mathrm{T}\nu_i{}^o} h_i'^{\mathrm{T}} = T^{\mathrm{T}} F^{\mathrm{T}\nu_i{}^o} h_i^{\mathrm{T}} \quad (i = 1, \ldots, m).$$

From (1.4.13) and (1.4.14) it is possible to write

$$F'^{\mathrm{T}} R' = T^{\mathrm{T}} F^{\mathrm{T}} R$$

and, consequently,

$$F'^{\mathrm{T}} = T^{\mathrm{T}} F^{\mathrm{T}} R R'^{-1} = T^{\mathrm{T}} F^{\mathrm{T}} (T^{\mathrm{T}})^{-1}$$

$$F' = T^{-1} F T.$$

From condition $b^o_{ijk} = b'^o_{ijk}$ it follows that

$$R'^{\mathrm{T}} G' = R^{\mathrm{T}} G$$

or also

$$G' = (R'^{\mathrm{T}})^{-1} R^{\mathrm{T}} G = T^{-1} G.$$

It has thus been proved that $(F, G, H)\, E\, (F', G', H')$ and, therefore, that the set $(f_i^{o\nu}, f_{ijk}^{o\alpha}, f_{ijk}^{ob})$ constitutes a complete invariant for E.

Independence of f_o

Let $(\nu_1^o, \ldots, \nu_m^o)$ be an arbitrary element of N^m with $n = \nu_1^o + \ldots + \nu_m^o$, $\nu_i^o \neq 0$, (α^o_{ijk}) an arbitrary element of $\mathscr{F}^l$ and (b^o_{ijk}) an arbitrary element of $\mathscr{F}^{(n \times r)}$. It will be proved that there exists an element $(F, G, H) \in \Sigma_o$ such that $f_o(F, G, H) = (\nu_i^o, \alpha^o_{ijk}, b^o_{ijk})$ i.e. that f_o is surjective with respect to $N^m \times \mathscr{F}^l \times \mathscr{F}^{(n \times r)}$. This will ensure the independence of the considered set of functions.

Choose an arbitrary basis, R, of $\mathscr{X}$ and denote its vectors as follows:

(1.4.15) $$R = [e_{11} \ldots e_{1\nu_1{}^o} \mid e_{21} \ldots e_{2\nu_2{}^o} \mid \ldots \mid e_{m1} \ldots e_{m\nu_m{}^o}].$$

Define now the rows of the $(m \times n)$ matrix H as

(1.4.16) $$h_i = e_{i1}^{\mathrm{T}} \quad (i = 1, \ldots, m)$$

while the columns of the $(n \times n)$ matrix $F^{\mathrm{T}} R$ are defined by the following relations

(1.4.17a) $$F^{\mathrm{T}} e_{ij} = e_{i(j+1)}$$

(1.4.17b) $$F^{\mathrm{T}} e_{i\nu_i{}^o} = \sum_{j=1}^{m} \sum_{k=1}^{\nu_{ij}{}^o} \alpha^o_{ijk} e_{jk}.$$

Since R is nonsingular, the n relations (1.4.17) define F uniquely. Similarly the columns of $R^{\mathrm{T}} G$ (and, consequently, of G) are defined by means of the relations

(1.4.18) $$R^{\mathrm{T}} g_i = [b_{1i1} \ldots b_{1i\nu_1{}^o} | \ldots | b_{mi1} \ldots b_{mi\nu_m{}^o}]^{\mathrm{T}}.$$

It is now necessary to verify that the image in f_o of (F, G, H) defined by relations (1.4.16), (1.4.17) and (1.4.18) is $(\nu_i^o, \alpha^o_{ijk}, b^o_{ijk})$. From (1.4.16), (1.4.17a) and (1.4.17b) it follows that

(1.4.19) $$e_{ij} = F^{\mathrm{T}} e_{i(j-1)} = \ldots = F^{\mathrm{T}(j-1)} e_{i1} = F^{\mathrm{T}(j-1)} h_i^{\mathrm{T}}.$$

Substitution of (1.4.19) in (1.4.17b), in (1.4.15) and, consequently, in (1.4.18) directly leads to relations (1.4.4) and (1.4.7). It is thus proved that $f^{o\alpha}_{ijk}(F, G, H) = (\alpha^{o}_{ijk})$, $f^{ob}_{ijk}(F, G, H) = (b^{o}_{ijk})$. Now let $\tilde{v}^{o}_{i} = f^{ov}_{i}(F, G, H)$; from the substitution of (1.4.19) in (1.4.17b) it follows that $\tilde{v}^{o}_{i} \leqq v^{o}_{i}$ but the substitution of (1.4.19) in (1.4.15) leads to relation $\tilde{v}^{o}_{1} + \ldots + \tilde{v}^{o}_{m} = n$ so that $\tilde{v}^{o}_{i} = v^{o}_{i}$ and $f^{ov}_{i}(F, G, H) = (v^{o}_{i})$. □

The following corollary directly follows from Property 1.1.

Corollary 1.2. Let g: $N^{m} \times \mathscr{F}^{l} \times \mathscr{F}^{(n \times r)} \to N^{m} \times \mathscr{F}^{l} \times \mathscr{F}^{(n \times r)}$ be a bijection. The function $g \cdot f_{o}$ is a complete set of independent invariants for E on Σ_{o}.

In [4] it is proved (in the dual case of completely reachable systems and with a weaker definition of independence) that $f'_{o} = (f^{ov}_{i}, f^{o\alpha}_{ijk})$ constitutes a complete set of independent invariants for equivalence relation (1.2.6) on the set of the pairs (F, H). The image of f'_{o}, however, does not allow to parametrize the quotient set Σ_{o}/E.

Canonical Forms on Σ_{o}

$f_{o} = (f^{ov}_{i}, f^{o\alpha}_{ijk}, f^{ob}_{ijk})$ is a complete set of independent invariants for E on Σ_{o}. The image of f_{o}, $(v^{o}_{i}, \alpha^{o}_{ijk}, b^{o}_{ijk})$ can therefore be used to parametrize Σ_{0}/E i.e. to construct a set of canonical forms for E on Σ_{o}.

