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CONDITIONS FOR A CONSTRAINED SYSTEM TO HAVE A SET OF IMPULSE ENERGY MEASURES

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Necessary and sufficient conditions are developed for the existence of an nth order single-input single-output linear time-invariant dynamical system without zeros to have a prespecified set of n impulse energy measures. A method is proposed for determining the parameters specifying the transfer function of such a system, if it exists.

1. INTRODUCTION

Given the transfer function $T(s)$ of a linear time-invariant dynamical system it is possible to determine the system's impulse energy measures, where the ith impulse energy measure

$$Y_i \triangleq \int_0^\infty \left(\frac{\mathrm{d}^i}{\mathrm{d}t^i} L^{-1}(T(s))\right)^2 \mathrm{d}t\,, \quad i = 0, 1, 2, \ldots$$

The problem posed and solved in this paper is: Given

$$\mathbf{Y} = [Y_0, Y_1, Y_2, \ldots, Y_{n-1}]^\mathrm{T}\,, \quad Y_i \text{ finite and real}\,, \quad i = 0, 1, 2, \ldots, n-1$$

does there exist a system whose transfer function is

$$T(s) = \frac{1}{s^n + p_{n-1}\, s^{n-1} + p_{n-2}\, s^{n-2} + \ldots + p_0}\,, \quad p_i \text{ real}\,, \quad i = 0, 1, 2, \ldots, n-1$$

such that the impulse energy measures of $T(s)$ are $Y_0, Y_1, Y_2, \ldots, Y_{n-1}$? If the answer is in the affirmative the paper proposes a method to determine $p_{n-1}, p_{n-2}, \ldots, p_0$.

Several results relating to the computation and application of the impulse response, impulse energy measures and quadratic moments of a system have been reported [1], [2], [3], [4], [5]. In this paper the technique used by Baklanov [2] to compute impulse energy measures has been used to advantage. At first sight, it may appear that the form of $T(s)$ indicated above is too constrained, there being no zeros and the feedforward gain being unity, for the solution to have any significant application.

Such, however, is not the case; the impulse energy measures of such a system have been employed to achieve exact and approximate model matching of multivariable systems by state feedback [6].

The structuring of this paper is as follows. First a set of linear algebraic equations are developed to determine the impulse energy measures of a given system which has a transfer function without any zeros. Next, the set of algebraic equations are taken as the starting point and conditions are derived for the solution to represent the impulse energy measures of an asymptotically stable system. This result paves the way for a theorem specifying the necessary and sufficient conditions for the existence of a $T(s)$ having a prespecified set of n impulse energy measures. An algorithm is then proposed to determine the parameters specifying $T(s)$, if it exists.

2. DETERMINATION OF IMPULSE ENERGY MEASURES

Lemma 1. If

$$T(s) = \frac{1}{s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \dots + p_0}, \quad p_i \text{ real for } i = 0, 1, 2, \dots, n-1,$$

is asymptotically stable and Y_i is its ith impulse energy measure,

$$\mathbf{FY} = \mathbf{Q},$$

where

$$(1) \qquad \mathbf{F} = \begin{bmatrix} p_0 & -p_2 & p_4 & \dots & 0 \\ 0 & p_1 & -p_3 & \dots & 0 \\ 0 & -p_0 & p_2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & \dots & & -1 \\ 0 & \dots & \dots & & p_{n-1} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_{n-2} \\ Y_{n-1} \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

Proof.

$$T(s) = \frac{1}{s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \dots + p_0}$$

Hence, in the time domain we can write

$$y^{(n)}(t) + p_{n-1}\, y^{(n-1)}(t) + p_{n-2}\, y^{(n-2)}(t) + \dots + p_0 y(t) = u(t)$$

where $y^{(i)}(t)$ is the ith derivative of the output $y(t)$ and $u(t)$ is the input. If $u(t)$ is a unit impulse and the initial conditions are zero, $y(t)$ becomes the impulse response. In such a case we have

$$(2) \qquad y^{(n)} + p_{n-1}\, y^{(n-1)} + p_{n-2}\, y^{(n-2)} + \dots + p_0\, y = 0$$

with

$$y(0) = y_0\,, \quad y^{(1)}(0) = y_1, \; \dots, \; y^{(n-1)}(0) = y_{n-1}\,.$$

The argument t has been dropped for convenience. Let us expand $T(s)$ as

$$T(s) = h_0/s + h_1/s^2 + h_2/s^3 + \dots$$

Then

(3) $$y(t) = L^{-1}(T(s)) = h_0 + h_1 t + h_2 t^2/2! + \dots$$

From (3) we have

$$y(0) = y_0 = h_0, \quad y^{(1)}(0) = y_1 = h_1, \quad \dots, \quad y^{(n-1)}(0) = y_{n-1} = h_{n-1}.$$

From the expansion of $T(s)$ we have

(4) $$1 = (s^n + p_{n-1}s^{n-1} + \dots + p_0)(h_0/s + h_1/s^2 + h_2/s^3 + \dots).$$

From (4)

(5) $$h_0 = y_0 = 0, \quad h_1 = y_1 = 0, \quad \dots, \quad h_{n-2} = 0, \quad h_{n-1} = y_{n-1} = 1.$$

Now multiply both sides of (2) by and integrate with respect to t from 0 to ∞. This yields

