SYSTEM SYNTHESIS FROM IMPULSE ENERGY MEASURES

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A method is proposed to determine whether there exist single input single output linear time-invariant dynamical systems without zeros which have a prespecified set of impulse energy measures. The mode of determining the parameters specifying the transfer function of the lowest order system satisfying the requirement, if such systems exist, is developed.

1. INTRODUCTION.

Given the transfer function $T(s)$ of a linear time-invariant dynamical system it is possible to determine the system's impulse energy measures, where the ith impulse energy measure

$$Y_i \triangleq \int_0^\infty \left(\frac{\mathrm{d}^i}{\mathrm{d}t^i} L^{-1}(T(s))\right)^2 \mathrm{d}t\,, \quad i = 0, 1, 2, \ldots.$$

The problem posed and solved in this paper is:

Given

$$\mathbf{Y} = [Y_0 \ Y_1 \ Y_2 \ldots Y_{n-1}]^{\mathrm{T}}\,, \quad Y_i \text{ finite and real}\,, \quad i = 0, 1, 2, \ldots, n-1$$

does there exist a system whose transfer function is

$$T(s) = \frac{1}{s^r + p_{r-1}s^{r-1} + p_{r-2}s^{r-2} + \ldots + p_0}\,, \quad p_i \text{ real}\,, \quad i = 0, 1, 2, \ldots, r-1$$

where r may be less than, equal to or greater than n? If several such systems exist, what are the parameters of the lowest order system satisfying the requirement?

Several results relating to the computation and application of the impulse response, impulse energy measures and quadratic moments of a system have been reported in the literature [1], [2], [3], [4], [5]. This paper leans heavily on some earlier results of the author relating to impulse energy measures [6], [7]; however, following the statement of the pertinent earlier results, this paper is self-contained. The relevant results are:

Lemma 1. If

$$T(s) = \frac{1}{s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \ldots + p_0}, \quad p_i \text{ real for } \quad i = 0, 1, 2, \ldots, n-1$$

is asymptotically stable and Y_i is its ith impulse energy measure, then $\mathbf{FY} = \mathbf{Q}$, where

(1)
$$\mathbf{F} = \begin{bmatrix} p_0 & -p_2 & p_4 & \cdots & 0 \\ 0 & p_1 & p_3 & \cdots & 0 \\ 0 & -p_0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & & \cdots & -1 \\ 0 & \cdots & & \cdots & p_{n-1} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_{n-2} \\ Y_{n-1} \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

Lemma 2. Given $\mathbf{FY} = \mathbf{Q}$, formed as in (1), and $\mathbf{Y}$ finite

$$T(s) = \frac{1}{s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \ldots + p_0}$$

has impulse energy measures $Y_0, Y_1, Y_2, \ldots, Y_{n-1}$ if and only if

$$\det \mathbf{F}\begin{bmatrix} 1 & 2 & 3 & \ldots & i \\ 1 & 2 & 3 & \ldots & i \end{bmatrix} > 0, \quad i = 1, 2, 3, \ldots, n$$

where

$$\mathbf{F}\begin{bmatrix} 1 & 2 & 3 & \ldots & i \\ 1 & 2 & 3 & \ldots & i \end{bmatrix}$$

is the matrix formed by rows $1, 2, 3, \ldots, i$ and columns $1, 2, 3, \ldots, i$ of $\mathbf{F}$ taken in that order.

Theorem 1. Given $\mathbf{Y} = [Y_0 \; Y_1 \; Y_2 \ldots Y_{n-1}]^T$, Y_i is finite and real for $i = 0, 1, 2, \ldots$ $\ldots, n-1$, there exists a system with a transfer function

$$T(s) = \frac{1}{s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \ldots + p_0}, \quad p_i \text{ finite and real,}$$

$$i = 0, 1, 2, \ldots, n-1$$

whose impulse energy measures are $Y_0, Y_1, Y_2, \ldots, Y_{n-1}$ if and only if $\mathbf{M}_n$ is positive definite, where

$$\mathbf{M}_n = \begin{bmatrix} Y_0 & 0 & -Y_1 & 0 & Y_2 & \cdots & \\ 0 & Y_1 & 0 & -Y_2 & 0 & \cdots & \\ -Y_1 & 0 & Y_2 & 0 & -Y_3 & \cdots & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & & 0 & Y_{n-1} \end{bmatrix}$$

