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ADDITION OF FUZZY QUANTITIES: DISJUNCTION-CONJUNCTION APPROACH

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This paper is subjected to the problem of algebraical operations over fuzzy quantities with rational values. The concept of the fuzzy quantity was introduced in [3] and [4] and applied e.g. in [5] and [6]. It represents one of possible models of uncertainty connected with rational numerical data. The elementary properties of the addition of fuzzy quantities were shown in [3] and further developed in [8]. It is also shown there that such conception of addition leads to some unconsistency with the usual notion of fuzzy sets. Another definition of the addition of fuzzy quantities, avoiding the mentioned discrepancies, is suggested and investigated below. It is shown there that the main properties of the addition of fuzzy quantities are analogous for both concepts of the operation. Namely, the fundamental group properties can be guaranteed only up to an equivalence relation.

0. INTRODUCTION

The concept of fuzzy quantities suggested in [3] is applicable in the branches in which vague numerical data appear. As it is often necessary not only to describe but also algebraically handle such vague numerical data, it is useful to know their algebraical properties.

This problem was investigated, besides [3] and [4], namely in [8]. Nevertheless the addition operation defined and investigated there is connected with some serious discrepancies. Namely, it could lead to some fuzzy quantities with unlimited values of their membership functions. This fact is an unavoidable consequence of the convolutionary approach to the addition operation.

The addition of fuzzy quantities suggested below is based on a completely different principle of conjunction and disjunction of the possible values of the considered fuzzy quantities. This approach, even if it is analytically less convenient than the previous one, much more respects the specific philosophy and interpretation of the fuzziness. In the following sections we verify that its fundamental algebraic properties are comparable with the algebraic properties of the convolutionary method.

1. FUZZY QUANTITIES AND THEIR SUMS

Due to [3] and [8] the fuzzy quantity represents the numerical data with vague values. In this paper we consider the fuzzy quantities the possible values of which are rational numbers.

If we denote by $\mathbb{Q}$ the set of all rational numbers then each *fuzzy quantity* $\boldsymbol{a}$ with rational values is fully described by a *membership function*

$$f_{\boldsymbol{a}}: \mathbb{Q} \to [0, 1]$$

with the usual fuzzy-theoretical interpretation. By $\mathbf{Q}$ we denote the set of all fuzzy quantities with rational values for which the set

$$\{i \in \mathbb{Q}: f_{\boldsymbol{a}}(i) > 0\}$$

is non-empty and bounded. This assumption is realistic if we consider the fact that each fuzzy quantity achieves some value and that the set of its possible values does not extend some acceptable finite limits.

If $\boldsymbol{a} \in \mathbf{Q}$ and $\boldsymbol{b} \in \mathbf{Q}$ are fuzzy quantities then we say that $\boldsymbol{a} = \boldsymbol{b}$ iff $f_{\boldsymbol{a}}(i) = f_{\boldsymbol{b}}(i)$ for all $i \in \mathbb{Q}$.

In this paper we are interested in the addition operation over the set $\mathbf{Q}$. The sum of two fuzzy quantities is also a fuzzy quantity represented by a membership function. One of possible approaches to its definition, based on the convolution of the membership functions, was investigated in [8]. Another one, better reflecting the typical character and interpretation of the fuzzy sets and fuzzy quantities, is suggested and investigated here.

Definition 1. If $\boldsymbol{a} \in \mathbf{Q}$, $\boldsymbol{b} \in \mathbf{Q}$ and $f_{\boldsymbol{a}}, f_{\boldsymbol{b}}$ are their membership functions then the fuzzy quantity $\boldsymbol{a} + \boldsymbol{b}$ represented by the membership function $f_{\boldsymbol{a}+\boldsymbol{b}}: \mathbb{Q} \to [0, 1]$ is called the *sum of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ iff

$$f_{\boldsymbol{a}+\boldsymbol{b}}(k) = \sup_i \left(\min\left(f_{\boldsymbol{a}}(i), f_{\boldsymbol{b}}(k-i)\right)\right), \quad k \in \mathbb{Q}, \tag{1}$$

where the supremum is considered over the whole set $\mathbb{Q}$.

Remark 1. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ then also

$$f_{\boldsymbol{a}+\boldsymbol{b}}(k) = \sup_j \left(\min\left(f_{\boldsymbol{a}}(k-j), f_{\boldsymbol{b}}(j)\right)\right), \quad k \in \mathbb{Q},$$

where the supremum is considered over the set $\mathbb{Q}$.

The definition can be interpreted as follows. The fuzzy quantity $\boldsymbol{a}$ achieves the value $i \in \mathbb{Q}$ and the fuzzy quantity $\boldsymbol{b}$ achieves the value $k - i \in \mathbb{Q}$ simultaneously. The coincidence of both values can be understood as the fuzzy logical conjunction

$$\min\left(f_{\boldsymbol{a}}(t), f_{\boldsymbol{b}}(k-i)\right)$$

of both events. The possibility that the fuzzy quantity $\boldsymbol{a} + \boldsymbol{b}$ achieves the value $k \in \mathbb{Q}$ is then a fuzzy logical disjunction of the existences of all such pairs $i \in \mathbb{Q}$,

$j = k - i \in \mathbb{Q}$ fulfilling the mentioned condition. This disjunction is represented by the supremum in (1).

The addition operation defined above possesses some of the group properties.

Lemma 1. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbf{Q}$ are fuzzy quantities then $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ and $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$.