(6) $$\int_0^\infty yy^{(n)}\,dt + \int_0^\infty p_{n-1}yy^{(n-1)}\,dt + \dots + \int_0^\infty p_0 yy\,dt = 0.$$

Consider a term of (6), say $\int_0^\infty yy^{(j-1)}\,dt$,

$$\int_0^\infty yy^{(j-1)}\,dt = \int_0^\infty y\,d(y^{(j-2)}) = yy^{(j-2)}\big|_0^\infty - \int_0^\infty y^{(1)}y^{(j-2)}\,dt.$$

Since $T(s)$ is asymptotically stable

$$y(\infty) = 0, \quad y^{(i)}(\infty) = 0, \quad i = 0, 1, 2, \dots$$

Thus

$$\int_0^\infty yy^{(j-1)}\,dt = -y_0 y_{n-2} + \int_0^\infty y^{(1)}y^{(j-2)}\,dt.$$

The second term can again be integrated by parts to yield a product term comprising initial condition and an integral involving the product of derivatives of y. Two possibilities exist with regard to the final result of such operations. Apart from terms explicitly dependent on the initial conditions, the term containing the integral can become $\int_0^\infty y^{(q)}y^{(q+1)}\,dt$ or $\int_0^\infty y^{(q)}y^{(q)}\,dt$. In the former case $y^{(q)^2}/2\big|_0^\infty$ is the result while in the latter case it is Y_q. Thus we can reduce the terms of (6) to product terms dependent on the known initial conditions and unknown terms involving the impulse energy measures. From (5) it follows that the terms dependent on the initial conditions are zero. Next (2) can be multiplied by $y^{(i)}$, $i = 1, 2, \dots, n-1$ and the process of integration by parts repeated. For $i = 1, 2, \dots, n-2$ all the terms dependent on the initial conditions will be zero, while for $i = n-1$ one term will be nonzero, this being $y^{(n-1)^2}/2\big|_0^\infty = 0{\cdot}5$.

In matrix form the concerned equations become

$$\begin{bmatrix} p_0 & -p_2 & p_4 & \dots & 0 \\ 0 & p_1 & -p_3 & \dots & 0 \\ 0 & -p_0 & p_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & & -1 \\ 0 & 0 & \dots & & p_{n-1} \end{bmatrix} \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_{n-2} \\ Y_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -0{\cdot}5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

□

3. VALIDITY OF THE SOLUTION AND STABILITY CONSTRAINTS

Lemma 2. Given $\mathbf{FY} = \mathbf{Q}$, formed as in (1) and $\mathbf{Y}$ finite,

$$T(s) = \frac{1}{s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \ldots + p_0}$$

has impulse energy measures $Y_0, Y_1, \ldots, Y_{n-1}$ if and only if

$$\det \mathbf{F}\begin{bmatrix} 1 & 2 & 3 & \ldots & i \\ 1 & 2 & 3 & \ldots & i \end{bmatrix} > 0\,, \quad i = 1, 2, 3, \ldots, n$$

where

$$\mathbf{F}\begin{bmatrix} 1 & 2 & 3 & \ldots & i \\ 1 & 2 & 3 & \ldots & i \end{bmatrix}$$

is the matrix formed by rows $1, 2, 3, \ldots, i$ and columns $1, 2, 3, \ldots, i$ of $\mathbf{F}$ taken in that order.

Proof. Suppose $Y_0, Y_1, \ldots, Y_{n-1}$ are the impulse energy measures of $T(s)$. Since Y_i, $i = 0, 1, 2, \ldots, n-1$ are finite $T(s)$ is asymptotically stable. Hence $p_0 + p_1 s + p_2 s^2 + \ldots + p_{n-1}s^{n-1} + s^n$ is a strictly Hurwitz polynomial. The necessary and sufficient conditions for a polynomial of this type to be strictly Hurwitz are known to be

$$p_0 > 0\,, \quad p_1 > 0 \quad \text{and} \quad \det \mathbf{D}_i > 0\,, \quad i = 2, 3, \ldots, n$$

where

$$\mathbf{D}_i = \begin{bmatrix} p_1 & p_0 & 0 & 0 & \ldots & 0 \\ p_3 & p_2 & p_1 & p_0 & \ldots & 0 \\ p_5 & p_4 & p_3 & p_2 & \ldots & 0 \\ \vdots & & \ldots & & & \vdots \\ p_{2i-1} & & \ldots & & & p_i \end{bmatrix}, \quad \text{with} \quad p_j = 0 \quad \text{for} \quad j > n\,, \quad \text{and} \quad p_j = 1 \quad \text{for} \quad j = n\,.$$

Since the coefficient of s^n is unity, $\det \mathbf{D}_n = 1 . \det \mathbf{D}_{n-1} = \mathbf{D}_{n-1}$. So $\mathbf{D}_i$, $i = 1, 2, 3, \ldots$ $\ldots, n-1$ alone need by considered.

Let $\mathbf{B}_i = \mathbf{A}_i^T \mathbf{D}_i^T \mathbf{A}_i$, where

$$\mathbf{A}_i = \begin{bmatrix} -1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \end{bmatrix}_{(i \times i)}$$

Now,

$$\det \mathbf{B}_i = \det \mathbf{A}_i^T . \det \mathbf{D}_i^T . \det \mathbf{A}_i = \det \mathbf{A}_i^T \mathbf{A}_i . \det \mathbf{D}_i = 1 . \det \mathbf{D}_i = \det \mathbf{D}_i\,.$$

In the light of the above, the stability conditions can be reformulated as $p_0 > 0$, $p_0 p_1 > 0$, $\det \mathbf{B}_i > 0$, $i = 2, 3, \ldots, n-1$.