With the above results assumed, this paper is organized as follows:

The first issue tackled is whether, in case an nth order system does not exist which has the prespecified n impulse energy measures, it may be that there exists a system of order greater than n which satisfies the requirements. Next, systems of lower order are analysed. After this the criteria for the existence of a system without zeros having the prespecified n impulse energy measures are stated. Then an algorithm is proposed for determining the parameters of the lowest order system satisfying the criteria. The procedure is illustrated by means of a numerical example.

2. SYSTEMS OF ORDER GREATER THAN n

If no system of order n exists which has the prespecified n impulse energy measures, that is $\mathbf{M}_n$ of Theorem 1 is not positive definite, the question arises whether there are systems of similar form but of higher order which satisfy the requirement. The present section tackles this issue.

Corollary 1. Given $Y_0, Y_1, \ldots, Y_{n-1}$, Y_i finite and real for $i = 0, 1, 2, \ldots, n - 1$, if no system with transfer function

$$T(s) = \frac{1}{s^r + p_{r-1}s^{r-1} + p_{r-2}s^{r-2} + \ldots + p_0}, \quad p_i \text{ finite and real,}$$

$$i = 0, 1, 2, \ldots, n - 1$$

exists for $r = n$ which has $Y_0, Y_1, \ldots, Y_{n-1}$ for its impulse energy measures, then no $T(s)$ exists with $r > n$ which has $Y_0, Y_1, \ldots, Y_{n-1}$ as its impulse energy measures.

Proof. Suppose no $T(s)$ exists for $r = n$ which has $Y_0, Y_1, \ldots, Y_{n-1}$ for its impulse energy measures.

Let us suppose that there does exist a system of order $q > n$ which has $Y_0, Y_1, \ldots$ $\ldots, Y_{n-1}$ for its impulse energy measures. Let its further $q - n$ impulse energy measures be $Y_n, Y_{n+1}, Y_{n+2}, \ldots, Y_q$. For this system, by Theorem 1, $\mathbf{M}_q$ must be positive definite. This implies

$$\det \mathbf{M}_q \begin{bmatrix} 1 & 2 & 3 & \ldots & i \\ 1 & 2 & 3 & \ldots & i \end{bmatrix} > 0, \quad i = 1, 2, 3, \ldots, q.$$

Since $q > n$, the above condition includes

$$\det \mathbf{M}_q \begin{bmatrix} 1 & 2 & 3 & \ldots & i \\ 1 & 2 & 3 & \ldots & i \end{bmatrix} > 0, \quad i = 1, 2, 3, \ldots, n. \tag{2}$$

But

$$\det \mathbf{M}_q \begin{bmatrix} 1 & 2 & \ldots & i \\ 1 & 2 & \ldots & i \end{bmatrix} = \det \mathbf{M}_n \begin{bmatrix} 1 & 2 & \ldots & i \\ 1 & 2 & \ldots & i \end{bmatrix}, \quad i = 1, 2, 3, \ldots, n. \tag{3}$$

From (2) and (3) it follows that

$$\det \mathbf{M}_n \begin{bmatrix} 1 & 2 & \dots & i \\ 1 & 2 & \dots & i \end{bmatrix} > 0\,, \quad i = 1, 2, \dots, n\,.$$

This means that $\mathbf{M}_n$ is positive definite. Then, by Theorem 1, there exists $T(s)$ for $r = n$ which has impulse energy measures $Y_0, Y_1, \dots, Y_{n-1}$. This contradicts the assumption we started with.

Thus there cannot exist a $T(s)$ with $r > n$ which has impulse energy measures $Y_0, Y_1, \dots, Y_{n-1}$. This completes the proof. □

3. SYSTEMS OF ORDER LESS THAN n

Having considered systems of order greater than n we can now adress ourselves to the question whether there exist systems of order less than n which meet the requirement. The following lemma helps to considerably reduce the search for lower order systems.