Proof. The first equality follows from Definition 1 immediately. The associativity property follows from the relations

$$
\begin{aligned}
f_{(a+b)+c}(m) &= \sup_k (\min (f_{a+b}(k), f_c(m - k))) = \\
&= \sup_k (\min (\sup_i (\min (f_a(i), f_b(k - i)))), f_c(m - k)) = \\
&= \sup_k (\sup_i (\min (\min (f_a(i), f_b(k - i)), f_c(m - k)))) = \\
&= \sup_i (\sup_k (\min (\min (f_b(k - i), f_c(m - k)), f_a(i)))) = \\
&= \sup_i (\min (f_a(i), \sup_k (\min (f_b(k - i), f_c(m - k))))) = \\
&= \sup_i (\min (f_a(i), \sup_j (\min f_b(j), f_c(m - i - j))))) = \\
&= \sup_i (\min (f_a(i), f_{b+c}(m - i))) = f_{a+(b+c)}(m) . \qquad \square
\end{aligned}
$$

Theorem 1. The set $\mathbf{Q}$ of fuzzy quantities with rational values is a commutative semigroup with respect to the addition operation (1).

Proof. The statement is an immediate consequence of Lemma 1. $\square$

2. OTHER GROUP PROPERTIES

Analogously to [8] we define the concepts of the zero element and inverse elements in the set $\mathbf{Q}$.

Definition 2. The fuzzy quantity $\boldsymbol{o} \in \mathbf{Q}$ such that $f_o(0) = 1$, $f_o(i) = 0$ for $i \in \mathbb{Q}$, $i \neq 0$, is called the *zero element* of the set $\mathbf{Q}$.

Definition 3. If $\boldsymbol{a} \in \mathbf{Q}$ then the fuzzy quantity $-\boldsymbol{a} \in \mathbf{Q}$ such that $f_{-a}(i) = f_a(-i)$ for all $i \in \mathbb{Q}$ is called the *opposite element* to $\boldsymbol{a}$.

Lemma 2. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ then for all $k \in \mathbb{Q}$

$$f_{a+(-b)}(k) = \sup_j (\min (f_a(k + j), f_b(j))) = \sup_i (\min (f_a(i), f_b(i - k))) .$$

Proof. For all $k \in \mathbb{Q}$

$$
\begin{aligned}
f_{a+(-b)}(k) &= \sup_i (\min (f_a(i), f_{-b}(k - i))) = \\
&= \sup_i (\min (f_a(i), f_b(i - k))) = \sup_j (\min (f_a(k + j), f_b(j))) ,
\end{aligned}
$$

where $j = i - k$ was substituted, and the suprema are considered over the set $\mathbb{Q}$. $\square$

Lemma 3. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ then $-(\boldsymbol{a}+\boldsymbol{b}) = (-\boldsymbol{a}) + (-\boldsymbol{b})$ and $-(-\boldsymbol{a}) = \boldsymbol{a}$.

Proof. For all $k \in \mathbb{Q}$

$$f_{-(a+b)}(k) = f_{a+b}(k) = \sup_i (\min(f_a(i), f_b(-k-i))) =$$

$$= \sup_j (\min(f_a(-j), f_b(-k+j))) = \sup_j (\min(f_{-a}(j), f_{-b}(k-j))) = f_{(-a)+(-b)}(k).$$

The equality $-(-\boldsymbol{a}) = \boldsymbol{a}$ follows from Definition 3 immediately. □

As the equality $\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{o}$ does not generally hold, the group properties are not generally fulfilled for the addition operation (1) on $\mathbf{Q}$ and with $\boldsymbol{o}$ as the zero element. However, it is possible to guarantee, analogously to [8], the group properties up to certain equivalence relation.

3. EQUIVALENCE OF FUZZY QUANTITIES

There exists a similarity between some fuzzy quantities. This similarity will be specified here, and its importance for the group properties of the set $\mathbf{Q}$ will be shown.

Definition 4. A fuzzy quantity $\boldsymbol{s} \in \mathbf{Q}$ is said to be symmetric iff $\boldsymbol{s} = -\boldsymbol{s}$. The class of all symmetric fuzzy quantities from $\mathbf{Q}$ will be denoted by $\boldsymbol{S} \subset \mathbf{Q}$.

Remark 2. It is obvious that the zero element is symmetric, $\boldsymbol{o} \in \boldsymbol{S}$.

Lemma 4. If $\boldsymbol{s} \in \boldsymbol{S}$ and $\boldsymbol{t} \in \boldsymbol{S}$ then also $\boldsymbol{s} + \boldsymbol{t} \in \boldsymbol{S}$.

Proof. For all $k \in \mathbb{Q}$

$$f_{s+t}(k) = \sup_i (\min(f_s(i), f_t(k-i))) =$$

$$= \sup_i (\min(f_s(-i), f_t(-k+i))) = \sup_j (\min(f_s(j), f_t(-k-j))) = f_{s+t}(-k),$$

where $j = -i$ was substituted, and the suprema are considered over the set $\mathbb{Q}$. □

Lemma 5. For any $\boldsymbol{a} \in \mathbf{Q}$ the relation $\boldsymbol{a} + (-\boldsymbol{a}) \in \boldsymbol{S}$ holds.

Proof. For all $k \in \mathbb{Q}$

$$f_{a+(-a)}(k) = \sup_i (\min(f_a(i), f_{-a}(k-i))) =$$

$$= \sup_i (\min(f_{-a}(-i), f_a(-k+i))) =$$

$$= \sup_j (\min(f_{-a}(j), f_a(-k-j))) = f_{a+(-a)}(-k).$$ □

Definition 5. If $\boldsymbol{a} \in \mathbf{Q}$ then the fuzzy quantity $\bar{\boldsymbol{a}} \in \mathbf{Q}$ such that

(2) $$f_{\bar{a}}(i) = f_a(i) \,.\, (\sup_j (f_a(j)))^{-1}, \quad i \in \mathbb{Q},$$

where the supremum is considered over the set $\mathbb{Q}$, is called the *normalized form* of $\boldsymbol{a}$. Any fuzzy quantity $\boldsymbol{a} \in \mathbf{Q}$ such that

$$\sup_j (f_{\boldsymbol{a}}(j)) = 1$$

is called *normalized.*

Remark 3. If $\boldsymbol{u} \in \boldsymbol{S}$ is symmetric then its normalized form $\bar{\boldsymbol{u}}$ is also symmetric.