Definition of the Set of Canonical Forms C_{o}

Very useful canonical forms are the multicompanion ones that can be directly obtained from the set of scalars $(v^{o}_{i}, \alpha^{o}_{ijk}, b^{o}_{ijk})$. This canonical subset of Σ_{o} will be denoted with C_{o}. The elements of C_{o} can be constructed by means of relations (1.4.15)–(1.4.18) when choosing the natural basis for $\mathscr{X}$. From $R = I_{n}$ in fact it follows that

(1.4.20)
$$\tilde{H} = \begin{bmatrix} 1\,0\,0 & \ldots & 0\,0\,0 & \ldots & 0\,0\,0 & \ldots & 0 \\ 0\,0\,0 & \ldots & 0\,1\,0 & \ldots & 0\,0\,0 & \ldots & 0 \\ \vdots & & & & & & \vdots \\ 0\,0\,0 & \ldots & 0\,0\,0 & \ldots & 0\,1\,0 & \ldots & 0 \end{bmatrix}$$
$$\begin{matrix} \uparrow & & \uparrow & & \uparrow \\ 1 & & (v_{1}^{o}+1) & & (v_{1}^{o}+\ldots+v_{m-1}^{o}+1) \end{matrix}$$

(1.4.21a)
$$\tilde{F} = [\tilde{F}_{ij}] \quad (i, j = 1, \ldots, m)$$

(1.4.21b)
$$\underset{(v_{i}^{o} \times v_{i}^{o})}{\tilde{F}_{ii}} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & 1 \\ \alpha^{o}_{ii1} & \alpha^{o}_{ii2} & \ldots & \alpha^{o}_{iiv_{i}^{o}} \end{bmatrix}$$

(1.4.21c)
$$\underset{(v_{i}^{o} \times v_{j}^{o})}{\tilde{F}_{ij}} = \begin{bmatrix} 0 & \ldots & \ldots & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \\ \alpha^{o}_{ij1} & \ldots & \alpha^{o}_{ijv_{ij}^{o}} & 0 \ldots & 0 \end{bmatrix}$$

$$(1.4.22) \qquad \tilde{G} = \begin{bmatrix} \tilde{G}_1 \\ \vdots \\ \tilde{G}_m \end{bmatrix} \qquad \tilde{G}_i = \begin{bmatrix} \tilde{g}_{i1}^{T} \\ \vdots \\ \tilde{g}_{iv_i^o}^{T} \end{bmatrix} = \begin{bmatrix} b_{i11}^o & \dots & b_{ir1}^o \\ \vdots & & \vdots \\ b_{i1v_i^o}^o & \dots & b_{irv_i^o}^o \end{bmatrix}_{(v_i^o \times r)}$$

It is well known how the canonical triple $(\tilde{F}, \tilde{G}, \tilde{H})$ is algebraically linked to a generic triple (F, G, H) equivalent under E. In fact $\tilde{F} = TFT^{-1}$, $\tilde{G} = TG$, $\tilde{H} = HT^{-1}$ where T is the transpose of the matrix of regular vectors (1.4.9).

Other canonical forms for E on Σ_o can be parametrized by means of sets of scalars bijectively obtained from $(v_i^o, \alpha_{ijk}^o, b_{ijk}^o)$ [2], [7].

Example 1.4.1

Let us consider the triple $(F, G, H) \in \Sigma_o$ given by

$$(1.4.23) \qquad F = \begin{bmatrix} -0{\cdot}5 & 1 & 0 & 1{\cdot}5 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0{\cdot}5 & 0 & 1 & -1{\cdot}5 & -1 \\ 0{\cdot}5 & 0 & 0 & -0{\cdot}5 & 0 \end{bmatrix}$$

$$(1.4.24) \qquad G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$(1.4.25) \qquad H = \begin{bmatrix} 0{\cdot}5 & 0 & 0 & 0{\cdot}5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The sequence of vectors (1.4.3) is given by:

$$(1.4.26) \qquad \left.\begin{matrix} 0{\cdot}5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0{\cdot}5 & 0 \\ 0 & 1 \\ \circ & \circ \end{matrix}\right| \begin{matrix} 0 & 0{\cdot}5 \\ 0{\cdot}5 & 0 \\ 0{\cdot}5 & 0 \\ 0 & -0{\cdot}5 \\ 0 & 0 \\ \circ & \circ \end{matrix} \left| \begin{matrix} 0 & -0{\cdot}5 \\ -0{\cdot}5 & 0{\cdot}5 \\ 0{\cdot}5 & -0{\cdot}5 \\ 0 & 1{\cdot}5 \\ 0 & 1 \\ \circ & \bullet \end{matrix} \right| \begin{matrix} 1 & 0{\cdot}5 \\ 0{\cdot}5 & -1 \\ -0{\cdot}5 & 2 \\ 0 & -3{\cdot}5 \\ -1 & -1 \\ \bullet & \bullet \end{matrix} \quad \dots$$

where the vectors linearly independent of their antecedents have been denoted with the abstract symbol $\circ$, the linearly dependent ones with the symbol $\bullet$. The scalars v_1^o and v_2^o are therefore given by $v_1^o = 3$ and $v_2^o = 2$.

The scalars α_{ijk}^o can be obtained by computing the dependence coefficients of the first dependent vectors in (1.4.26), i.e. $(F^T)^2 h_2^T$ and $(F^T)^3 h_1^T$ from their antecedents. The values obtained are

$$\alpha_{211}^o = 1 \qquad \alpha_{212}^o = 0 \qquad \alpha_{213}^o = -1 \qquad \alpha_{221}^o = 1 \qquad \alpha_{222}^o = -2$$

$$\alpha^o_{111} = 1 \qquad \alpha^o_{112} = 0 \qquad \alpha^o_{113} = -1$$
$$\alpha^o_{121} = -1 \qquad \alpha^o_{122} = 1$$

The scalars b^o_{ijk} can then be determined as scalar products of the columns of G with the regular vectors in sequence (1.4.26). The values obtained are

$$b^o_{111} = 0 \qquad b^o_{112} = 1 \qquad b^o_{113} = 0$$
$$b^o_{211} = 0 \qquad b^o_{212} = 0$$
$$b^o_{121} = 1 \qquad b^o_{122} = 0 \qquad b^o_{123} = 0$$
$$b^o_{221} = 1 \qquad b^o_{222} = 0$$

The scalars computed in this way are the image $f_o(F, G, H)$. The canonical form (1.4.20)–(1.4.22) directly parametrized by this image is thus given by the following triple.

(1.4.27)
$$\tilde{F} = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & -2 \end{array}\right]$$

(1.4.28)
$$\tilde{G} = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \end{array}\right]$$

(1.4.29)
$$\tilde{H} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Remark 1.7. The canonical forms (1.4.20)–(1.4.22) that have been considered on Σ_o can obviously be considered also on Σ_m since Σ_m is a subset of Σ_o, which is closed with respect to equivalence relation (1.2.6).

5. CANONICAL FORMS ON S_o AND S_{oc}

In this section a subset, K_o, of S_o is defined. It is then proved that K_o is a set of canonical forms for equivalence relation (1.2.7) on S_o. The transformation of a generic element of S_o to the corresponding canonical form is then considered and a transformation algorithm given. The invariance properties of this transformation are then investigated and a numerical example is proposed.