Lemma 3. Given $Y_0, Y_1, \dots, Y_{n-1}$ with Y_i finite and real, $i = 0, 1, 2, \dots, n-1$, there exists a system of order $r < n$, with transfer function

$$T(s) = \frac{1}{s^r + p_{r-1}s^{r-1} + p_{r-2}s^{r-2} + \dots + p_0}\,, \quad p_i \text{ finite and real,}$$

$$i = 0, 1, 2, \dots, n-1$$

only if $\mathbf{M}_r$ is positive definite and

$$\det \left[\begin{array}{c|c} & \vdots \\ & -Y_{r-3} \\ & 0 \\ \mathbf{M}_r & Y_{r-2} \\ & 0 \\ & -Y_{r-1} \\ & -\frac{1}{2} \\ \hline \dots\; -Y_{r-3}\;\; 0\;\; Y_{r-2}\;\; 0\;\; -Y_{r-1}\;\; -\frac{1}{2} & Y_r \end{array}\right]_{(r+1 \times r+1)} = 0$$

Proof. Suppose $T(s)$ has impulse energy measures $Y_0, Y_1, Y_2, \dots, Y_{r-1}$. Then, by Theorem 1, $\mathbf{M}_r$ is positive definite. Obviously this condition holds even if the higher impulse energy measures are required to be $Y_r, Y_{r+1}, \dots, Y_{n-1}$. It follows that if $\mathbf{M}_r$ is not positive definite then $T(s)$ cannot have the prespecified n impulse energy measures. Thus the first part of the lemma stands proved.

To prove the second part we will start with the assumption that $T(s)$ has impulse energy measures $Y_0, Y_1, \dots, Y_{n-1}$. Since Y_i, $i = 0, 1, 2, \dots, r-1$ are finite, n being greater than r, it follows that $T(s)$ is asymptotically stable. So, by Lemma 1

$$\begin{bmatrix} p_0 & -p_2 & p_4 & \cdots & 0 \\ 0 & p_1 & -p_3 & \cdots & 0 \\ 0 & -p_0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & & \cdots & -1 \\ 0 & \cdots & & \cdots & p_{r-1} \end{bmatrix} \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_{r-2} \\ Y_{r-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

Rearranging, we get

$$\begin{bmatrix} Y_0 & 0 & -Y_1 & 0 & Y_2 & \cdots & \\ 0 & Y_1 & 0 & -Y_2 & 0 & \cdots & \\ -Y_1 & 0 & Y_2 & 0 & -Y_3 & & \\ 0 & -Y_2 & 0 & Y_3 & 0 & & \\ \vdots & & & \ddots & & \ddots & \\ & & & & & & 0 \\ & & & & 0 & & Y_{r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{r-2} \\ p_{r-1} \end{bmatrix} = \begin{bmatrix} \vdots \\ Y_{r-3} \\ 0 \\ -Y_{r-2} \\ 0 \\ Y_{r-1} \\ \frac{1}{2} \end{bmatrix} \tag{4}$$

(4) readily yields

$$\begin{bmatrix} Y_0 & 0 & -Y_1 & 0 & \cdots & & & \\ 0 & Y_1 & 0 & -Y_2 & \cdots & & & \\ -Y_1 & 0 & Y_2 & 0 & \cdots & & & \\ & & & & & & & \\ & & & & & -Y_{r-2} & 0 & \\ & & & & & 0 & -Y_{r-1} & \\ & & & & 0 & Y_{r-1} & -0{\cdot}5 & \end{bmatrix}_{(r\times r+1)} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{r-2} \\ p_{r-1} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