Remark 4. If $\boldsymbol{a} \in \mathbf{Q}$ is a fuzzy quantity and $-\boldsymbol{a}$ is its opposite element then the normalized form of $-\boldsymbol{a}$ is the opposite element to the normalized form of $\boldsymbol{a}$; in symbols $\overline{(-\boldsymbol{a})} = -(\bar{\boldsymbol{a}})$.

Lemma 6. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ are fuzzy quantities and $\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}$ are their normalized forms then the sum $\bar{\boldsymbol{a}} + \bar{\boldsymbol{b}}$ is also normalized.

Proof. If $\bar{\boldsymbol{a}} + \bar{\boldsymbol{b}}$ is not normalized then there exists $\varepsilon > 0$ such that

$$\sup_i (f_{\bar{\boldsymbol{a}}+\bar{\boldsymbol{b}}}(i)) = 1 - \varepsilon \,. \tag{3}$$

On the other hand,

$$\sup_i (f_{\bar{\boldsymbol{a}}}(i)) = 1 = \sup_i (f_{\bar{\boldsymbol{b}}}(i))$$

and there exist $i, j \in \mathbb{Q}$ such that

$$f_{\bar{\boldsymbol{a}}}(i) > 1 - \varepsilon \,, \quad f_{\bar{\boldsymbol{b}}}(j) > 1 - \varepsilon \,.$$

It means that for $k = i + j$

$$f_{\bar{\boldsymbol{a}}+\bar{\boldsymbol{b}}}(k) \geqq \min (f_{\bar{\boldsymbol{a}}}(i) \,, \quad f_{\bar{\boldsymbol{b}}}(j)) > 1 - \varepsilon$$

which is a contradiction to (3). □

Lemma 7. If $\boldsymbol{a} \in \mathbf{Q}$, $\boldsymbol{b} \in \mathbf{Q}$ and if $\bar{\boldsymbol{a}}$ is the normalized form of $\boldsymbol{a}$ then $\bar{\boldsymbol{a}} + \boldsymbol{b} = \bar{\boldsymbol{a}}$ iff $\boldsymbol{b} = \boldsymbol{o}$.

Proof. If $\boldsymbol{b} = \boldsymbol{o}$ then $\bar{\boldsymbol{a}} + \boldsymbol{o} = \bar{\boldsymbol{a}}$ as follows from (1) immediately. Let $\boldsymbol{b} \neq \boldsymbol{o}$. Then either there exists $j \neq 0$ such that $f_{\boldsymbol{b}}(j) > 0$, or $f_{\boldsymbol{b}}(0) < 1$. If $f_{\boldsymbol{b}}(j) > 0$ for some $j \neq 0$ and $\bar{\boldsymbol{a}} + \boldsymbol{b} = \bar{\boldsymbol{a}}$ then for all $k \in \mathbb{Q}$

$$f_{\bar{\boldsymbol{a}}}(k) = f_{\bar{\boldsymbol{a}}+\boldsymbol{b}}(k) = \sup_i (\min (f_{\bar{\boldsymbol{a}}}(k - i), f_{\boldsymbol{b}}(i))) \,.$$

As the set $\{i \in \mathbb{Q}\colon f_{\bar{\boldsymbol{a}}}(i) > 0\}$ is bounded, there exists $k_0 \in \mathbb{Q}$ such that $f_{\bar{\boldsymbol{a}}}(k_0) = 0$ but $f_{\bar{\boldsymbol{a}}}(k_0 - j) > 0$ and also $f_{\boldsymbol{b}}(j) > 0$. It means that for k_0, $f_{\bar{\boldsymbol{a}}}(k_0) \neq f_{\bar{\boldsymbol{a}}+\boldsymbol{b}}(k_0)$. If $f_{\boldsymbol{b}}(0) < 1$ then $f_{\bar{\boldsymbol{a}}+\boldsymbol{b}}(j) < 1$ for all $j \in \mathbb{Q}$ and consequently $\bar{\boldsymbol{a}} + \boldsymbol{b}$ cannot be equal to $\bar{\boldsymbol{a}}$. □

Definition 6. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ are fuzzy quantities then we say that $\boldsymbol{a}$ is *equivalent* to $\boldsymbol{b}$ and write $\boldsymbol{a} \sim \boldsymbol{b}$, iff there exist $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{S}$ such that $\bar{\boldsymbol{a}} + \bar{\boldsymbol{u}}_1 = \bar{\boldsymbol{b}} + \bar{\boldsymbol{u}}_2$, where $\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}, \bar{\boldsymbol{u}}_1, \bar{\boldsymbol{u}}_2$ are normalized forms of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{u}_1, \boldsymbol{u}_2$, respectively.

Theorem 2. The equivalence relation defined above is reflexive, symmetric and transitive.

Proof. If $a \in \mathbf{Q}$ then $\bar{a} + o = \bar{a} + o$ and consequently $a \sim a$ as $o \in \mathbf{S}$ is normalized. If $a \in \mathbf{Q}$, $b \in \mathbf{Q}$ then $a \sim b$ iff $b \sim a$ as follows from the symmetry of Definition 6. Let $a, b, c \in \mathbf{Q}$, and let $a \sim b$ and $b \sim c$. Then there exist $u_1, u_2, u_3, u_4 \in \mathbf{S}$ such that

$$\bar{a} + \bar{u}_1 = \bar{b} + \bar{u}_2 \quad \text{and} \quad \bar{b} + \bar{u}_3 = \bar{c} + \bar{u}_4 .$$

It means that also

$$\bar{a} + \bar{u}_1 + \bar{u}_3 = \bar{b} + \bar{u}_2 + \bar{u}_3 \quad \text{and} \quad \bar{b} + \bar{u}_2 + \bar{u}_3 = \bar{c} + \bar{u}_2 + \bar{u}_4 ,$$

and consequently $\bar{a} + \bar{u}_1 + \bar{u}_3 = \bar{c} + \bar{u}_2 + \bar{u}_4$. Lemma 6 and Lemma 4 imply $a \sim c$. □

Remark 5. If $u_1, u_2 \in \mathbf{S}$ then $u_1 \sim u_2$ as $\bar{u}_1 + \bar{u}_2 = \bar{u}_2 + \bar{u}_1$.