Definition of the Set of Canonical Forms K_o

Consider a subset, K_o, of S_o whose elements $(\tilde{P}(z), \tilde{Q}(z))$ are characterized by the following conditions:

1) The polynomials on the main diagonal of $\tilde{P}(z)$ are monic;
2) The relations among the degrees of the entries of $\tilde{P}(z)$ are

(1.5.1a) $$\deg\{\tilde{p}_{ii}(z)\} \geqq \deg\{\tilde{p}_{ij}(z)\} \quad \text{if} \quad i > j$$

(1.5.1b) $$\deg\{\tilde{p}_{ii}(z)\} > \deg\{\tilde{p}_{ij}(z)\} \quad \text{if} \quad i < j$$

(1.5.1c) $$\deg\{\tilde{p}_{ii}(z)\} > \deg\{\tilde{p}_{ji}(z)\} \quad \text{if} \quad i \neq j\,;$$

3) The relation between the degrees of the entries of $\tilde{P}(z)$ and $\tilde{Q}(z)$ is

(1.5.2) $$\deg\{\tilde{p}_{ii}(z)\} > \deg\{\tilde{q}_{ij}(z)\}\,.$$

The entries of the elements of K_o will be denoted as follows:

(1.5.3) $$\tilde{P}(z) = \begin{bmatrix} \tilde{p}_{11}(z) & \dots & \tilde{p}_{1m}(z) \\ \vdots & & \vdots \\ \tilde{p}_{m1}(z) & \dots & \tilde{p}_{mm}(z) \end{bmatrix}$$

(1.5.4) $$\tilde{Q}(z) = \begin{bmatrix} \tilde{q}_{11}(z) & \dots & \tilde{q}_{1r}(z) \\ \vdots & & \vdots \\ \tilde{q}_{m1}(z) & \dots & \tilde{q}_{mr}(z) \end{bmatrix}$$

(1.5.5a) $$\tilde{p}_{ii}(z) = z^{\nu_i{}^o} - \alpha^o_{ii\nu_i{}^o} z^{(\nu_i{}^o - 1)} - \dots - \alpha^o_{ii2} z - \alpha^o_{ii1}$$

(1.5.5b) $$\tilde{p}_{ij}(z) = -\alpha^o_{ij\nu_{ij}{}^o} z^{(\nu_{ij}{}^o - 1)} - \dots - \alpha^o_{ij2} z - \alpha^o_{ij1}$$

(1.5.6) $$\tilde{q}_{ij}(z) = \beta^o_{ij\nu_i{}^o} z^{(\nu_i{}^o - 1)} + \dots + \beta^o_{ij2} z + \beta^o_{ij1}\,.$$

Remark 1.8. Because of relations (1.5.1) it follows that the row degrees in $\tilde{P}(z)$ are the degrees of $\tilde{p}_{11}(z), \dots, \tilde{p}_{mm}(z)$, i.e. $\nu^o_1, \dots, \nu^o_m$. Moreover it holds that

(1.5.7) $$\deg\det\{\tilde{P}(z)\} = \sum_{i=1}^{m} \nu^o_i = n\,.$$

Remark 1.9. The total number of significant coefficients in the entries of $\tilde{P}(z)$ is given by

(1.5.8) $$l = \sum_{i=1}^{m} \sum_{j=1}^{m} \nu^o_{ij} \quad (\nu^o_{ii} = \nu^o_i)$$

while the total number of coefficients in the entries of $\tilde{Q}(z)$ is given by

(1.5.9) $$\sum_{i=1}^{m} \sum_{j=1}^{r} \nu^o_i = \sum_{j=1}^{r} n = n \times r\,.$$

Theorem 1.3. K_o constitutes a set of canonical forms for E on S_o.

Proof. The proof will be decomposed into the following steps:

a) *For every element of S_0, $(P(z), Q(z))$ there exists an element of K_o, $(\tilde{P}(z), \tilde{Q}(z))$, equivalent to $(P(z), Q(z))$.*

A constructive proof of the existence of this element is given by Algorithm 1.1.

b) *The element of K_o equivalent to a given element of S_o, is unique.*

Assume that for a given element $(P(z), Q(z))$ of S_0 there exist two different elements of K_o, $(\tilde{P}'(z), \tilde{Q}'(z))$ and $(\tilde{P}''(z), \tilde{Q}''(z))$ equivalent to $(P(z), Q(z))$. From this assumption it immediately follows that $(\tilde{P}'(z), \tilde{Q}'(z))\, E\, (\tilde{P}''(z), \tilde{Q}''(z))$, i.e. that there exists a unimodular matrix $M(z)$ such that $P''(z) = M(z)\,\tilde{P}'(z)$ and $\tilde{Q}''(z) = M(z)\,\tilde{Q}'(z)$. Let us now consider the ith row of $\tilde{P}''(z)$; this row is a linear combination of the rows of $\tilde{P}'(z)$ multiplied by the elements of the ith row of $M(z)$. Since $(\tilde{P}''(z), \tilde{Q}''(z))$ is an element of K_o, the elements of the ith row of $\tilde{P}''(z)$ must satisfy conditions (1.5.1a) and (1.5.1b). Since, however, the element of the ith row of $M(z)$ are polynomials in z (and not rational functions), and the elements of $\tilde{P}'(z)$ satisfy conditions (1.5.1a) and (1.5.1b) it follows that, necessarily, $m_{ii}(z) \neq 0$ and that the row degree of this row is $\nu_i^{o\prime\prime} = \nu_i^{o\prime} + \deg\{m_{ii}(z)\}$. Since $M(z)$ is unimodular, $\deg\det\{\tilde{P}'(z)\} = \deg\det\{\tilde{P}''(z)\}$ i.e. $\sum_{i=1}^{m} \nu_i^{o\prime} = \sum_{i=1}^{m} \nu_i^{o\prime\prime}$ and therefore $\deg\{m_{ii}(z)\} = 0$, $(i = 1, \ldots, m)$. It has thus been established that $\tilde{P}'(z)$ and $\tilde{P}''(z)$ share the same ordered set of row degrees. The elements of the ith row of $\tilde{P}''(z)$ must also satisfy column conditions (1.5.1c) with respect to the on-diagonal elements of the subsequent rows; this necessarily leads to the conditions $m_{ij}(z) = 0$ for $i < j$ on the ith row of $M(z)$. Similarly the elements of the ith row of $\tilde{P}''(z)$ must satisfy column condition (1.5.1c) with respect to the on-diagonal elements of the preceding rows and this leads to the conditions $m_{ij}(z) = 0$ for $i > j$ on the ith row of $M(z)$. It has thus been established that $M(z) = \mathrm{diag}\{m_{ii}(z)\}$ with $\deg\{m_{ii}(z)\} = 0$; $M(z)$ is therefore a diagonal real matrix. Since the polynomials on the main diagonal of $\tilde{P}'(z)$ and $\tilde{P}''(z)$ are monic it follows that $M(z) = I$ and, consequently, $\tilde{P}'(z) = \tilde{P}''(z)$.

c) *Elements of S_o in the same equivalence class (with respect to E) are equivalent to the same element of K_0.*