Now

$$T(s) = \frac{1}{s^r + p_{r-1}\, s^{r-1} + p_{r-2} s^{r-2} + \ldots + p_0}$$

Hence in the time domain we have

$$y^{(r)}(t) + p_{r-1}\, y^{(r-1)}(t) + p_{r-2}\, y^{(r-2)}(t) + \ldots + p_1\, y^{(1)}(t) + p_0 y = u(t)$$

where $y^{(i)}(t)$ stands for the ith derivative of the output and $u(t)$ is the input. For the case when $u(t)$ is a unit impulse, $y(t)$ is the impulse response. Replacing the impulse response by an appropriate initial condition we get (this aspect is further clarified in the proof of Theorem 2)

$$y^{(r)} + p_{r-1}y^{(r-1)} + p_{r-2}y^{(r-2)} + \ldots + p_1 y^{(1)} + p_0 y = 0 \tag{6}$$

$$y(0) = 0\,, \quad y^{(1)}(0) = 0, \ldots, y^{(r-2)}(0) = 0\,, \quad y^{(r-1)}(0) = 1\,. \tag{7}$$

The argument t has been dropped for convenience. Multiplying (6) by $y^{(r)}$ and integrating between 0 and ∞ yields

$$\int_0^\infty (y^{(r)2})\, dt + p_{r-1} \int_0^\infty y^{(r-1)} y^{(r)}\, dt + \ldots + p_0 \int_0^\infty y y^{(r)}\, dt = 0\,.$$

The first term yields Y_r. The second term simplifies to $p_{r-1}(y^{(r-1)}(0))^2/(2)\,|_0^\infty =$ $= -0{\cdot}5p_{r-1}$; the upper limit yields zero for stable $T(s)$. Simplifying the other terms using $\int u \, dv = uv - \int v \, du$ and (7), we have

$$Y_r - 0{\cdot}5p_{r-1} - p_{r-2}Y_{r-1} + p_{r-4}Y_{r-2} - p_{r-6}Y_{r-3} \dots = 0 \tag{8}$$

Incorporating (8) in (5) yields

$$\begin{bmatrix} & & & \vdots & \\ & \mathbf{M}_r & & 0 & \\ & & & -Y_{r-1} & \\ & & & -\frac{1}{2} & \\ \dots & 0 & -Y_{r-1} & -\frac{1}{2} & Y_r \end{bmatrix}_{(r+1 \times r+1)} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{r-1} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

Since the vector on the right hand side is a null vector, the $(r + 1 \times r + 1)$ matrix on the left hand side is singular. This completes the proof. □

We can use this lemma to advantage as follows. First $\mathbf{M}_r$, $r = 1, 2, \dots, n$ are successively checked for positive definiteness. Suppose r_1 is that value of r such that $\mathbf{M}_{r_1}$ is not positive definite but $\mathbf{M}_r$, $r = 1, 2, \dots, r_1 - 1$ are. The lemma assures us that no system of order r_1 and above can meet our requirement and so limits our search to systems of order less than r_1. On the other hand $\mathbf{M}_n$ itself may be positive definite rather than just $\mathbf{M}_{r_1-1}$, where $r_1 - 1 < n$. In such a case Theorem 1 guarantees the existence of a system of order n which has the necessary impulse energy measures. But we have to examine whether a lower order system is available. In this case also Lemma 3 is helpful for it requires us to search for only those orders of systems for which the determinant specified therein is zero. Further, an examination of the determinant specified in the lemma, (4) and (9) makes it clear that if $T(s)$ has impulse energy measures $Y_0, Y_1, \dots, Y_{r-1}$ then the next impulse energy measure Y_r is that which satisfies the condition laid down in the lemma.

While Lemma 3 helps in reducing our search, it is not a sufficiency condition. The following theorem embodies the necessary and sufficient conditions for $T(s)$ of order $r < n$ to have a set of n given impulse energy measures.

Theorem 2. Given $Y_0, Y_1, \dots, Y_{n-1}$ with Y_i finite and real for $i = 0, 1, \dots, n - 1$ there exists a system of order $r < n$ with a transfer function

$$T(s) = \frac{1}{s^r + p_{r-1}s^{r-1} + p_{r-2}s^{r-2} + \dots + p_0}, \quad p_i \text{ real for } i = 0, 1, 2, \dots, r - 1$$

which has $Y_0, Y_1, \dots, Y_{n-1}$ as its impulse energy measures if and only if

$$\sum_{i=0}^{r/2}(-1)^i p_{r-2i}Y_{r-i+k} - 0{\cdot}5 \sum_{i=0}^{(r-2)/2}(-1)^i p_{r-1-2i} h_{r-1-i+k}^2 - \tag{10}$$