Lemma 8. If $a, b \in \mathbf{Q}$ then $a \sim b$ iff $(-a) \sim (-b)$.

Proof. Let $\bar{a} + \bar{u}_1 = \bar{b} + \bar{u}_2$ for some $u_1, u_2 \in \mathbf{S}$. By Lemma 4 and Definition 4

$$-(\bar{a} + \bar{u}_1) = -\bar{a} + \bar{u}_1 \quad \text{and} \quad -(\bar{b} + \bar{u}_2) = -\bar{b} + \bar{u}_2 .$$

As by Definition 3 $-(\bar{a} + \bar{u}_1) = (\bar{b} + \bar{u}_2)$ iff $\bar{a} + \bar{u}_1 = \bar{b} + \bar{u}_2$, the statement is proved. □

Theorem 3. If $a, b \in \mathbf{Q}$ are fuzzy quantities such that $\bar{a} + (-\bar{b}) \in \mathbf{S}$ then a and b are equivalent; $a \sim b$.

Proof. If $\bar{a} + (-\bar{b}) = u_2 \in \mathbf{S}$ then $(-\bar{b}) + \bar{b} = u_1 \in \mathbf{S}$ and $\bar{a} + u_1 = \bar{b} + u_2$. Lemma 6 implies that u_1 and u_2 are normalized and consequently $a \sim b$. □

Theorem 4. If $a, b \in \mathbf{Q}$ are fuzzy quantities and if there exists a normalized symmetric fuzzy quantity $\bar{u} \in \mathbf{S}$ such that $\bar{a} + \bar{u} = \bar{b}$ then a and b are equivalent; $a \sim b$.

Proof. The validity of the theorem is obvious if we put $\bar{u}_1 = \bar{u}$ and $\bar{u}_2 = o$ in Definition 6. □

Theorem 5. If $\bar{a}, \bar{b}, \bar{c} \in \mathbf{Q}$ are normalized fuzzy quantities then $\bar{a} + \bar{c} \sim \bar{b} + \bar{c}$ if and only if $\bar{a} \sim \bar{b}$.

Proof. If $\bar{a} + \bar{c} \sim \bar{b} + \bar{c}$ then $\bar{a} + \bar{c} + \bar{u}_1 = \bar{b} + \bar{c} + \bar{u}_2$ for some $u_1, u_2 \in \mathbf{S}$ and

$$\bar{a} + \bar{c} + (-\bar{c}) + \bar{u}_1 = \bar{b} + \bar{c} + (-\bar{c}) + \bar{u}_2 .$$

Lemma 5 and Lemma 6 imply that $\bar{a} + \bar{u}_3 = \bar{b} + \bar{u}_4$ where $\bar{u}_3 = \bar{c} + (-\bar{c}) + \bar{u}_1$ and $\bar{u}_4 = \bar{c} + (\bar{c}) + \bar{u}_2$ are normalized and symmetric. Hence $\bar{a} \sim \bar{b}$. If, on the other hand, $\bar{a} \sim \bar{b}$ then there exist $\bar{u}_1, \bar{u}_2 \in \mathbf{S}$ such that $\bar{a} + \bar{u}_1 = \bar{b} + \bar{u}_2$. It means that also $\bar{a} + \bar{c} + \bar{u}_1 = \bar{b} + \bar{c} + \bar{u}_2$. □

Similarly to [8], the equivalence relation presented above can be effectively used for the verification of some form of the group properties valid for the disjunction-conjustion concept of the addition over the set $\mathbf{Q}$. Comparing the next Theorem 6 with its analogy (also Theorem 6) in [8], we can see that they are not completely analogous. The difference is caused by the different analytical properties of the two addition concepts defined in [8] and here. These different analytical properties cause the necessity to define different concepts of the equivalence relation over the set **Q** and the consequent difference in the form of some group properties.

Theorem 6. The addition operation defined by (1) is a group operation over the set **Q** of fuzzy quantities up to the equivalence relation introduced by Definition 5: namely

$$\boldsymbol{a} + \boldsymbol{b} \sim \boldsymbol{b} + \boldsymbol{a}$$

$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} \sim \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$$

$$\boldsymbol{a} + \boldsymbol{o} \sim \boldsymbol{a}$$

$$\boldsymbol{a} + (-\boldsymbol{a}) \sim \boldsymbol{o}$$

for any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbf{Q}$, $(-\boldsymbol{a})$ being the opposite element to $\boldsymbol{a}$ and $\boldsymbol{o}$ being the zero element from **Q**.

Proof. The first two relations follow from Lemma 1 and Theorem 2 immediately. Relation $\boldsymbol{a} + \boldsymbol{o} \sim \boldsymbol{a}$ is an immediate consequence of Lemma 7, and relation $\boldsymbol{a} + (-\boldsymbol{a}) \sim \boldsymbol{o}$ follows from Lemma 5 and Remark 5 as $\boldsymbol{o} \in \mathbf{S}$. □

4. DETERMINISTIC MULTIPLICATION

It was shown in [8] already that the product of a fuzzy quantity $\boldsymbol{a} \in \mathbf{Q}$ and a rational number $r \in \mathbb{Q}$ has sense. Here we briefly remember its definition and show that its distributivity with the disjunction-conjunction type of addition of fuzzy quantities is valid analogously to the case mentioned in [8].