Here

$$\mathbf{B}_i = \begin{bmatrix} p_1 & -p_3 & p_5 & \cdots \\ -p_0 & p_2 & -p_4 & \cdots \\ 0 & -p_1 & p_3 & \cdots \\ 0 & p_0 & -p_2 & \cdots \\ \vdots & & & \vdots \end{bmatrix}_{(i \times i)}$$

It can be readily seen that the above n conditions are equivalent to

$$\det \mathbf{F}\begin{bmatrix} 1 & 2 & 3 & \ldots & i \\ 1 & 2 & 3 & \ldots & i \end{bmatrix} > 0\,, \quad i = 1, 2, 3, \ldots, n\,.$$

Thus the necessity part of the lemma stands proved.

Now suppose that we start with

$$\det \mathbf{F}\begin{bmatrix} 1 & 2 & 3 & \ldots & i \\ 1 & 2 & 3 & \ldots & i \end{bmatrix} > 0\,, \quad i = 1, 2, 3, \ldots, n\,.$$

Then, proceeding as before, but backwards, we have

$$p_0 > 0\,, \quad p_1 > 0 \quad \text{and} \quad \det \mathbf{D}_i > 0\,, \quad i = 2, 3, \ldots, n\,.$$

It follows that $p_0 + p_1 s + p_2 s^2 + \ldots + p_{n-1} s^{n-1} + s^n$ is strictly Hurwitz. So $T(s)$ is asymptotically stable. By Lemma 1, the impulse energy measures of $T(s)$ are such that the vector $\mathbf{Y}$ generated from them satisfies $\mathbf{FY} = \mathbf{Q}$. Now

$$\det \mathbf{F}\begin{bmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 2 & 3 & \ldots & n \end{bmatrix} = \det \mathbf{F} > 0\,.$$

So the solution of $\mathbf{FY} = \mathbf{Q}$ is unique. It follows that

$$\det \mathbf{F}\begin{bmatrix} 1 & 2 & 3 & \ldots & i \\ 1 & 2 & 3 & \ldots & i \end{bmatrix} > 0\,, \quad i = 1, 2, 3, \ldots, n$$

implies that $Y_0, Y_1, Y_2, \ldots, Y_{n-1}$ are the impulse energy measures of $T(s)$. □

4. CONSTRAINED INVERSE PROBLEM

The matrix equation $\mathbf{FY} = \mathbf{Q}$ and the conditions on $\mathbf{F}$ for the solution to be relevant provide the basis for developing explicit conditions to be satisfied by $Y_0, Y_1, Y_2, \ldots, Y_{n-1}$ to guarantee the existence of a $T(s)$ which has them as its impulse energy measures. The following theorem embodies a major result in this direction.

Theorem 1. Given $\mathbf{Y} = [Y_0 \;\; Y_1 \;\; \ldots \;\; Y_{n-1}]^{\mathrm{T}}$, Y_i is finite and real for $i = 0, 1, 2, \ldots$ $\ldots, n-1$, there exists a system with a transfer function

$$T(s) = \frac{1}{s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \ldots + p_0} \qquad p_i \text{ finite and real, } i = 0, 1, \ldots, n-1$$

whose impulse energy measures are $Y_0, Y_1, Y_2, \ldots, Y_{n-1}$ if and only if $\mathbf{M}_n$ is positive definite where

$$\mathbf{M}_n \triangleq \begin{bmatrix} Y_0 & 0 & -Y_1 & 0 & Y_2 & \cdots & & \\ 0 & Y_1 & 0 & -Y_2 & 0 & & & \\ -Y_1 & 0 & Y_2 & 0 & -Y_3 & \cdots & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \\ & & & & & & 0 & Y_{n-1} \end{bmatrix}.$$

Proof. Suppose that there exists a system with transfer function $T(s)$,

$$T(s) = \frac{1}{s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \ldots + p_0}, \quad p_i \text{ real for } i = 0, 1, 2, \ldots, n-1,$$

which has impulse energy measures $Y_0, Y_1, \ldots, Y_{n-1}$ which are finite. Then $T(s)$ is asymptotically stable. Hence, by Lemma 1,

$$\begin{bmatrix} p_0 & -p_2 & p_4 & \cdots & 0 \\ 0 & p_1 & -p_3 & \cdots & 0 \\ 0 & -p_0 & p_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & & \cdots & -1 \\ 0 & 0 & & \cdots & p_{n-1} \end{bmatrix} \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_{n-2} \\ Y_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0{\cdot}5 \end{bmatrix}$$

That is

(7) $$\mathbf{FY} = \mathbf{Q}.$$

Next, by Lemma 2,

(8) $$\det \mathbf{F}\begin{bmatrix} 1 & 2 & 3 & \cdots & i \\ 1 & 2 & 3 & \cdots & i \end{bmatrix} > 0, \quad i = 1, 2, 3, \ldots, n.$$