$$-\sum_{i=1}^{r/2}\sum_{j=1}^{i-1}(-1)^j p_{r-2i}h_{r-1-j+k}h_{r-2i+j+k} - \sum_{i=0}^{(r-4)/2}\sum_{j=0}^{i}(-1)^j p_{r-3-2i}h_{r-1-j+k} \cdot$$

$$\cdot\, h_{r-3-2i+j+k} = 0\,, \quad k = 1, 2, 3, \dots, n - r - 1\,; \quad r \text{ even}$$

$$(11)\qquad \sum_{i=0}^{(r-1)/2}(-1)^i p_{r-2i}Y_{r-i+k} - 0{\cdot}5\sum_{i=0}^{(r-1)/2}(-1)^i p_{r-1-2i}h^2_{r-1-i+k} -$$

$$-\sum_{i=1}^{(r-1)/2}\sum_{j=0}^{i-1}(-1)^j p_{r-2i}h_{r-1-j+k}h_{r-2i+j+k} - \sum_{i=0}^{(r-3)/2}\sum_{j=0}^{i}(-1)^j p_{r-3-2i}h_{r-1-j+k}\cdot$$

$$\cdot\, h_{r-3-2i+j+k} = 0\,, \quad k = 1, 2, 3, \ldots, n-r-1\,; \quad r \text{ odd}$$

with

$$(12)\qquad \begin{bmatrix} -Y_0 & 0 & -Y_1 & 0 & \ldots & & \\ 0 & Y_1 & 0 & -Y_2 & \ldots & & \\ -Y_1 & 0 & Y_2 & 0 & \ldots & & \\ & & & & & & \\ & & & & & & 0 \\ & & & & & 0 & Y_{r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{r-2} \\ p_{r-1} \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ -Y_{r-2} \\ 0 \\ Y_{r-1} \\ 0{\cdot}5 \end{bmatrix}$$

(13) $\mathbf{M}_r$, the $(r \times r)$ matrix of (12), is positive definite

$$(14)\qquad p_r = 1$$

and

$$(15)\qquad \begin{aligned} h_i &= 0\,, && i = 0, 1, 2, \ldots, r-2 \\ &= 1\,, && i = r-1 \\ &= -\sum_{j=0}^{r-1} p_j h_{i-r+j}\,, && i = r\,,\quad r+1\,,\quad r+2, \ldots \end{aligned}$$

Proof. Suppose $T(s)$ has finite impulse energy measures $Y_0, Y_1, \ldots, Y_{n-1}$. It is then asymptotically stable. Hence, by Lemma 1,

$$\mathbf{FY} = \mathbf{Q}\,,$$

where $\mathbf{F}$, $\mathbf{Y}$ and $\mathbf{Q}$ are as defined in (1). (12) is just a rearranged form of this equation and hence becomes true. By Theorem 1, $\mathbf{M}_n$ is positive definite. Since $r < n$, $\mathbf{M}_r$ is positive definite and so (13) is true. Now

$$(16)\qquad T(s) = \frac{1}{s^r + p_{r-1}s^{r-1} + p_{r-2}s^{r-2} + \ldots + p_0}\,.$$

$T(s)$ can be expanded as

$$(17)\qquad T(s) = h_0/s + h_1/s^2 + h_2/s^3 + \ldots.$$

When the input is a unit impulse, the output $y(t)$ is the impulse response and is given, in the light of (17), by

$$(18)\qquad y(t) = L^{-1}(T(s)) = h_0 + h_1 t + h_2 t^2/2! + h_3 t^3/3! + \ldots.$$

From (18) we have

$$y(0) = h_0\,,\quad y^{(1)}(0) = h_1\,,\quad y^{(2)}(0) = h_2\,, \ldots$$

Thus h_i represents the value of the ith derivative of the impulse response at $t = 0_+$. We will now show that the h_i of (15) is the same as this h_i. Combining (16) and (17) we have

$$(19) \quad (s^r + p_{r-1}s^{r-1} + p_{r-2}s^{r-2} + \ldots + p_0)(h_0/s + h_1/s^2 + h_2/s^3 + \ldots) = 1\,.$$