Definition 7. Let $\boldsymbol{a} \in \mathbf{Q}$ be a fuzzy quantity with rational values, let $\boldsymbol{o} \in \mathbf{Q}$ be the zero element of **Q** and let $r \in \mathbb{Q}$ be a rational number. Then the fuzzy quantity $r \,.\, \boldsymbol{a} \in \mathbf{Q}$ such that for all $i \in \mathbb{Q}$

$$f_{r.\boldsymbol{a}}(i) = f_{\boldsymbol{a}}(i/r) \quad \text{if} \quad r \neq 0\,,$$
$$= f_{\boldsymbol{o}}(i) \quad \text{if} \quad r = 0$$

is called the *product by r* of the fuzzy quantity $\boldsymbol{a}$.

Theorem 7. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ are fuzzy quantities and if $r \in \mathbb{Q}$, then

$$r \,.\, (\boldsymbol{a} + \boldsymbol{b}) = r \,.\, \boldsymbol{a} + r \,.\, \boldsymbol{b}\,.$$

Proof. If $r \neq 0$ then for any $k \in \mathbb{Q}$

$$f_{r.(\boldsymbol{a}+\boldsymbol{b})}(k) = f_{\boldsymbol{a}+\boldsymbol{b}}(k/r) = \sup_j \left(\min\left(f_{\boldsymbol{a}}(j), f_{\boldsymbol{b}}((k/r) - j)\right)\right) =$$

$$= \sup_i \left(\min\left(f_{\boldsymbol{a}}(i/r), f_{\boldsymbol{b}}((k - i)/r)\right)\right) = \sup_i \left(\min\left(f_{r.\boldsymbol{a}}(i), f_{r.\boldsymbol{b}}(k - i)\right)\right) = f_{r.\boldsymbol{a}+r.\boldsymbol{b}}(k)\,.$$

If $r = 0$ then $r \,.\, (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{o} = \boldsymbol{o} + \boldsymbol{o} = r \,.\, \boldsymbol{a} + r \,.\, \boldsymbol{b}\,.$ □

Other properties of the *r*th product of fuzzy quantities from **Q** shown in [8] do not depend on the type of addition and keep unchanged even in this case. It concerns namely also the fact that the conjugate distributivity law $(r + s) \,.\, \boldsymbol{a} = r \,.\, \boldsymbol{a} + s \,.\, \boldsymbol{a}$ is not generally fulfilled for $\boldsymbol{a} \in \mathbf{Q}$, $r, s \in \mathbb{Q}$.

5. FUZZY QUANTITIES WITH FINITE SUPPORT

The fuzzy quantities which can acquire only finite number of possible values represent a special but interesting type of fuzzy quantities. Their applicability in many practical situations deserves special interest. Here we shall see that, as well as in case of convolutionary addition in [8], it is possible to prove some significant properties of those fuzzy quantities.

Definition 8. We say that a fuzzy quantity $\boldsymbol{a} \in \mathbf{Q}$ *has a finite support* iff there exists a finite set of rational numbers

$$\{i_1, i_2, \ldots, i_n\}\,, \quad i_k \in \mathbb{Q}\,, \quad k = 1, 2, \ldots, n\,,$$

such that $\{i \in \mathbb{Q} : f_{\boldsymbol{a}}(i) > 0\} = \{i_k\}_{k=1}^{n}$.

Remark 6. If $\boldsymbol{a} \in \mathbf{Q}$ has a finite support and $\boldsymbol{b} \in \mathbf{Q}$ then for all $m \in \mathbb{Q}$

$$f_{\boldsymbol{a}+\boldsymbol{b}}(m) = \max_i (\min (f_{\boldsymbol{a}}(i)\,,\, f_{\boldsymbol{b}}(m - i))) =$$
$$= \max_{k=1,\ldots,n} (\min (f_{\boldsymbol{a}}(i_k), f_{\boldsymbol{b}}(m - i_k)))\,,$$

where $\{i_k\}_{k=1}^{n}$ is the finite support of the fuzzy quantity $\boldsymbol{a}$.

Remark 7. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ have finite supports and $r \in \mathbb{Q}$ then also $\boldsymbol{a} + \boldsymbol{b}$, $-\boldsymbol{a}$ and $r \,.\, \boldsymbol{a}$ have finite supports as follows from the corresponding definitions.

Lemma 9. Let $\boldsymbol{a} \in \mathbf{Q}$ have a finite support, let

$$\{i \in \mathbb{Q} : f_{\boldsymbol{a}}(i) > 0\} = \{i_k\}_{k=1}^{n}\,, \quad i_0 \in \mathbb{Q}\,, \quad i_0 \neq i_k\,, \quad k = 1, \ldots, n\,,$$

and let $\boldsymbol{a}_0$ be a fuzzy quantity from $\mathbf{Q}$ such that

$$f_{\boldsymbol{a}_0}(i) = f_{\boldsymbol{a}}(i) \quad \text{for} \quad i \in \mathbb{Q}\,, \quad i_0 \neq i \neq -i_0\,,$$
$$f_{\boldsymbol{a}_0}(i_0) = f_{\boldsymbol{a}_0}(-i_0) \geqq f_{\boldsymbol{a}}(i) \quad \text{for all} \quad i \in \mathbb{Q}\,.$$

Then $\boldsymbol{a} \in \mathbf{S}$ iff $\boldsymbol{a}_0 \in \mathbf{S}$. If $\boldsymbol{u} \in \mathbf{S}$ then $\boldsymbol{a} + \bar{\boldsymbol{u}} \in \mathbf{S}$ iff $\boldsymbol{a}_0 + \bar{\boldsymbol{u}} \in \mathbf{S}$.