Define

$$\mathbf{P} \triangleq \begin{bmatrix} (-1)^{(n-1)/2} Y_{(n-1)/2} & \cdots & -Y_1 & -Y_1 & Y_0 & Y_0 \\ (-1)^{(n-1)/2} Y_{(n+1)/2} & \cdots & -Y_2 & -Y_2 & Y_1 & Y_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -Y_{n-2} & -Y_{n-2} & Y_{n-3} & Y_{n-3} \\ 0 & \cdots & 0 & -Y_{n-1} & Y_{n-2} & Y_{n-2} \\ 0 & \cdots & 0 & 0 & 0 & Y_{n-1} \end{bmatrix}, \quad n \text{ odd}.$$

$$\triangleq \begin{bmatrix} (-1)^{n/2-1} Y_{n/2-1} & \cdots & -Y_1 & -Y_1 & Y_0 & Y_0 \\ (-1)^{n/2-1} Y_{n/2} & \cdots & -Y_2 & -Y_2 & Y_1 & Y_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -Y_{n-2} & -Y_{n-2} & Y_{n-3} & Y_{n-3} \\ 0 & \cdots & 0 & -Y_{n-1} & Y_{n-2} & Y_{n-2} \\ 0 & \cdots & 0 & 0 & 0 & Y_{n-1} \end{bmatrix}_{(n\times n)}, \quad n \text{ even}.$$

The structure of **P** is apparent when its columns are examined commencing with the nth. Use of (7) and expansion, for different n, readily yield

$$\text{(9)} \qquad \mathbf{FP} = \begin{bmatrix} e_{11} & \dots & 0 & 0 & 0 & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ \times & \dots & Y_{n-1}. & 0 & 0 & 0 \\ \times & \dots & \times & 0{\cdot}5 & 0 & 0 \\ \times & \dots & \times & \times & Y_{n-1} & 0 \\ \times & \dots & \times & \times & \times & 0{\cdot}5 \end{bmatrix}_{(n \times n)}$$

Here $\times$ stands for entries with which we shall not be concerned at present and

$$\begin{aligned} e_{11} &= Y_{n-1}, \quad n \quad \text{even} \\ &= 0{\cdot}5, \quad n \quad \text{odd.} \end{aligned}$$

In the light of (8), **F** can be represented as [7]

$$\text{(10)} \qquad \mathbf{F} = \mathbf{LDU}.$$

Here **L** is a lower triangular matrix with unity as the diagonal elements while **U** is an upper triangular matrix with unity as the diagonal elements. **D** is a diagonal matrix such that

$$\mathbf{D}(1, 1) = \alpha_1 = \det \mathbf{F}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{D}(i, i) = \alpha_i = \frac{\det \mathbf{F}\begin{bmatrix} 1 & 2 & 3 & \dots & i \\ 1 & 2 & 3 & \dots & i \end{bmatrix}}{\det \mathbf{F}\begin{bmatrix} 1 & 2 & 3 & \dots & i-1 \\ 1 & 2 & 3 & \dots & i-1 \end{bmatrix}}, \quad i = 1, 2, \dots, n.$$

By (8) α_i, $i = 1, 2, \dots, n$ are all positive.

Combining (9) and (10) and noting that $\mathbf{D}^{-1}$ is a diagonal matrix with diagonal entries $1/\alpha_i$, $i = 1, 2, \dots, n$ and that $\mathbf{L}^{-1}$ is a lower triangular matrix with unity as the diagonal elements ,we get

$$\text{(11)} \qquad \mathbf{UP} = \mathbf{D}^{-1}\mathbf{L}^{-1}\mathbf{FP} = \begin{bmatrix} q_{11} & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & & \\ \times & \dots & \dfrac{1}{2\alpha_{n-2}} & 0 & 0 \\ \times & \dots & \times & \dfrac{Y_{n-1}}{\alpha_{n-1}} & 0 \\ \times & \dots & \times & \times & \dfrac{1}{2\alpha_n} \end{bmatrix}$$

$$\begin{aligned} q_{11} &= Y_{n-1}/\alpha_1, \quad n \quad \text{even}, \\ &= 1/(2\alpha_1), \quad n \quad \text{odd}. \end{aligned}$$

Y_{n-1} is positive since, by assumption, it is an impulse energy measure of $T(s)$. Thus, from (11) and (8) we have

$$(12)\quad \det\left(\mathbf{UP}\begin{bmatrix} i & i+1 & i+2 & \dots & n \\ i & i+1 & i+2 & \dots & n \end{bmatrix}\right) > 0\,, \quad i = n, \quad n-1\,, \quad n-2, \dots, 1\,.$$

Since $\mathbf{U}$ is an upper triangular matrix with unity as the diagonal elements, its effect is to leave the nth row of $\mathbf{P}$ unaltered and to make the ith row a linear combination of the $i+1, i+2, i+3, \dots, n$th rows without multiplying the ith row by a constant that is not unity. Hence

$$(13)\qquad \det \mathbf{P}\begin{bmatrix} i & i+1 & i+2 & \dots & n \\ i & i+1 & i+2 & \dots & n \end{bmatrix} = \det\left(\mathbf{UP}\begin{bmatrix} i & i+1 & i+2 & \dots & n \\ i & i+1 & i+2 & \dots & n \end{bmatrix}\right)$$

$$i = n\,,\ n-1\,,\ n-2, \dots, 1\,.$$

Combining (12) and (13) we have

$$(14)\quad \det \mathbf{P}\begin{bmatrix} i & i+1 & i+2 & \dots & n \\ i & i+1 & i+2 & \dots & n \end{bmatrix} > 0\,, \quad i = n\,,\ n-1\,,\ n-2, \dots, 1\,.$$

We shall now use (14) as the starting point for what follows.