Here we can treat the coefficient of s^r as p_r, where $p_r = 1$. With this consideration we can now compare the coefficients of s^i, $i = r-1, r-2, \ldots$ on both sides of (19). Such comparison yields

$$\begin{aligned} &p_r h_0 = 0 && \text{and so} \quad h_0 = 0 \\ &p_{r-1}h_0 + p_r h_1 = 0 && \text{and so} \quad h_1 = 0 \\ &\vdots \\ &p_2 h_0 + p_3 h_1 + \ldots + p_r h_{r-2} = 0 && \text{and so} \quad h_{r-2} = 0 \\ &p_1 h_0 + p_2 h_1 + \ldots + p_r h_{r-1} = 1 && \text{and so} \quad h_{r-1} = 1\,. \end{aligned}$$

Further

$$p_0 h_i + p_1 h_{i+1} + p_2 h_{i+2} + \ldots + p_{r-1}h_{i+r-1} + p_r h_{i+r} = 0\,, \quad i = 0, 1, 2, \ldots$$

Thus, p_r being 1,

$$h_{i+r} = -\sum_{j=0}^{r-1} p_j h_{i+j}\,, \quad i = 0, 1, 2, \ldots.$$

In the light of the above equations it can be seen that h_i of (15) stands for the ith derivative of the impulse response of $T(s)$ at time 0_+. Further the conditions specified by (15) are been to hold.

To complete the first part of the proof we have to show that (10) and (11) also hold. We will do this by showing that (10) and (11) are expressions which express the $(r+i)$th impulse energy measure of $T(s)$ in terms of the $(r+i-1)$th, $(r+i-$ $-2)$th ..., impulse energy measures and the impulse response and its derivatives at time 0_+.

From (16), we have, in the time domain,

$$(20) \quad y^{(r)}(t) + p_{r-1}\,y^{(r-1)}(t) + p_{r-2}\,y^{(r-2)}(t) + \ldots + p_1\,y^{(1)}(t) + p_0\,y(t) = u(t).$$

Here $u(t)$ is a unit impulse, $y(t)$ the output. The initial conditions are zero. When $u(t)$ is a unit impulse, $y(t)$ becomes the impulse response. In this case we can replace the input by a set of initial conditions which, as seen earlier, are $h_0, h_1, h_2, \ldots$. Thus we have, from (20),

$$(21) \quad \begin{gathered} y^{(r)} + p_{r-1}y^{(r-1)} + p_{r-2}y^{(r-2)} + \ldots + p_1 y^{(r-2)} + \ldots + p_1 y^{(1)} + \\ + p_0 y = 0 \\ y(0) = h_0\,, \quad y^{(i)}(0) = h_i\,, \quad i = 1, 2, 3, \ldots. \end{gathered}$$

In (21) the argument t has been dropped for convenience. To get Y_{r+i} we first differentiate both sides of (21) i times with respect to time. We then have

$$(22) \quad y^{(r+i)} + p_{r-1}y^{(r+i-1)} + p_{r-2}y^{(r+i-2)} + \ldots + p_1 y^{(1+i)} + p_0 y^{(i)} = 0\,.$$

Multiply (22) by $y^{(r+i)}$ and integrate between 0 and ∞. This gives

$$(23) \quad \int_0^\infty y^{(r+i)}y^{(r+i)}\,dt + p_{r-1}\int_0^\infty y^{(r+i-1)}y^{(r+i)}\,dt + \ldots + p_0\int_0^\infty y y^{(r+i)}\,dt = 0\,.$$

The first term yields Y_{r+i}. The second reduces to

$$p_{r-1}(y^{(r+i-1)^2}/2)\,|_0^{\infty} = -p_{r-1}(h_{r+i-1}^2/2)\,.$$

The upper limit yields zero because of the assumption we started with that $T(s)$ has finite impulse energy measures. The other terms can be reduced using the well-known identity $\int u\,dv = uv - \int v\,du$. Simplification, taking the coefficients of the first term of (21) to be p_r (this is done for convenience of expression), readily yields (10) or (11) depending on whether r is even or odd. This completes the necessity part of the proof.