Proof. The first statement is evident. Concerning the second one, for every $m \in \mathbb{Q}$

$$f_{\boldsymbol{a}_0+\bar{\boldsymbol{u}}}(m) = \max_i (\min (f_{\boldsymbol{a}_0}(i)\,,\, f_{\bar{\boldsymbol{u}}}(m - i))) =$$
$$= \max_{k=1,\ldots,n} (\min (f_{\boldsymbol{a}_0}(i_k), f_{\bar{\boldsymbol{u}}}(m - i_k)))\,,$$
$$f_{\boldsymbol{a}_0+\bar{\boldsymbol{u}}}(-m) = \max_j (\min (f_{\boldsymbol{a}_0}(j)\,,\, f_{\bar{\boldsymbol{u}}}(-m - j))) =$$
$$= \max_j (\min (f_{\boldsymbol{a}_0}(j), f_{\bar{\boldsymbol{u}}}(m + j))) = \max_i (\min (f_{\boldsymbol{a}_0}(-i), f_{\bar{\boldsymbol{u}}}(m - i)))\,,$$

where the maxima are considered over the set $\mathbb{Q}$ if not specified otherwise. Let us choose $m_0 \in \mathbb{Q}$ such that

$$f_{a_0+\bar{u}}(m_0) = \min\left(f_{a_0}(i_0), f_{\bar{u}}(m_0 - i_0)\right) = f_{a_0}(i_0).$$

Such m_0 exists according to Definition 5. The equation

$$f_{a_0}(-i_0) = f_{a_0}(i_0)$$

implies

$$\min\left(f_{a_0}(-i_0), f_{\bar{u}}(m_0 - i_0)\right) = f_{a_0}(-i_0),$$

and as

$$f_{a_0}(i_0) \geqq f_{a_0}(i) = f_{a}(i) \quad \text{for all} \quad i \neq i_0, -i_0,$$

then also

$$f_{a_0+\bar{u}}(-m_0) = \max_i\left(\min\left(f_{a_0}(-i_0), f_{\bar{u}}(m_0 - i)\right)\right) = f_{a_0}(i_0) = f_{a_0+\bar{u}}(m_0).$$

If $m \in \mathbb{Q}$ is such that $f_{a_0+\bar{u}}(m) \neq f_{a_0}(i_0)$ then the inequality $f_{a_0+\bar{u}}(-m) \neq f_{a_0}(i_0)$ follows from the previous steps of this proof. Hence

$$f_{a_0+\bar{u}}(m) = f_{a+\bar{u}}(m) \quad \text{and} \quad f_{a_0+\bar{u}}(-m) = f_{a+\bar{u}}(-m),$$

and consequently $\boldsymbol{a}_0 + \bar{\boldsymbol{u}} \in \boldsymbol{S}$ iff $\boldsymbol{a} + \bar{\boldsymbol{u}} \in \boldsymbol{S}$. □

Lemma 10. Let the fuzzy quantity $\boldsymbol{a} \in \mathbf{Q}$ have finite support and let $\boldsymbol{u} \in \boldsymbol{S}$ be symmetric. If $\bar{\boldsymbol{a}} + \bar{\boldsymbol{u}} \in \boldsymbol{S}$ then also $\boldsymbol{a} \in \boldsymbol{S}$.

Proof. Let $\{i \in \mathbb{Q}: f_a(i) > 0\} = \{i_1, \ldots, i_n\}$. Then

$$f_{\bar{a}+\bar{u}}(m) = \max_{k=1,\ldots,n}\left(\min\left(f_{\bar{a}}(i_k), f_{\bar{u}}(m - i_k)\right)\right),$$

$$f_{\bar{a}+\bar{u}}(-m) = \max_j\left(\min\left(f_{\bar{a}}(j), f_{\bar{u}}(-m - j)\right)\right) =$$

$$= \max_j\left(\min\left(f_{\bar{a}}(j), f_{\bar{u}}(m + j)\right)\right) = \max_i\left(\min\left(f_{\bar{a}}(-i), f_{\bar{u}}(m - i)\right)\right),$$

where the maxima are considered over the set $\mathbb{Q}$ if not specified otherwise. Let us denote $j_1 \in \{i_1, \ldots, i_n\}$ such that

$$f_{\bar{a}}(j_1) \geqq f_{\bar{a}}(i) \quad \text{for all} \quad i \in \mathbb{Q},$$

and

$$|j_1| \geqq |k| \quad \text{for all} \quad k \in \mathbb{Q} \quad \text{such that} \quad f_{\bar{a}}(k) = \max_{l=1,\ldots,n} f_{\bar{a}}(i_l).$$

If $j_1 \geqq 0$ then we choose $m_0 \in \mathbb{Q}$ such that

$$f_{\bar{u}}(m_0 - j_1) = 1, \quad f_{\bar{u}}(m_0 - i) < f_{\bar{a}}(j_1) \quad \text{for} \quad i > -j_1$$

which exists as the sets $\{i \in \mathbb{Q}: f_a(i) > 0\}$ and $\{i \in \mathbb{Q}: f_u(i) > 0\}$ are bounded. Then

$$f_{\bar{a}+\bar{u}}(m_0) = f_{\bar{a}}(j_1), \quad f_{\bar{a}+\bar{u}}(-m_0) = \max_i\left(\min\left(f_{\bar{a}}(-i), f_{\bar{u}}(m_0 - i)\right)\right).$$

For $|i| > j_1$, $f_{\bar{a}}(-i) < f_{\bar{a}}(j_1)$ by assumption and consequently

$$\min\left(f_{\bar{a}}(-i), f_{\bar{u}}(m_0 - i)\right) < f_{\bar{a}}(j_1).$$

For $i > -j$, $f_{\bar{u}}(m_0 - i) < f_{\bar{a}}(j_1)$ and consequently

$$\min(f_{\bar{a}}(-i), f_{\bar{u}}(m_0 - i)) < f_{\bar{a}}(j_1).$$

As

$$f_{\bar{a}+\bar{u}}(m_0) = f_{\bar{a}+\bar{u}}(-m_0) \quad \text{and} \quad f_{\bar{a}+\bar{u}}(m_0)) = f_{\bar{a}}(j_1),$$