Let $(P'(z), Q'(z))$, $(P''(z), Q''(z))$ be two equivalent elements of S_0, $(\tilde{P}'(z), \tilde{Q}'(z))$, $(\tilde{P}''(z), \tilde{Q}''(z))$ the two corresponding equivalent elements of K_o. Because of the equivalence between $(P'(z), Q'(z))$ and $(P''(z), Q''(z))$ it also follows that $(\tilde{P}'(z), \tilde{Q}'(z))\, E\, (\tilde{P}''(z), \tilde{Q}''(z))$ and since, because of step b), the equivalence classes with respect to E in K_0 have a single element, it follows that $(\tilde{P}'(z), \tilde{Q}'(z)) = (\tilde{P}''(z), \tilde{Q}''(z))$.

d) *Elements of S_o which do not belong to the same equivalence class are equivalent to distinct elements of K_o.*

Let $(P'(z), Q'(z))$ and $(P''(z), Q''(z))$ be two elements of S_o belonging to distinct equivalence classes. It follows that $P'(z) \neq M(z)\,P''(z)$, $Q'(z) \neq M(z)\,Q''(z)$ for every unimodular matrix $M(z)$. If there exists an element of K_o, $(\tilde{P}(z), \tilde{Q}(z))$ equivalent to $(P'(z), Q'(z))$ and to $(P''(z), Q''(z))$ then $\tilde{P}(z) = M'(z)\,P'(z) = M''(z)\,P''(z)$, $\tilde{Q}(z) = M'(z)\,Q'(z) = M''(z)\,Q''(z)$ and, consequently, $P'(z) = M'^{-1}(z)\,M''(z)\,P''(z)$, $Q'(z) = M'^{-1}(z)\,M''(z)\,Q''(z)$. $(P'(z), Q'(z))$ and $(P''(z), Q''(z))$ are therefore equivalent to distinct elements of K_o.

According to Definition 1.7 it has thus been proved that K_o is a set of canonical forms for equivalence relation (1.2.7) on S_o.

Transformation to the Canonical Forms on S_o

Step a) in the proof of Theorem 1.3 will be constructively established by means of the following algorithm which allows, given a generic element $(P(z), Q(z))$ of S_o, the transformation to the corresponding canonical form $(\tilde{P}(z), \tilde{Q}(z))$ of K_o to be performed.

Algorithm 1.1. (cf. [8], [9])

STEP 1. The matrices $P(z)$ and $Q(z)$ are premultiplied for a suitable unimodular matrix $M(z)$ such that $M(z)\,P(z)$ is row-proper. A detailed description of this step can be found in [1].

STEP 2. *Achievement of row condition* (1.5.1a). By means of exchanges of rows, in every row of $P(z)$ polynomials whose degree equals the row degree are moved on the main diagonal. The same row exchanges are performed on $Q(z)$. This operation is always possible if $P(z)$ is row-proper [8].

STEP 3. *Achievement of row condition* (1.5.1b). The entries $p_{m-1,m}(z)$, $p_{m-2,m}(z)$, ..., $p_{1,m}(z)$, $p_{m-2,m-1}(z)$, ..., $p_{1,m-1}(z)$, ..., $p_{1,2}(z)$ are tested in the given order with respect to row condition (1.5.1b). If $\deg\{p_{ij}(z)\} < \deg\{p_{ii}(z)\}$ no operation is performed. When $\deg\{p_{ij}(z)\} = \deg\{p_{ii}(z)\}$ and $\deg\{p_{ij}(z)\} = \mu_{ij} \geqq \deg\{p_{jj}(z)\} = = \mu_{jj}$ the degree of $p_{ij}(z)$ is lowered by subtracting from the ith row of $P(z)$ the jth row multiplied by $\alpha z^{\mu_{ij}-\mu_{jj}}$ where α is the ratio of the maximal degree coefficients in $p_{ij}(z)$ and $p_{jj}(z)$. If $\deg\{p_{ij}(z)\} = \deg\{p_{ii}(z)\}$ and $\deg\{p_{ij}(z)\} = \mu_{ij} < < \deg\{p_{jj}(z)\} = \mu_{jj}$ it is sufficient to exchange the ith row of $P(z)$ with the difference of the jth row and of the ith row multiplied by $\alpha z^{\mu_{jj}-\mu_{ij}}$ where α is the ratio of the maximal degree coefficients in $p_{jj}(z)$ and $p_{ij}(z)$. The same elementary operations performed on $P(z)$ are obviously performed also on $Q(z)$. It is important to note that this step does not change *all* conditions obtained in previous steps.

STEP 4. *Achievement of column condition* (1.5.1c) *in the right-upper triangular part of* $P(z)$. The polynomials considered in Step 3 are tested in the same order with respect to column condition (1.5.1c). If $\mu_{ij} < \mu_{jj}$ no operation is performed. When $\mu_{ij} \geqq \mu_{jj}$ the degree of $p_{ij}(z)$ is lowered by subtracting from the ith row of $P(z)$ the jth row multiplied by $\alpha z^{\mu_{ij}-\mu_{jj}}$ where α is the ratio of the maximal degree coefficients in $p_{ij}(z)$ and $p_{jj}(z)$. After the described operation the next polynomial in the given sequence must be tested even if condition (1.5.1c) with respect to $p_{ij}(z)$ has not been achieved. The entire step is repeated until condition (1.5.1c) on the right upper triangular part of $P(z)$ is achieved. The same elementary row operations are performed on $Q(z)$. Note that the operations performed at this step to reduce the order of $p_{ij}(z)$ do not change the row conditions obtained at Step 3 or the column condition (1.5.1c) on the polynomials tested before $p_{ij}(z)$.

STEP 5. Achievement of column condition (1.5.1c) *in the left lower triangular part of* $P(z)$. The entries $p_{2,1}(z), p_{3,1}(z), \ldots, p_{m,1}(z), p_{3,2}(z), \ldots, p_{m,2}(z), \ldots, p_{m,m-1}(z)$ are tested in the given order with respect to column condition (1.5.1c). If $\mu_{ij} < \mu_{jj}$ no operation is performed. When $\mu_{ij} \geqq \mu_{jj}$ the degree of $p_{ij}(z)$ is lowered by subtracting from the ith row of $P(z)$ the jth row multiplied by $\alpha z^{\mu_{ij}-\mu_{jj}}$ where α is the ratio of the maximal degree coefficients in $p_{ij}(z)$ and $p_{jj}(z)$. After this operation the next polynomial in the given sequence must be tested even if condition (1.5.1c) with respect to $p_{ij}(z)$ has not been achieved. The same elementary row operations are performed on $Q(z)$. The entire step is repeated until condition (1.5.1c) is achieved on the left lower triangular part of $P(z)$.

This step does not change all the conditions obtained in previous steps.

STEP 6. Adjustment of the coefficients on the main diagonal of $P(z)$. The first, second, ..., mth rows of $P(z)$ and $Q(z)$ are divided for the maximal degree coefficients in $p_{11}(z), p_{22}(z), \ldots, p_{mm}(z)$ respectively. After this step the polynomials on the main diagonal of $P(z)$ are monic.