To prove sufficiency we shall begin with the assumption that (10), (11), (12), (13), (14) and (15) hold, with $Y_0, Y_1, \ldots, Y_{n-1}$ finite and real. Since (13) is true, by Theorem 1 there exists a $T(s)$ of order r which has $Y_0, Y_1, Y_2, \ldots, Y_{r-1}$ as its impulse energy measures. Since $\mathbf{M}_r$ the $(r \times r)$ matrix of (12) is nonsingular by virtue of (13), it follows that the parameters of $T(s)$ are unique; $p_0, p_1, \ldots, p_{r-1}$ are determined from (12). (10) and (11), as noted earlier, enable the impulse energy measures $Y_r, Y_{r+1}, \ldots,$ to be recursively computed from the earlier impulse energy measures. Thus if the given $Y_r, Y_{r+1}, \ldots, Y_{n-1}$ satisfy (10) and (11) it follows that they are the $(r+1)$th, $(r+2)$th, $\ldots$, nth impulse energy measures of $T(s)$; a system's impulse energy measures are unique. Thus the $T(s)$ of order r has the required n impulse energy measures. This completes the proof. □

4. CRITERIA FOR EXISTENCE

In the light of the results obtained so far, the criteria for the existence of a $T(s)$ which has a prespecified set of n impulse energy measures may be stated as follows:

Given $Y_0, Y_1, \ldots, Y_{n-1}$, Y_i finite and real for $i = 0, 1, 2, \ldots, n-1$, there exists a system with transfer function

$$T(s) = \frac{1}{s^r + p_{r-1}s^{r-1} + p_{r-2}s^{r-2} + \ldots + p_0}\,, \quad p_i \text{ finite and real,}$$
$$i = 0, 1, 2, \ldots, r-1$$

having $Y_0, Y_1, \ldots, Y_{n-1}$ as its impulse energy measures if and only if $\mathbf{M}_n$ is positive definite or the conditions of Theorem 2 hold.

Algorithm. We now propose an algorithm for determining whether a $T(s)$ exists which has a prespecified set on n impulse energy measures and if so for determining the parameters of the lowest order $T(s)$ satisfying the requirement.

1. Successively form and evaluate det $\mathbf{M}_i$, $i = 1, 2, \ldots, n$. If for any value, say i_1, the determinant becomes negative or zero for the first time, go to Step 2. Else, that is if det $\mathbf{M}_i > 0$, $i = 1, 2, \ldots, n$ and so $\mathbf{M}_n$ is positive definite, go to Step 3.
2. Apply Lemma 3 for $r = 1, 2, 3, \ldots, i_1 - 1$. Only the second condition of the lemma need be checked as the first condition is ensured in Step 1. If for no value of

r is the second condition of Lemma 3 satisfied, no $T(s)$ satisfying the requirement exists and the process stops. If Lemma 3 is satisfied for some values of r, say $r_1, r_2, \ldots, r_q$ then for these values of r apply (10) and (11) of Theorem 2 starting with the smallest value of r under consideration. If for $r_1, r_2, \ldots, r_q$ the conditions of Theorem 2 are not satisfied, no $T(s)$ exists and the process stops. Else use (12) to determine the parameters of $T(s)$ for the lowest tested value of r for which (10) or (11) holds. $T(s)$ thus stands determined and the process stops.

3. Apply Lemma 3 for $r = 1, 2, \ldots, n - 1$. Only the second condition of the lemma has to be checked. If for no such r is Lemma 3 satisfied then the lowest order of $T(s)$ is n. Its parameters are determined from (12), with $r = n$, and the process stops. On the other hand if Lemma 3 is satisfied for some values of r say $r_1, r_2, \ldots$ $\ldots, r_q$, apply (10) and (11) of Theorem 2 starting with the smallest value of r being tested. If for none of $r_1, r_2, \ldots, r_q$ Theorem 2 is satisfied then again the lowest order of $T(s)$ is n and its parameters are determined from (12). The process then ends. Else use (12) to determine the parameters of the system with the smallest among $r_1, r_2, \ldots, r_q$ which satisfies (10) or (11). $T(s)$ is thus determined and the process stops.