then necessarily $f_{\bar{a}+\bar{u}}(-m_0) = f_{\bar{a}}(j_1)$, and

$$(4) \qquad f_{\bar{a}}(-j_1) = f_{\bar{a}}(j_1).$$

If $j_1 < 0$ then we choose $m_0 \in \mathbb{Q}$ such that

$$f_{\bar{u}}(m_0 - j_1) = 1, \quad f_{\bar{u}}(m_0 - i) < f_{\bar{a}}(j_1) \quad \text{for} \quad i < j_1,$$

and analogously to the previous procedure we prove that (4) holds. Let us suppose that

$$f_{\bar{a}}(i) = f_{\bar{a}}(-i) \quad \text{for} \quad i = j_1, \ldots, j_m, \quad m < n,$$
$$j_k \in \{i_i, \ldots, i_n\} \quad \text{for} \quad k = 1, \ldots, m,$$

and that there still exists $i \in \mathbb{Q}$ such that $i \neq j_1, \ldots, j_m$, $f_a(i) > 0$. Let us construct $\boldsymbol{a}_m \in \mathbf{Q}$ such that

$$f_{a_m}(i) = f_{\bar{a}}(i) \quad \text{for} \quad i \in \mathbb{Q}, \quad i \notin \{j_1, \ldots, j_m, -j_1, \ldots, -j_m\},$$
$$= 0 \quad \text{for} \quad i = j_1, \ldots, j_m, -j_1, \ldots, -j_m.$$

Lemma 9 implies that $\boldsymbol{a}_m \in \boldsymbol{S}$ iff $\bar{\boldsymbol{a}} \in \boldsymbol{S}$ and $\boldsymbol{a}_m + \bar{\boldsymbol{u}} \in \boldsymbol{S}$ iff $\bar{\boldsymbol{a}} + \bar{\boldsymbol{u}} \in \boldsymbol{S}$. It means that the proof procedure used above can be repeated. Let us choose $j_{m+1} \in \mathbb{Q}$ such that

$$f_{a_m}(j_{m+1}) \geqq f_{a_m}(i) \quad \text{for all} \quad i \in \mathbb{Q}$$
$$|j_{m+1}| \geqq |k| \quad \text{for all} \quad k \in \mathbb{Q} \quad \text{such that} \quad f_{a_m}(k) = \max_i f_{a_m}(i).$$

Applying the procedure used for j_1 it is possible to prove that

$$f_{\bar{a}}(j_{m+1}) = f_{a_m}(j_{m+1}) = f_{a_m}(-j_{m+1}) = f_{\bar{a}}(-j_{m+1}).$$

After a finite number of such steps the equality $f_{\bar{a}}(j) = f_{\bar{a}}(-j)$ and consequently also $f_a(j) = f_a(-j)$ will be proved for all $j \in \mathbb{Q}$. □

Theorem 8. If the fuzzy quantities $\boldsymbol{a} \in \mathbf{Q}$ and $\boldsymbol{b} \in \mathbf{Q}$ have a finite support then $\boldsymbol{a} \sim \boldsymbol{b}$ if and only if $\bar{\boldsymbol{a}} + (-\bar{\boldsymbol{b}})$ is symmetric.

Proof. The implication $\bar{\boldsymbol{a}} + (-\bar{\boldsymbol{b}}) \in \boldsymbol{S} \Rightarrow \boldsymbol{a} \sim \boldsymbol{b}$ was already proved in Theorem 3 for more general case. On the other hand, if $\boldsymbol{a} \sim \boldsymbol{b}$ then there exist $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{S}$ such that $\bar{\boldsymbol{a}} + \bar{\boldsymbol{u}}_1 = \bar{\boldsymbol{b}} + \bar{\boldsymbol{u}}_2$. Then

$$\bar{\boldsymbol{a}} + (-\bar{\boldsymbol{b}}) + \bar{\boldsymbol{u}}_1 = \bar{\boldsymbol{b}} + (-\bar{\boldsymbol{b}}) + \bar{\boldsymbol{u}}_2 = \bar{\boldsymbol{u}}_3 + \bar{\boldsymbol{u}}_2 \in \boldsymbol{S},$$

where Lemma 6 was used. Lemma 10 implies that $\bar{\boldsymbol{a}} + (-\bar{\boldsymbol{b}}) \in \boldsymbol{S}$. □

Theorem 9. Let $\boldsymbol{a} \in \mathbf{Q}$ have a finite support. Then $\boldsymbol{a} \sim \boldsymbol{o}$ iff $\boldsymbol{a} \in \boldsymbol{S}$ where $\boldsymbol{o}$ is the zero element of $\mathbf{Q}$.

Proof. The statement follows from Theorem 8 and Remarks 2 and 5. □

Corollary. Let us denote by $\mathbf{Q}_0 \subset \mathbf{Q}$ the class of all fuzzy quantities from $\mathbf{Q}$ having finite supports. Then $\mathbf{S} \cap \mathbf{Q}_0$ is one of the equivalence classes generated by the relation $\sim$ in the set $\mathbf{Q}_0$.

6. CONCLUSIVE REMARKS

The operation of addition considered over the set $\mathbf{Q}$ can be defined in more ways. One of them, based on the supremum and minimum operations, was suggested and investigated here. Another one, based on the convolution of the membership functions was suggested in [3] and investigated in [8], too. Let us remember its definition. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ then the membership function f_{a+b} of their sum is defined by

$$f_{a+b}(k) = \sum_i f_a(i) f_b(k-i) = \sum_j f_a(k-j) f_b(j), \quad k \in \mathbb{Q}, \tag{5}$$

where the sums are considered over $\mathbb{Q}$. If we take into consideration that both, the maximum (supremum) and the sum, can in some sense represent the disjunction of the existences of some values of the considered fuzzy quantities, and both, the minimum and the product, can analogously represent their conjunction, then we see that there exists an essential inner similarity between (1) and (5). However, the analytical difference of the formal tools applied in (1) and (5) demands rather different formulation of some concepts connected with both definitions of f_{a+b}. The main difference seems to be in the equivalence definition, and in the set of admissible values of the membership function f_a, $\boldsymbol{a} \in \mathbf{Q}$. The convolutionary approach can be applied without other problems only if we accept the assumption that $f_a: \mathbb{Q} \to \to [0, \infty)$, where the values of $f_a(i) > 1$ are interpreted in the same way as $f_a(i) = = 1$ (cf. [8]). Under this assumption the equivalence concept introduced by Definition 6 of this paper implies, in case of the convolutionary approach, also the equivalence used in [8]. It follows from the next statement.