Given a generic element $(P(z), Q(z))$ of S_o, Algorithm 1.1 leads (by means of steps equivalent to the premultiplication of $P(z)$ and $Q(z)$ for unimodular matrices) to the equivalent canonical pair $(\tilde{P}(z), \tilde{Q}(z))$. The algorithm is based on the fact that every step does not change *all* previously obtained conditions.

By means of Algorithm 1.1 a function $\phi^o = (\phi_i^{ov}, \phi_{ijk}^{o\alpha}, \phi_{ijk}^{o\beta})$: $S_0 \to N^m \times \mathscr{F}^l \times \mathscr{F}^{(n\times r)}$ has been implicitly defined. The image $\phi^o(P(z), Q(z)) = (v_i^o, \alpha_{ijk}^o, \beta_{ijk}^o)$ has been used for the parametrization of the elements of K_o, i.e. for the parametrization of the canonical forms on S_o. The following theorem can therefore be established.

Theorem 1.4. $\phi^o = (\phi_i^{ov}, \phi_{ijk}^{o\alpha}, \phi_{ijk}^{o\beta})$ constitutes a complete set of independent invariants for equivalence relation (1.2.7) on S_o.

Proof.

Invariance of ϕ^o

Let $(P'(z), Q'(z))$ and $(P''(z), Q''(z))$ be two elements of S_0 with $(P'(z), Q'(z))$ $E(P''(z), Q''(z))$. It must be proved that $\phi^o(P'(z), Q'(z)) = \phi^o(P''(z), Q''(z))$. This has already been done in step c of the proof of Theorem 1.3.

Completeness of ϕ^o

Let $(P'(z), Q'(z))$ and $(P''(z), Q''(z))$ be two elements of S_0 such that $\phi^o(P'(z), Q'(z)) = \phi^o(P''(z), Q''(z)) = (v_i^o, \alpha_{ijk}^o, \beta_{ijk}^o)$. It must be proved that $(P'(z), Q'(z))$ $E(P''(z), Q''(z))$. Since ϕ^o: $S_o \to K_o$, the pairs $(P'(z), Q'(z))$ and $(P''(z), Q''(z))$ have the same canonical form. Because of steps c) and d) in the proof of Theorem 1.3 it follows that $(P'(z), Q'(z))$ and $(P''(z), Q''(z))$ belong to the same equivalence class of S_o.

Independence of ϕ^o

Let $(\nu_1^o, \ldots, \nu_m^o)$ be an arbitrary element of N^m with $\nu_i^o \neq 0$ and $n = \nu_1^o + \ldots + \nu_m^o$, (α_{ijk}^o) an arbitrary element of $\mathscr{F}^l$ and (β_{ijk}^o) an arbitrary element of $\mathscr{F}^{(n \times r)}$. It must be proved that there exists an element of S_o, $(P(z), Q(z))$, such that $\phi^o(P(z), Q(z)) = (\nu_i^o, \alpha_{ijk}^o, \beta_{ijk}^o)$ i.e. that ϕ^o is surjective with respect to $N^m \times \mathscr{F}^l \times \mathscr{F}^{(n \times r)}$. This will assure the independence of the considered set of functions. Using relations (1.5.3), (1.5.4), (1.5.5) and (1.5.6) an element $(\tilde{P}(z), \tilde{Q}(z))$ of K_o can be obtained such that $\phi^o(\tilde{P}(z), \tilde{Q}(z)) = (\nu_i^o, \alpha_{ijk}^o, \beta_{ijk}^o)$ and since K_o is a subset of S_o this completes the proof. □

Remark 1.10. Theorem 1.4 can be obtained as a corollary of Theorems 1.3 and 1.1. Similarly, Theorem 1.3 could be considered as a corollary of Theorems 1.4 and 1.1. The way this material has been presented allows either of these two ways to be selected.

Example 1.5.1

A numerical example regarding the application, step by step, of Algorithm 1.1 to an element of S_o so as to obtain the equivalent canonical form, is now proposed. Let us consider the pair $(P(z), Q(z))$ given by

$$P(z) = \begin{bmatrix} z^2 - 1 & z^2 + 2z - 1 \\ 2z^3 + 2z^2 - z - 2 & z^3 + 3z^2 \end{bmatrix} \quad Q(z) = \begin{bmatrix} 1 & 2z + 2 \\ 2z + 2 & 3z^2 + 5z + 1 \end{bmatrix}$$

STEP 1. $P(z)$ is already row-proper since the real matrix, whose rows are obtained from the coefficients of the terms in the rows of $P(z)$, whose degree equals the row degree is the nonsingular matrix

$$K = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

STEP 2. Row condition (1.5.1a) is already satisfied.

STEP 3. The only element to be tested is $p_{12}(z)$, Since $\deg\{p_{12}(z)\} = \deg\{p_{11}(z)\} = 2$ and $\deg\{p_{11}(z)\} < \deg\{p_{22}(z)\} = 3$ rows 1 and 2 of $P(z)$ are exchanged and the degree of $p_{12}(z)$ is *now* lowered by subtracting from the first row the second one multiplied by z. The same operations performed on the rows of $P(z)$ are performed also on $Q(z)$. The matrices obtained are

$$P_1(z) = M_1(z)\, P(z) = \begin{bmatrix} z^3 + 2z^2 - 2 & z^2 + z \\ z^2 - 1 & z^2 + 2z - 1 \end{bmatrix}$$

$$Q_1(z) = M_1(z)\, Q(z) = \begin{bmatrix} z + 2 & z^2 + 3z + 1 \\ 1 & 2z + 2 \end{bmatrix}$$

where

$$M_1(z) = \begin{bmatrix} -z & 1 \\ 1 & 0 \end{bmatrix}.$$

STEP 4. The only element to be tested is again $p_{12}(z)$. Since $\deg\{p_{12}(z)\} =$

$= \deg\{p_{22}(z)\}$ the degree of $p_{12}(z)$ is lowered by subtracting the second row from the first. The matrices obtained are the following.

$$P_2(z) = M_2(z)\,P_1(z) = \begin{bmatrix} z^3 + z^2 - 1 & -z + 1 \\ z^2 - 1 & z^2 + 2z - 1 \end{bmatrix} \tag{1.5.10}$$

$$Q_2(z) = M_2(z)\,Q_1(z) = \begin{bmatrix} z + 1 & z^2 + z - 1 \\ 1 & 2z + 2 \end{bmatrix} \tag{1.5.11}$$

where

$$M_2(z) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Since condition (1.5.1c) is now achieved the next step be considered.

STEP 5. The only element to be tested is $p_{21}(z)$. Since $\deg\{p_{21}(z)\} < \deg\{p_{11}(z)\}$ no operations are performed.

STEP 6. The polynomials $p_{11}(z)$ and $p_{22}(z)$ are already monic so no operations must be performed.