Case 1: $(n-i+1)$ odd.

Define

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 1 \end{bmatrix}_{((n-i+1)\times(n-i+1))}$$

Since $\det \mathbf{C} = 1$, we have

$$(15)\quad \det \mathbf{P}\begin{bmatrix} i & i+1 & i+2 & \dots & n \\ i & i+1 & i+2 & \dots & n \end{bmatrix} = \det\left(\mathbf{PC}\begin{bmatrix} i & i+1 & i+2 & \dots & n \\ i & i+1 & i+2 & \dots & n \end{bmatrix}\right) > 0\,.$$

In the second determinant of (15) columns $n-i+1, n-i-1, n-i-3, \dots, 1$ are identical to the corresponding columns of the first determinant. Columns $n-i$, $n-i-2$, $n-i-4, \dots, 2$ however contain one entry, $\pm\, Y_{n+1}$, each in rows $n-i+1$, $n-i$, $n-i-1, \dots, (n-i+4)/2$ respectively. Evaluating the second determinant of (15) with respect to the columns containing the sole entry $\pm Y_{n-1}$ we have

$$\det \mathbf{P}\begin{bmatrix} i & i+1 & i+2 & \dots & n \\ i & i+1 & i+2 & \dots & n \end{bmatrix} =$$

$$Y_{n-1}^{(n-i)/2} \det \begin{bmatrix} (-1)^{(n-i)/2} Y_{(n+i-2)/2} & (-1)^{(n-i+2)/2} Y_{(n+i-4)/2} & \cdots & Y_{i-1} \\ (-1)^{(n-i)/2} Y_{(n+i)/2} & (-1)^{(n-i+2)/2} Y_{(n+i-2)/2} & \cdots & Y_i \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{(n-i)/2} Y_{n-2} & (-1)^{(n-i+2)/2} Y_{n-3} & \cdots & Y_{(n+i-4)/2} \\ (-1)^{(n-i)/2} Y_{n-1} & (-1)^{(n-i+2)/2} Y_{n-2} & \cdots & Y_{(n+i-2)/2} \end{bmatrix}_{((n-i+2)/2)\times((n-i+2)/2)} =$$

$$= (Y_{n-1})^{(n-i)/2} \det \mathbf{E} > 0$$

$\mathbf{E}$ is the $((n-i+2)/2) \times ((n-i+2)/2)$ matrix indicated above. Now define

$$\mathbf{C}_1 \triangleq \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{((n-i+2)/2)\times((n-i+2)/2)}, \quad (n-i+2)/2 \text{ even}$$

$$\triangleq \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{((n-i+2)/2)\times((n-i+2)/2)}, \quad (n-i+2)/2 \text{ odd}$$

Since $\det \mathbf{C}_1 = 1$, premultiplying $\mathbf{E}$ by $\mathbf{C}_1$ will not alter the value of the determinant of $\mathbf{E}$. Thus we have

(16) $$\det \mathbf{P}\begin{bmatrix} i & i+1 & i+2 & \dots & n \\ i & i+1 & i+2 & \dots & n \end{bmatrix} > 0 \Leftrightarrow Y_{n-1}^{(n-i)/2} \det(\mathbf{C}_1\mathbf{E})\, 0 \Leftrightarrow$$

$$\Leftrightarrow Y_{n-1}^{(n-i)/2} \det \begin{bmatrix} Y_{n-1} & -Y_{n-2} & \cdots & -Y_{(n+i-2)/2} \\ -Y_{n-2} & Y_{n-3} & \cdots & Y_{(n+i-4)/2} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{(n+i)/2} & -Y_{(n+i-2)/2} & \cdots & -Y_i \\ -Y_{(n+i-2)/2} & Y_{(n+i-4)/2} & \cdots & Y_{i-1} \end{bmatrix} > 0,$$

$$(n-i+2)/2 \quad \text{even},$$

and

$$Y_{n-1}^{(n-i)/2} \det \begin{bmatrix} Y_{n-1} & -Y_{n-2} & \cdots & Y_{(n+i-2)/2} \\ -Y_{n-2} & Y_{n-3} & \cdots & -Y_{(n+i-4)/2} \\ Y_{n-3} & -Y_{n-4} & \cdots & -Y_{(n+i-6)/2} \\ \vdots & \vdots & \vdots & \vdots \\ -Y_{(n+i)/2} & Y_{(n+i-2)/2} & \cdots & -Y_i \\ Y_{(n+i-2)/2} & Y_{(n+i-4)/2} & \cdots & Y_{i-1} \end{bmatrix} > 0,$$

$$(n-i+2)/2 \quad \text{odd}.$$

Case 2: $(n - i + 1)$ even.