Example. Given $Y_0 = 1/120$, $Y_1 = 1/120$, $Y_2 = 11/120$, $Y_3 = 481/120$, $Y_4 = 9971/120$

$$\det \mathbf{M}_1 = \det(1/120) > 0\,; \quad \det \mathbf{M}_2 = \begin{bmatrix} 1/120 & 0 \\ 0 & 1/120 \end{bmatrix} > 0$$

$$\det \mathbf{M}_3 = \det \begin{bmatrix} 1/120 & 0 & -1/120 \\ 0 & 1/120 & 0 \\ -1/120 & 0 & 11/120 \end{bmatrix} > 0$$

$$\det \mathbf{M}_4 = \det \begin{bmatrix} 1/120 & 0 & -1/120 & 0 \\ 0 & 1/120 & 0 & -11/120 \\ -1/120 & 0 & 11/120 & 0 \\ 0 & -11/120 & 0 & 481/120 \end{bmatrix} > 0$$

$$\det \mathbf{M}_5 = \det \begin{bmatrix} 1/120 & 0 & -1/120 & 0 & 11/120 \\ 0 & 1/120 & 0 & -11/120 & 0 \\ -1/120 & 0 & 11/120 & 0 & -481/120 \\ 0 & -11/120 & 0 & 481/120 & 0 \\ 11/120 & 0 & 481/120 & 0 & 9971/120 \end{bmatrix} > 0$$

Hence $\mathbf{M}_n$, $n = 5$, is positive definite. Thus the existence of $T(s)$ of order 5 having the prespecified 5 impulse energy measures is assured. We shall now use Lemma 3 to check for the possibility of a lower order system meeting the requirement.

$$r = 1 \quad \det \begin{bmatrix} 1/120 & -1/2 \\ -1/2 & 1/120 \end{bmatrix} \neq 0\,. \quad \text{Lemma 3 not satisfied.}$$

$$r = 2 \quad \det \begin{bmatrix} 1/120 & 0 & -1/120 \\ 0 & 1/120 & -1/2 \\ -1/120 & -1/2 & 11/120 \end{bmatrix} \neq 0 . \quad \text{Lemma 3 is not satisfied.}$$

$$r = 3 \quad \det \begin{bmatrix} 1/120 & 0 & -1/120 & 0 \\ 0 & 1/120 & 0 & -11/120 \\ -1/120 & 0 & 11/120 & -1/2 \\ 0 & -11/120 & -1/2 & 481/120 \end{bmatrix} = 0 .$$

$$r = 4 \quad \det \begin{bmatrix} 1/120 & 0 & -1/120 & 0 & 11/120 \\ 0 & 1/120 & 0 & -11/120 & 0 \\ -1/120 & 0 & 11/120 & 0 & -481/120 \\ 0 & -11/120 & 0 & 481/120 & -1/2 \\ 11/120 & 0 & -481/120 & -1/2 & 9971/120 \end{bmatrix} \neq 0 . \quad \text{Lemma 3 not satisfied.}$$

Hence we need search only for a system of order 3. (12), for $r = 3$, yields

$$\begin{bmatrix} 1/120 & 0 & -1/120 \\ 0 & 1/120 & 0 \\ -1/120 & 0 & 11/120 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 11/120 \\ 1/2 \end{bmatrix}$$

So $p_0 = 6, p_1 = 11, p_2 = 6$.

Using (15) we have

$$\begin{aligned} h_1 &= 0 , \quad i = 0, 1 \\ &= 1 , \quad i = 2 \\ &= -6 , \quad i = 3 \\ &= 25 , \quad i = 4 \end{aligned}$$

Since $r = 3$ is odd we test using (11). It is satisfied. So the third order system is the lowest order system satisfying the requirement. The required

$$T(s) = \frac{1}{s^3 + 6s^2 + 11s + 6} .$$

5. CONCLUSION

Criteria were developed for determining whether there exists a system with a transfer function without zeros which has a prespecified set of impulse energy measures. An algorithm was proposed for determining the parameters of the lowest order system satisfying the requirement, if such systems exist.

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