The canonical pair $(\tilde{P}(z), \tilde{Q}(z))$ is therefore given by $\tilde{P}(z) = P_2(z)$ and $\tilde{Q}(z) = Q_2(z)$.

Remark 1.11. Note that trasnformation to the canonical form of the given pair $(P(z), Q(z))$ has been performed by premultiplying $P(z)$ and $Q(z)$ for the nonsingular unimodular matrix

$$M(z) = M_2(z)\,M_1(z) = \begin{bmatrix} -z - 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Remark 1.12. The image $\phi^o(P(z), Q(z))$ is given by

$$\begin{array}{lll}
\nu_1^o = 3 & & \nu_2^o = 2 \\
\alpha_{113}^o = -1 & & \\
\alpha_{112}^o = 0 & & \alpha_{122}^o = 1 \\
\alpha_{111}^o = 1 & & \alpha_{121}^o = -1 \\
\alpha_{213}^o = -1 & & \\
\alpha_{212}^o = 0 & & \alpha_{222}^o = -2 \\
\alpha_{211}^o = 1 & & \alpha_{221}^o = 1 \\
\beta_{113}^o = 0 & & \beta_{123}^o = 1 \\
\beta_{112}^o = 1 & & \beta_{122}^o = 1 \\
\beta_{111}^o = 1 & & \beta_{121}^o = -1 \\
\beta_{212}^o = 0 & & \beta_{222}^o = 2 \\
\beta_{211}^o = 1 & & \beta_{221}^o = 2
\end{array}$$

Remark 1.13. The canonical forms that have been considered on S_0 can also be considered on S_{oc} since S_{oc} is closed with respect to E.

6. ALGEBRAICAL LINKS BETWEEN CANONICAL FORMS ON Σ_o AND S_o

In this section the strict equivalence between triples (F, G, H) on Σ_0 and pairs $(P(z), Q(z))$ of S_o will be defined. Subsequently, the algebraical links between the elements of the canonical set C_o and the strictly equivalent elements of K_o will be deduced and a numerical example proposed. The section ends with a discussion of the invariance properties of the considered transformations.

In the following the state, output and input vectors will be denoted as

$$x(t) = [x_1(t), \ldots, x_n(t)]^T, \quad y(t) = [y_1(t), \ldots, y_m(t)]^T, \quad u(t) = [u_1(t), \ldots, u_r(t)]^T$$

Strict Equivalence between Elements of Σ_o and S_o

Definition 1.9. Let (F, G, H) be an element of Σ_o with $n = \dim\{F\}$. An element $(P(z), Q(z))$ of S_0 with $\deg\det\{P(z)\}) = n$ will be called *strictly equivalent* to (F, G, H) iff for every $x(t_0) \in \mathscr{X}$ and for every possible input sequence $u(\cdot)$ there exist n scalars of $\mathscr{F}$, $y_1(t_0), \ldots, y_m(t_0), \ldots, y_1(t_0 + n_1), \ldots, y_m(t_0 + n_m)$ such that model (1.2.1) with initial state $x(t_0)$ and the input-output model (1.2.4) with initial conditions $y_1(t_0), \ldots, y_m(t_0 + n_m)$ generate the same output sequence $y(\cdot)$, with the considered input sequence $u(\cdot)$, for every $t \geqq t_0$.

Remark 1.14. From Definion 1.9 it follows that, because of equivalence relation (1.2.6), every element $(P(z), Q(z))$ of S_o strictly equivalent to an element (F, G, H) of Σ_o is also strictly equivalent to all elements of Σ_o equivalent to (F, G, H) under (1.2.6).

Algebraical Links between Canonical Triples $(\tilde{F}, \tilde{G}, \tilde{H})$ and Canonical Pairs $(\tilde{P}(z), \tilde{Q}(z))$

Theorem 1.5. For every canonical triple $(\tilde{F}, \tilde{G}, \tilde{H})$ of C_o there exists a strictly equivalent canonical pair $(\tilde{P}(z), \tilde{Q}(z))$ of K_o.

Proof. Let $(\tilde{F}, \tilde{G}, \tilde{H})$ be a canonical multicompanion triple with the structure (1.4.20)–(1.4.22) and with $\dim\{\tilde{F}\} = n$. In this representation the system is decomposed into m interconnected subsystems. The states of these subsystems are given by the components with position $1, \ldots, \nu_1^o; \ldots; \nu_1^o + \ldots + \nu_{m-1}^o + 1, \ldots, \nu_1^o + \ldots + \nu_m^o$ of the system state vector. Moreover, the state of the jth subsystem can be completely observed from the jth component of the output vector. Thanks to the particularly simple structure of $\tilde{F}$ and $\tilde{H}$, it is in fact very easy to obtain the following relations

$$\begin{aligned}
x_{\nu_1{}^o + \ldots + \nu_{j-1}{}^o + 1}(t) &= y_j(t) \\
x_{\nu_1{}^o + \ldots + \nu_{j-1}{}^o + 2}(t) &= z\, y_j(t) - \tilde{g}_{j1}^T\, u(t) \\
x_{\nu_1{}^o + \ldots + \nu_{j-1}{}^o + 3}(t) &= z^2\, y_j(t) - \tilde{g}_{j2}^T\, u(t) - \tilde{g}_{j1}^T z\, u(t) \\
&\vdots \\
x_{\nu_1{}^o + \ldots + \nu_j{}^o}(t) &= z^{\nu_j - 1^o}\, y_j(t) - \tilde{g}_{j(\nu_j{}^o - 1)}^T\, u(t) - \ldots - \tilde{g}_{j1}^T z^{\nu_j{}^o - 2}\, u(t)\,.
\end{aligned} \tag{1.6.1}$$

Relations (1.6.1), written for $j = 1, \ldots, m$, allow the state vector $x(t)$ to be expressed as a function of the input-output sequences. These n relations can be written more concisely in the form

(1.6.2) $$x(t) = V(z)\, y(t) + W Z(z)\, u(t)$$

where

(1.6.3) $$V(z) = \underset{(n \times m)}{\begin{bmatrix} 1 & \ldots & 0 \\ z & & 0 \\ \vdots & & \vdots \\ z^{\nu_1^\circ - 1} & & 0 \\ \vdots & & \vdots \\ 0 & & 1 \\ 0 & & z \\ \vdots & & \vdots \\ 0 & \ldots & z^{\nu_m^\circ - 1} \end{bmatrix}}$$

(1.6.4) $$W = \underset{(n \times r(\nu_M^\circ - 1))}{\begin{bmatrix} 0 & \ldots & \ldots & \ldots & 0 \\ \tilde{g}_{11}^T & 0 & \ldots & \ldots & 0 \\ \vdots & & & & \vdots \\ \tilde{g}_{1(\nu_1^\circ - 1)}^T & \ldots & \tilde{g}_{11}^T & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \\ \tilde{g}_{m1}^T & 0 & \ldots & \ldots & 0 \\ \vdots & & & & \vdots \\ \tilde{g}_{m(\nu_m^\circ - 1)}^T & \ldots & \tilde{g}_{m1}^T & \ldots & 0 \end{bmatrix}}$$