Proceeding as before we get

$$\text{(17)} \qquad \det \mathbf{P}\begin{bmatrix} i & i+1 & i+2 & \dots & n \\ i & i+1 & i+2 & \dots & n \end{bmatrix} > 0 \Leftrightarrow$$

$$\Leftrightarrow Y_{n-1}^{(n-i+1)/2} \det \begin{bmatrix} Y_{n-2} & -Y_{n-3} & \dots & -Y_{(n+i-3)/2} \\ -Y_{n-3} & Y_{n-4} & \dots & Y_{(n+i-5)/2} \\ Y_{n-4} & -Y_{n-5} & \dots & -Y_{(n+i-7)/2} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{(n+i-1)/2} & -Y_{(n+i-3)/2} & \dots & -Y_i \\ -Y_{(n+i-3)/2} & Y_{(n+i-5)/2} & \dots & Y_{i-1} \end{bmatrix} > 0,$$

$$(n - i + 1)/2 \quad \text{even},$$

and

$$Y_{n-1}^{(n-i+1)/2} \det \begin{bmatrix} Y_{n-2} & -Y_{n-3} & \dots & Y_{(n+i-3)/2} \\ -Y_{n-3} & Y_{n-4} & \dots & -Y_{(n+i-5)/2} \\ Y_{n-4} & -Y_{n-5} & \dots & Y_{(n+i-7)/2} \\ \vdots & \vdots & \vdots & \vdots \\ -Y_{(n+i-1)/2} & Y_{(n+i-3)/2} & \dots & -Y_i \\ Y_{(n+i-3)/2} & -Y_{(n+i-5)/2} & \dots & Y_{i-1} \end{bmatrix} > 0,$$

$$(n - i + 1)/2 \quad \text{odd}.$$

The next step will be to show that the conditions (16) and 17 ensure the positive definiteness of $\mathbf{M}_n$. For this we will start with $\mathbf{M}_n$ and note the conditions for its positive definiteness.

$\mathbf{M}_n$ is positive definite is equivalent to $\hat{\mathbf{M}}_n$ being positive definite, where

$$\hat{\mathbf{M}}_n = \mathbf{C}_2^{\mathrm{T}} \mathbf{M}_n \mathbf{C}_2$$

$$\mathbf{C}_2 \triangleq \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{(n \times n)}$$

$\mathbf{C}_2$ is nonsingular; else the positive definiteness of $\mathbf{M}_n$ would not have implied the positive definiteness of $\hat{\mathbf{M}}_n$.

Now

$$\hat{\mathbf{M}}_n = \begin{bmatrix} Y_{n-1} & 0 & -Y_{n-2} & 0 & & \\ 0 & Y_{n-2} & 0 & -Y_{n-3} & & \\ -Y_{n-2} & 0 & Y_{n-3} & 0 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & 0 \\ & & & 0 & & Y_0 \end{bmatrix}$$

$\hat{\mathbf{M}}_n$ is positive definite if and only if

$$\det \hat{\mathbf{M}}_n \begin{bmatrix} 1 & 2 & 3 & \dots & i \\ 1 & 2 & 3 & \dots & i \end{bmatrix} > 0, \quad i = 1, 2, 3, \dots, n.$$

Case 1: i even.

$$(18)\quad \det \mathbf{M}_n \begin{bmatrix} 1 & 2 & 3 & \dots & i \\ 1 & 2 & 3 & \dots & i \end{bmatrix} = \det \begin{bmatrix} Y_{n-1} & 0 & -Y_{n-2} & 0 & & \\ 0 & Y_{n-2} & 0 & -Y_{n-3} & & \\ -Y_{n-2} & 0 & -Y_{n-3} & 0 & & \\ \vdots & & \ddots & & \ddots & \\ \vdots & & & \ddots & & 0 \\ 0 & & \dots & & 0 & Y_{n-i} \end{bmatrix} =$$

$$= \det \begin{bmatrix} Y_{n-1} & -Y_{n-2} & \dots & (-1)^{(i+2)/2}\, Y_{n-(i/2)} \\ -Y_{n-2} & Y_{n-3} & \dots & (-1)^{i+4/2}\, Y_{n-(i+2)/2} \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{(i+2)/2}\, Y_{n-i/2} & (-1)^{(i+4)/2}\, Y_{n-(i+2)/2} & \dots & Y_{n-i+1} \end{bmatrix} \times$$

$$\det \begin{bmatrix} Y_{n-2} & -Y_{n-3} & \dots & (-1)^{(i+2)/2}\, Y_{n-(i+2)/2} \\ -Y_{n-3} & Y_{n-4} & \dots & (-1)^{(i+4)/2}\, Y_{n-(i+4)/2} \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{(i+2)/2}\, Y_{n-(i+2)/2} & (-1)^{(i+4)/2}\, Y_{n-(i+4)/2} & \dots & Y_{n-i} \end{bmatrix}_{(i/2 \times i/2)} > 0$$

Case 2: i odd.

$$(19)\quad \det \mathbf{M}_n = \begin{bmatrix} 1 & 2 & 3 & \dots & i \\ 1 & 2 & 3 & \dots & i \end{bmatrix} =$$

$$= \det \begin{bmatrix} Y_{n-1} & 0 & \dots & (-1)^{(i-1)/2}\, Y_{n-(i+1)/2} \\ 0 & Y_{n-2} & \dots & 0 \\ -Y_{n-2} & 0 & \dots & (-1)^{(i+1)/2}\, Y_{n-(i+3)/2} \\ \vdots & \vdots & \cdot & \vdots \\ (-1)^{(i-1)/2}\, Y_{n-(i+1)/2} & 0 & \dots & Y_{n-i} \end{bmatrix}$$

$$= \det \begin{bmatrix} Y_{n-1} & -Y_{n-2} & \dots & (-1)^{(i-1)/2}\, Y_{n-(i+1)/2} \\ -Y_{n-2} & Y_{n-3} & \dots & (-1)^{(i+1)/2}\, Y_{n-(i+3)/2} \\ \vdots & \vdots & \dots & \vdots \\ (-1)^{(i-1)/2}\, Y_{n-(i+1)/2} & (-1)^{(i+1)/2}\, Y_{n-(i+3)/2} & \dots & Y_{n-i} \end{bmatrix} \times$$

$$\det \begin{bmatrix} Y_{n-2} & -Y_{n-3} & \dots & -(1)^{(i+1)/2}\, Y_{n-(i+1)/2} \\ -Y_{n-3} & Y_{n-4} & \dots & (-1)^{(i+3)/2}\, Y_{n-(i+3)/2} \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{(i+1)/2}\, Y_{n-(i+1)/2} & (-1)^{(i+3)/2}\, Y_{n-(i+3)/2} & \dots & Y_{n-i+1} \end{bmatrix} > 0$$