Theorem 10. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ and let the addition operation (5) on the set $\mathbf{Q}$ be considered. Then there exist $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbf{S}$ such that $f_{s_1}: \mathbb{Q} \to [0, \infty), f_{s_2}: \mathbb{Q} \to [0, \infty)$ and $\boldsymbol{a} + \boldsymbol{s}_1 = \boldsymbol{b} + \boldsymbol{s}_2$ if and only if there exist $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbf{S}$ such that $f_{u_1}: \mathbb{Q} \to [0, 1]$, $f_{u_2}: \mathbb{Q} \to [0, 1]$ and $\bar{\boldsymbol{a}} + \bar{\boldsymbol{u}}_1 = \bar{\boldsymbol{b}} + \bar{\boldsymbol{u}}_2$, where $\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}, \bar{\boldsymbol{u}}_1, \bar{\boldsymbol{u}}_2$ are the normalized forms of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{u}_1, \boldsymbol{u}_2$, respectively.

Proof. Let us denote for arbitrary $\boldsymbol{x} \in \mathbf{Q}$

$$\sigma_x = \sup_i (f_x(i)) .$$

If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$, and if $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbf{S}$ are such that

$$f_{u_1}: \mathbb{Q} \to [0, 1], \quad f_{u_2}: \mathbb{Q} \to [0, 1],$$

then $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbf{Q}$ such that for all $i \in \mathbb{Q}$

$$f_{s_1}(i) = f_{u_1}(i) \,.\, (\sigma_a \,.\, \sigma_{u_1})^{-1}, \quad f_{s_2}(i) = f_{u_2}(i) \,.\, (\sigma_b \,.\, \sigma_{u_2})^{-1}$$

fulfil the following conditions: $s_1 \in \mathbf{S}$, $s_2 \in \mathbf{S}$,

$$f_{s_1}: \mathbb{Q} \to [0, \infty), \quad f_{s_2}: \mathbb{Q} \to [0, \infty).$$

Moreover

$$f_{\bar{a}+\bar{u}_1}(k) = \sum_i f_{\bar{a}}(i) f_{\bar{u}_1}(k-i) =$$

$$= (\sigma_a \cdot \sigma_{u_1})^{-1} \sum_i f_a(i) f_{u_1}(k-i) = \sum_i f_a(i) f_{s_1}(k-i) = f_{a+s_1}(k),$$

$$f_{\bar{b}+\bar{u}_2}(k) = \sum_i f_{\bar{b}}(i) f_{\bar{u}_2}(k-i) =$$

$$= (\sigma_b \cdot \sigma_{u_2})^{-1} \sum_i f_b(i) f_{u_2}(k-i) = \sum_i f_b(i) f_{s_2}(k-i) = f_{b+s_2}(k),$$

for all $k \in \mathbb{Q}$. It means that $\bar{a} + \bar{u}_1 = \bar{b} + \bar{u}_2$ iff $a + s_1 = b + s_2$. The inverse implication is evident. □

The results presented in the previous sections and in [8] are applicable not only for single fuzzy quantities but also for their more-dimensional modifications. If we consider the vectors of fuzzy quantities $(a_1, \ldots, a_n)$, $a_i \in \mathbf{Q}$, $i = 1, \ldots, n$, then the methods of addition (1) and (5) can be used even for the addition of such vectors,

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n),$$

multiplication by a deterministic rational number $r \in \mathbb{Q}$,

$$r \cdot (a_1, \ldots, a_n) = (r \cdot a_1, \ldots, r \cdot a_n),$$

or scalar product with a deterministic rational vector $(r_1, \ldots, r_n) \in \mathbb{Q}^n$,

$$(r_1, \ldots, r_n) \cdot (a_1, \ldots, a_n) = r_1 \cdot a_1 + \ldots + r_n \cdot a_n,$$

and the usual algebraical properties of such operations hold in the sense shown here and in [8], i.e. up to the corresponding equivalence relation. The same conclusions keep true if we consider fuzzy quantities with more-dimensional values in $\mathbb{Q}^n$, i.e. such that $f_a: \mathbb{Q}^n \to [0, 1]$, respectively $f_a: \mathbb{Q}^n \to [0, \infty)$. Then the methods and results presented above and in [8] are valid. For example, instead of (5) we can consider

$$f_{a+b}(k_1, \ldots, k_n) = \sum_{(i_1, \ldots, i_n)} f_a(i_1, \ldots, i_n) \cdot f_b(k_1 - i_1, \ldots, k_n - i_n),$$

instead of (1) we put

$$f_{a+b}(k_1, \ldots, k_n) = \sup_{(i_1, \ldots, i_n)} (\min (f_a(i_i, \ldots, i_n), f_b(k_1 - i_i, \ldots, k_n - i_n))),$$

e.t.c. It is obvious that the results derived for the one-dimensional case can be proved even for such quantities with n-dimensional rational values.

It can be also useful to note that, in spite of the essential difference between the formal tools, both the convolutionary and the disjunction-conjunction, approaches to the addition of rational fuzzy quantities, the results obtained in [8] and here are remarkably similar. This similarity is, at least partly, caused by the principal similarity between the motivation of both approaches mentioned in the first paragraph of these conclusive remarks. Nevertheless, it seems to indicate that the properties of the addition of fuzzy quantities (including the approach to its definition) reflect some essential qualities of the rational fuzzy quantities invariant to the chosen manner of the formal manipulation with them.

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