$\nu_M^\circ = \max_i \{\nu_i^\circ\}$

(1.6.5) $$Z(z) = \underset{(r(\nu_M^\circ - 1) \times r)}{\begin{bmatrix} I \\ zI \\ \vdots \\ z^{(\nu_M^\circ - 2)} I \end{bmatrix}}$$

Substituting expression (1.6.2) for $x(t)$ in equation (1.2.1a) we obtain the following input-output relation

(1.6.6) $$[(zI - \tilde{F})\, V(z)]\, y(t) = [(zI - \tilde{F})\, W Z(z) + \tilde{G}]\, u(t)\,.$$

Among the n relations (1.6.6), however, only the ν_1°th, $(\nu_1^\circ + \nu_2^\circ)$th, ..., nth are significant, since the remaining ones are simple identities. Deleting non-significant relations, (1.6.6) can be written in the form

(1.6.7) $$\tilde{P}(z)\, y(t) = \tilde{Q}(z)\, u(t)$$

where

$$(1.6.8) \qquad \tilde{P}(z) = \begin{bmatrix} \tilde{p}_{11}(z) & \dots & \tilde{p}_{1m}(z) \\ \vdots & & \vdots \\ \tilde{p}_{m1}(z) & \dots & \tilde{p}_{mm}(z) \end{bmatrix}$$

$$(1.6.9) \qquad \tilde{Q}(z) = \begin{bmatrix} \tilde{q}_{11}(z) & \dots & \tilde{q}_{1r}(z) \\ \vdots & & \vdots \\ \tilde{q}_{m1}(z) & \dots & \tilde{q}_{mr}(z) \end{bmatrix}$$

The entries of $\tilde{P}(z)$ can be directly obtained from (1.6.6). In fact, from the structure of $\tilde{F}$ it follows that

$$(1.6.10) \qquad \tilde{p}_{ii}(z) = z^{\nu_i^o} - \alpha^o_{ii\nu_i^o} z^{\nu_i^o - 1} - \dots - \alpha^o_{ii2} z - \alpha^o_{ii1}$$

$$(1.6.11) \qquad \tilde{p}_{ij}(z) = -\alpha^o_{ij\nu_{ij}^o} z^{\nu_{ij}^o - 1} - \dots - \alpha^o_{ij2} z - \alpha^o_{ij1} .$$

The entries of $\tilde{Q}(z)$ can be obtained by computing the right side of expression (1.6.6). Simple passages lead to

$$(1.6.12) \qquad \tilde{q}_{ij}(z) = \beta^o_{ij\nu_i^o} z^{\nu_i^o - 1} + \dots + \beta^o_{ij2} z + \beta^o_{ij1}$$

where the scalars β^o_{ijk} are linked to the scalars b^o_{ijk}, i.e. to the entries of $\tilde{G}$, by the bijection

$$(1.6.13) \qquad \bar{G} = M\tilde{G}$$

$$(1.6.14) \qquad \bar{G} = \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_m \end{bmatrix} \quad \bar{G}_i = \begin{bmatrix} \beta^o_{i11} & \dots & \beta^o_{ir1} \\ \vdots & & \vdots \\ \beta^o_{i1\nu_i} & \dots & \beta^o_{ir\nu_i} \end{bmatrix}$$

$$(1.6.15a) \qquad M = [M_{ij}] \quad (i, j = 1, \dots, m)$$

$$(1.6.15b) \qquad M_{ii} = \underset{(\nu_i^o \times \nu_i^o)}{\begin{bmatrix} -\alpha^o_{ii2} & -\alpha^o_{ii3} & \dots & -\alpha^o_{ii\nu_i^o} & 1 \\ -\alpha^o_{ii3} & -\alpha^o_{ii4} & \dots & 1 & \\ \vdots & & & & \\ -\alpha^o_{ii\nu_i^o} & 1 & & & \\ 1 & & & & \end{bmatrix}}$$

$$(1.6.15c) \qquad M_{ij} = \underset{(\nu_i^o \times \nu_j^o)}{\begin{bmatrix} -\alpha^o_{ij2} & -\alpha^o_{ij3} & \dots & -\alpha^o_{ij\nu_{ij}^o} & 0 \\ -\alpha^o_{ij3} & -\alpha^o_{ij4} & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ -\alpha^o_{ij\nu_{ij}^o} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}}$$

Matrix M is structurally nonsingular for every set $(\alpha^o_{ijk}) \in \mathscr{F}^l$ since, in every case, $\det\{M\} = 1$.

From relations (1.4.5) it follows that the degrees of the polynomials of $\tilde{P}(z)$ and

$\tilde{Q}(z)$ satisfy the following conditions

(1.6.16a) $$\deg\{\tilde{p}_{ii}(z)\} \geqq \deg\{\tilde{p}_{ij}(z)\} \quad \text{if} \quad i > j$$

(1.6.16b) $$\deg\{\tilde{p}_{ii}(z)\} > \deg\{\tilde{p}_{ij}(z)\} \quad \text{if} \quad i < j$$

(1.6.16c) $$\deg\{\tilde{p}_{ii}(z)\} > \deg\{\tilde{p}_{ji}(z)\} \quad \text{if} \quad i \neq j$$

(1.6.17) $$\deg\{\tilde{p}_{ii}(z)\} > \deg\{\tilde{q}_{ij}(z)\}$$

The n initial conditions on the output vector components required by the definition of strict equivalence between state-space and input-output models, are given by relation (1.6.2) written for $t = t_0$. It can be noted that the conditions requested on the first component of the output vector are v_1^o, those on the second $v_2^o, \ldots,$ those on the mth component v_m^o.

Relation (1.6.10) shows that the diagonal elements of $\tilde{P}(z)$ are monic and, since the obtained conditions (1.6.16a), (1.6.16b), (1.6.16c) and (1.6.17) are coincident with conditions (1.5.1a), (1.5.1b), (1.5.1c) and (1.5.2), it follows that the obtained pair $(\tilde{P}(z), \tilde{Q}(z))$ is canonical. This completes the proof of the theorem. □

Corollary 1.3. For every element of Σ_o/E there exists a strictly equivalent element of K_o.

Example 1.6.1

Let us consider the canonical triple $(\tilde{F}, \tilde{G}, \tilde{H})$ of Σ_o given by (1.4.27)–(1.4.29), the initial state

(1.6.18) $$x(0) = [0 \ 0 \ 1 \ 0 \ 0]^{\mathrm{T}}$$

and the input sequence

(1.6.19) $$u(0) = [1, 0]^{\mathrm{T}}, \quad u(1) = [0, 1]^{\mathrm{T}}, \quad u(2) = [1, 1]^{\mathrm{T}} \ldots$$

A strictly equivalent pair $(\tilde{P}(z), \tilde{Q}(z))$ of S_o as well as the associated initial conditions on the output components will be determined.