A careful comparison of (16) and (17) with (18) and (19) reveals that the conditions are equivalent. Hence

$$\det \mathbf{P} \begin{bmatrix} i & i+1 & \dots & n \\ i & i+1 & \dots & n \end{bmatrix} > 0,$$

$i = n, n-1, \dots, 1 \Leftrightarrow \hat{\mathbf{M}}_n$ positive definite $\Leftrightarrow \mathbf{M}_n$ positive definite.

Thus if a $T(s)$ exists with finite impulse energy measures $Y_0, Y_1, \ldots, Y_{n-1}$ then $\mathbf{M}_n$ is positive definite. The necessity part of the theorem stands proved.

To prove sufficiency let us start with the assumption that $Y_0, Y_1, Y_2, \ldots, Y_{n-1}$ are such that $\mathbf{M}_n$ is positive definite. $p_0, p_1, \ldots, p_{n-1}$ can then be determined from

$$(20)\qquad \begin{bmatrix} Y_{n-1} & 0 & -Y_{n-2} & 0 & & \\ 0 & Y_{n-2} & 0 & -Y_{n-3} & & \\ -Y_{n-2} & 0 & Y_{n-3} & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & 0 \\ & & & 0 & & Y_0 \end{bmatrix} \begin{bmatrix} p_{n-1} \\ p_{n-2} \\ p_{n-3} \\ \vdots \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 0{\cdot}5 \\ Y_{n-1} \\ 0 \\ -Y_{n-2} \\ 0 \\ \vdots \end{bmatrix}$$

(20) has a unique solution because the positive definiteness of $\mathbf{M}_n$ ensures the positive definiteness of $\hat{\mathbf{M}}_n$ and so the $(n \times n)$ matrix on the left hand side of (20) is non-singular. Rearranging the terms of (20) we get

$$\begin{bmatrix} p_0 & -p_2 & p_4 & \cdots & 0 \\ 0 & p_1 & -p_3 & \cdots & 0 \\ 0 & -p_0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & -1 \\ 0 & \cdots & \cdots & \cdots & p_{n-1} \end{bmatrix} \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_{n-2} \\ Y_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0{\cdot}5 \end{bmatrix}$$

or

$$(21)\qquad \mathbf{FY} = \mathbf{Q}\,.$$

Let us form $\mathbf{P}$, defined as before, from $Y_0, Y_1, \ldots, Y_{n-1}$. Using (20) we get

$$(22)\qquad \mathbf{FP} = \begin{bmatrix} z_{11} & 0 & \cdots & 0 & 0 & 0 & 0 \\ x_1 & & & & & & \\ \vdots & & & & \vdots & \vdots & \vdots \\ x_1 & & \cdots & Y_{n-1} & 0 & 0 & 0 \\ x_1 & & \cdots & x_1 & 0{\cdot}5 & 0 & 0 \\ x_1 & & \cdots & x_1 & x_1 & Y_{n-1} & 0 \\ x_1 & & \cdots & x_1 & x_1 & x_1 & 0{\cdot}5 \end{bmatrix}$$

x_1 stands for entries with which we shall not be concerned at present,

$$z_{11} = Y_{n-1}\,, \quad n \quad \text{even}\,,$$
$$= 0{\cdot}5\,, \quad n \quad \text{odd}\,.$$

The $(n \times n)$ matrix $\mathbf{F}$ can be uniquely represented as [7]

$$(23)\qquad \mathbf{F} = \mathbf{LDU}\,.$$

where $\mathbf{L}$ is a lower triangular matrix with unity as the diagonal elements, $\mathbf{U}$ is an upper triangular matrix with unity as the diagonal elements and $\mathbf{D}$ is a diagonal matrix with the ith diagonal entry being α_i. We will show that α_i, $i = 1, 2, \ldots, n$ are all positive. Noting that $\mathbf{L}^{-1}$ is a lower triangular matrix with unity as the diagonal elements and using (22) and (23) we have

$$\text{(24)} \qquad \mathbf{DUP} = \begin{bmatrix} z_{11} & 0 & \dots & 0 & 0 & 0 & 0 \\ x_2 & & & & & & \\ \vdots & & & & \vdots & \vdots & \vdots \\ x_2 & & \dots & Y_{n-1} & 0 & 0 & 0 \\ x_2 & & \dots & x_2 & 0{\cdot}5 & 0 & 0 \\ x_2 & & \dots & x_2 & x_2 & Y_{n-1} & 0 \\ x_2 & & \dots & x_2 & x_2 & x_2 & 0{\cdot}5 \end{bmatrix}$$

x_2 stands for entries with which we shall not be concerned at present, z_{11} is as indicated earlier. The positive definiteness of $\mathbf{M}_n$ ensures that $Y_{n-1} > 0$. So from (24) we have

$$\text{(25)} \qquad \det\left(\mathbf{DUP}\begin{bmatrix} i & i+1 & \dots & n \\ i & i+1 & \dots & n \end{bmatrix}\right) > 0\,, \quad i = n\,, \quad n-1, \dots, 1\,.$$

Since $\mathbf{U}$ is an upper triangular matrix with unity as the diagonal elements

$$\text{(26)} \qquad \det\left(\mathbf{UP}\begin{bmatrix} i & i+1 & \dots & n \\ i & i+1 & \dots & n \end{bmatrix}\right) = \det \mathbf{P}\begin{bmatrix} i & i+1 & \dots & n \\ i & i+1 & \dots & n \end{bmatrix},$$
$$i = n\,, \quad n-1, \dots, 1\,.$$

The effect of premultiplication of $\mathbf{UP}$ by $\mathbf{D}$ can, in the light of (26), be seen to yield

$$\text{(27)} \qquad \det\left(\mathbf{DUP}\begin{bmatrix} n \\ n \end{bmatrix}\right) = \alpha_n \det\left(\mathbf{UP}\begin{bmatrix} n \\ n \end{bmatrix}\right) = \alpha_n \det \mathbf{P}\begin{bmatrix} n \\ n \end{bmatrix}$$

$$\text{(28)} \qquad \det\left(\mathbf{DUP}\begin{bmatrix} n-1 & n \\ n-1 & n \end{bmatrix}\right) = \alpha_{n-1}\,\alpha_n \det\left(\mathbf{UP}\begin{bmatrix} n-1 & n \\ n-1 & n \end{bmatrix}\right) =$$
$$= \alpha_{n-1}\alpha_n \det \mathbf{P}\begin{bmatrix} n-1 & n \\ n-1 & n \end{bmatrix}$$
$$\vdots$$

$$\text{(29)} \qquad \det\left(\mathbf{DUP}\begin{bmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{bmatrix}\right) = \alpha_1\alpha_2 \dots \alpha_n \det \mathbf{P}\begin{bmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{bmatrix}$$

As was seen earlier

$$\text{(30)} \qquad \mathbf{M}_n \text{ positive definite} \Leftrightarrow \det \mathbf{P}\begin{bmatrix} i & i+1 & \dots & n \\ i & i+1 & \dots & n \end{bmatrix} > 0\,, \quad i = n\,,\ n-1, \dots, 1.$$

In the light of (25) and (30) it can be seen from (27), (28) and (29) that

$$\text{(31)} \qquad \alpha_1 > 0\,, \quad \alpha_2 > 0, \dots, \alpha_n > 0\,.$$

The unique form of (23) and (31) readily yield

$$\det \mathbf{F}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \alpha_1 > 0$$

$$\det \mathbf{F}\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix} = \alpha_2 \det \mathbf{F}\begin{bmatrix}1\\ 1\end{bmatrix} = \alpha_1\alpha_2 > 0$$

$$\vdots$$

$$\det \mathbf{F}\begin{bmatrix}1 & 2 & \dots & n\\ 1 & 2 & \dots & n\end{bmatrix} = \alpha_n\alpha_{n-1}\alpha_{n-2}\dots\alpha_1 > 0\,.$$

So

$$\det \mathbf{F}\begin{bmatrix}1 & 2 & \dots & i\\ 1 & 2 & \dots & i\end{bmatrix} > 0\,, \quad i = 1, 2, \dots, n\,.$$

By Lemma 2, a $T(s)$ exists with impulse energy measures $Y_0, Y_1, \dots, Y_{n-1}$. This completes the proof. □

Algorithm.

1. Given $Y_0, Y_1, \dots, Y_{n-1}$, form $\mathbf{M}_n$.
2. Check if $\mathbf{M}_n$ is positive definite. If it is not then $T(s)$ does not exist and the procedure terminates. If $\mathbf{M}_n$ is positive definite go to Step 3.
3. Determine $p_{n-1}, p_{n-2}, \dots, p_0$ using (20) and form $T(s)$.

Example. Given $Y_0 = 2$, $Y_1 = 1$ and $Y_2 = 3$.

Then

$$\mathbf{M}_n = \begin{bmatrix} 2 & 0 & -1\\ 0 & 1 & 0\\ -1 & 0 & 3\end{bmatrix}$$

This is positive definite. So the required $T(s)$ exists. From (20)

$$\begin{bmatrix} 3 & 0 & -1\\ 0 & 1 & 0\\ -1 & 0 & 2\end{bmatrix}\begin{bmatrix}p_2\\ p_1\\ p_0\end{bmatrix} = \begin{bmatrix}0{\cdot}5\\ 3\\ 0\end{bmatrix}$$

Hence $p_2 = 0{\cdot}2$, $p_1 = 3$ and $p_0 = 0{\cdot}1$. It follows that

$$T(s) = \frac{1}{s^3 + 0{\cdot}2s^2 + 3s + 0{\cdot}1}$$

5. CONCLUSION

Necessary and sufficient conditions were derived for the existence of a system with a transfer function which has no zeros and n poles having a prespecified set of n impulse energy measures. A method was proposed to determine the parameters of the system, if one exists. The procedure was illustrated by means of a numerical example.

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