The matrix $\tilde{P}(z)$ can be written by direct inspection of $\tilde{F}$. In fact, from (1.6.10) and (1.6.11) it follows that

(1.6.20) $$\tilde{P}(z) = \begin{bmatrix} z^3 + z^2 - 1 & -z + 1 \\ z^2 - 1 & z^2 + 2z - 1 \end{bmatrix}.$$

Determination of $\tilde{Q}(z)$ requires the prior construction of matrix M (1.6.15). This matrix can be written by direct inspection of $\tilde{F}$ on the basis of (1.6.15).

(1.6.21) $$M = \left[\begin{array}{ccc|cc} 0 & 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

and

$$(1.6.22) \qquad \bar{G} = M\tilde{G} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \\ \hline 1 & 2 \\ 0 & 2 \end{bmatrix}$$

From (1.6.22) the scalars β^{o}_{ijk} are given by

$$\begin{array}{ll} \beta^{o}_{111} = 1 & \\ \beta^{o}_{112} = 1 & \beta^{o}_{211} = 1 \\ \beta^{o}_{113} = 0 & \beta^{o}_{212} = 0 \\ \beta^{o}_{121} = -1 & \\ \beta^{o}_{122} = 1 & \beta^{o}_{221} = 2 \\ \beta^{o}_{123} = 1 & \beta^{o}_{222} = 2 \end{array}$$

Matrix $\tilde{Q}(z)$ is thus given by

$$(1.6.23) \qquad \tilde{Q}(z) = \begin{bmatrix} z+1 & z^2+z-1 \\ 1 & 2z+2 \end{bmatrix}$$

The initial conditions on the output components are given by $y_1(0)$, $y_1(1)$, $y_1(2)$, $y_2(0)$ and $y_2(1)$. With the initial state (1.6.18) and the input sequence (1.6.19) it follows that

$$\begin{array}{ll} y_1(0) = 0 & \\ y_1(1) = 0 & y_2(0) = 0 \\ y_1(2) = 3 & y_2(1) = 0 \end{array}$$

It can be noted from the comparison of (1.6.20)–(1.6.23) with (1.5.10)–(1.5.11) that the obtained canonical pair $(\tilde{P}(z), \tilde{Q}(z))$ is the same as that considered in Example 1.5.1.

Invariance Properties of the Transformations to the Canonical Forms on Σ_o and on S_o

The parametrization of the elements of C_o has been performed by means of the image $(v^{o}_{i}, \alpha^{o}_{ijk}, b^{o}_{ijk})$ of a complete set of independent invariants, f_o for E on Σ_o. Similarly, the parametrization of the elements of K_o has been performed by means of the image $(v^{o}_{i}, \alpha^{o}_{ijk}, \beta^{o}_{ijk})$ of a complete set of independent invariants, ϕ_o, for E on S_0.

The map g_o: $\mathscr{F}^{(n\times r)} \to \mathscr{F}^{(n\times r)}$ described by relation (1.6.13) which transforms the set of scalars (b^{o}_{ijk}) onto the set (β^{o}_{ijk}) is, because of the structural nonsingularity of matrix M (1.6.15), one to one. Also function c_o: $N^m \times \mathscr{F}^l \times \mathscr{F}^{(n\times r)} \to N^m \times \mathscr{F}^l \times \mathscr{F}^{(n\times r)}$ defined by $c_o(v^{o}_{i}, \alpha^{o}_{ijk}, b^{o}_{ijk}) = (v^{o}_{i}, \alpha^{o}_{ijk}, \beta^{o}_{ijk})$ is, therefore, a bijection.

Because of Property 1.1 it follows that function δ_o: $\Sigma_0 \to N^m \times \mathscr{F}^l \times \mathscr{F}^{(n\times r)}$ given by δ_o: $c_o \cdot f_o$ constitutes a complete set of independent invariants of E on Σ_0.

Similarly, function $d_o: S_o \to N^m \times \mathscr{F}^l \times \mathscr{F}^{(n \times r)}$ given by $d_o = c_o^{-1} \cdot \phi_o$ constitutes a complete set of independent invariants for E on S_o.

The following theorems have thus been proved.

Theorem 1.6. Every canonical form $(\tilde{F}, \tilde{G}, \tilde{H})$ of C_o is parametrized by the image in d_o of any strictly equivalent element, $(P(z), Q(z))$, of S_o.

Theorem 1.7. Every canonical form $(\tilde{P}(z), \tilde{Q}(z))$ of K_o is parametrized by the image in δ_o of any strictly equivalent element, (F, G, H), of Σ_o.

Remark 1.15. In Sections 4 and 5 all the algorithms for the construction of functions f_o, ϕ_o, d_o and δ_o have been described. This allows every transformation between state-space observable and input-output models to be performed. In [10] an algorithm to obtain the set of scalars $(v_i^o, \alpha_{ijk}^o, \beta_{ijk}^o)$ directly from input-output sequences has been described.

The considered transformations between state-space and input-output canonical forms are summarized by the commutative diagram of Figure 1.2 where Π_o and Π'_o are sets whose elements are all the sets of scalars $(v_i^o, \alpha_{ijk}^o, b_{ijk}^o)$ and $(v_i^o, \alpha_{ijk}^o, \beta_{ijk}^o)$ respectively. Let us now denote with C_{om} the subset of C_o whose elements are the

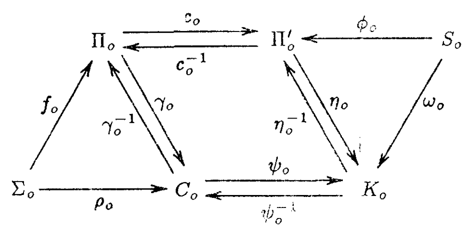

Fig. 1.2.

canonical forms of the equivalence classes of Σ_m, and with K_{om} the subset of K_o whose elements are the canonical forms of the equivalence classes on S_{oc}. The following theorem, analogous to Theorem 1.5, can be stated.

Theorem 1.8. For every canonical triple $(\tilde{F}, \tilde{G}, \tilde{H})$ of C_{om} there exists a strictly equivalent canonical pair $(\tilde{P}(z), \tilde{Q}(z))$ of K_{om}.

The proof follows from the properties of the elements of S_{oc} [1] and from Theorem 1.5.

7. CONCLUDING REMARKS

This first part of the paper has introduced three classes of models for multivariable systems and associated equivalence relations. After some recall on invariant functions for equivalence relations, canonical forms for state-space observable and input-output models parametrized by the image in a complete set of independent invariants of the elements belonging to the same equivalence class have been introduced. Finally the algebraic links between the previous formally different representations have been deduced.

The models considered previously refer to purely dynamical systems; the extension of the given results to systems where an algebraic input-output link is present si very simple and can be performed according to the lines followed, for instance, in [11].

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