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ADDITION OF RATIONAL FUZZY QUANTITIES: CONVOLUTIVE APPROACH

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Some of the algebraical group properties over the set of fuzzy quantities were briefly investigated in Mareš [4] and [5]. Further properties, concerning the addition operation defined over fuzzy quantities with rational values, are developed in the presented paper and in its continuation [11]. This addition can be defined in two different ways. The first one, based on the convolution of the membership functions, is investigated below. The second one, based on the fuzzy logical conjunctions and disjunctions of possible values of the considered fuzzy quantities, is described in the forthcoming paper [11].

The mathematical model of vague numbers considered here is based on the concept of fuzzy quantities suggested and investigated in [4] and [5]. The algebraical properties of the addition operation over such fuzzy numbers are verified and compared with the analogous properties of the classical addition over deterministic rationals. It is shown that some of the additive group axioms are fulfilled only up to certain equivalence relation between the fuzzy quantities. The general definition of the addition of fuzzy quantities used here as well as in [4] and [5] is discussed in the conclusive section of this paper.

0. INTRODUCTION

The concept of fuzzy quantities was suggested to describe the uncertainty connected with some numerical data existing in practical applications as shown e.g. in [6], [7] or [8]. The elementary results on fuzzy quantities presented in [4] and [5] proved to be useful but insufficient if more advanced and more complex models of vague numerical data are developed. Namely, it would be extremally useful to know whether the group properties are fulfilled in case of the addition of fuzzy quantities suggested in [4].

The set of all rational numbers was taken for the basic set of the possible values of the considered fuzzy quantities. It is large and dense enough to satisfy the demands of the typical possible applications of the presented theory, and on the other hand its analytical properties are simple and lucid enough to be theoretically managed by acceptably brief and effective methods. It concerns also the concept of the zero element of the set of fuzzy quantities which can be defined (cf. Definition 3) in an easier way than in case of real-valued fuzzy quantities.

The addition of fuzzy quantities suggested in [4] and developed here is based on the convolution of their membership functions. This method has certain advantages but also disadvantages discussed in the conclusive section. Even if it seems to be practical enough to offer useful tools for the solution of interesting problems, it is not the single possible method how to manage the addition of the fuzzy quantities. Another one is briefly mentioned in the conclusive comments of this paper.

The main goal of this paper is to verify the fundamental algebraical properties of the considered addition operation over the fuzzy quantities. If the set of possible values of the fuzzy quantities forms an additive group then the fuzzy quantities form a semigroup as proved in [4] already. Here we are interested in the existence of the zero element and opposite elements in the class of fuzzy quantities. It is shown in this paper that the sum of a fuzzy quantitity and its opposite need not be the zero element. However, if a natural equivalence relation between fuzzy quantities is considered instead of the equality then the desired group properties are fulfilled up to that equivalence.

1. FUZZY QUANTITIES AND THEIR SUMMATION

In the following sections we denote by $\mathbb{N}$ the set of all natural numbers, by $\mathbb{Q}$ the set of all rational numbers and by $\mathbb{R}_+$ the set of all non-negative real numbers.

Let us suppose that there exist some quantities with non-exactly known rational values. The set of possible values of each such vague quantity can be described by a nonempty fuzzy subset of the set of rational numbers $\mathbb{Q}$. Due to [4] or [3] such non-exact quantities described by fuzzy subsets will be called *fuzzy quantities*.

In symbols; if $\boldsymbol{a}$ is a fuzzy quantity then the fuzzy set of the possible values of $\boldsymbol{a}$ will be described by the *membership function* $f_{\boldsymbol{a}}$: $\mathbb{Q} \to \mathbb{R}_+$ with the usual fuzzy set theoretical interpretation. If, for some $x \in \mathbb{Q}$, $f_{\boldsymbol{a}}(x) = 0$ then the quantity $\boldsymbol{a}$ surely does not achieve the value x. The greater the value $f_{\boldsymbol{a}}(x)$ is the higher is the possibility that the quantity $\boldsymbol{a}$ can achieve the value x. If $f_{\boldsymbol{a}}(x) \geqq 1$ then the value x is considered to be surely achievable.

It is usual to define the membership function $f_{\boldsymbol{a}}$ of a fuzzy set or fuzzy quantity as a mapping from the basic set to the closed and bounded interval $[0, 1]$, and we shall respect this definition even here as much as possible. However, some of the calculations over the values $f_{\boldsymbol{a}}(x)$, $x \in \mathbb{Q}$, performed in the following sections can result in some values $f_{\boldsymbol{a}}(x) > 1$. The interpretation of such values suggested above is practically equivalent to the minimization procedure $\min (f_{\boldsymbol{a}}(x), 1)$ used in [4] but the preservation of the values $f_{\boldsymbol{a}}(x) > 1$ simplifies the formal analytical procedures and notations used below.

In the following sections we denote by $\mathbf{Q}$ the set of all fuzzy quantities with rational values and such that for any $\boldsymbol{a} \in \mathbf{Q}$ the set $\{x \in \mathbb{Q} : f_{\boldsymbol{a}}(x) > 0\}$ is non-empty and bounded.

This assumption about the fuzzy quantities from $\mathbf{Q}$ is natural if we consider the fact that any fuzzy quantity $\boldsymbol{a} \in \mathbf{Q}$ and its membership function $f_{\boldsymbol{a}}$ represent the set of possible values of some existing rational quantity, that the quantity can achieve at least some values, and that the extent of the really possible values of such quantity is not unlimited.

The fuzzy quantities from $\mathbf{Q}$ are fully defined by their membership functions. It means that for any pair of fuzzy quantities $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$

$$(1) \qquad \boldsymbol{a} = \boldsymbol{b} \quad \text{iff} \quad f_{\boldsymbol{a}}(i) = f_{\boldsymbol{b}}(i) \quad \text{for all} \quad i \in \mathbb{Q}\,.$$

The definition of fuzzy quantities would be of no use if we cannot realize some of the usual algebraic operations over them. Here we are interested in the addition operation which was in general form suggested in [4] and [5].

Let us suppose that $\boldsymbol{a}$ and $\boldsymbol{b}$ are fuzzy quantities from $\mathbf{Q}$ with rational possible values and with the membership functions $f_{\boldsymbol{a}}: \mathbb{Q} \to \mathbb{R}_+$, $f_{\boldsymbol{b}}: \mathbb{Q} \to \mathbb{R}_+$. Then their sum $\boldsymbol{a} + \boldsymbol{b}$ is also a fuzzy quantity with rational values and with the membership function $f_{\boldsymbol{a}+\boldsymbol{b}}: \mathbb{Q} \to \mathbb{R}_+$.

Definition 1. If $\boldsymbol{a} \in \mathbf{Q}$, $\boldsymbol{b} \in \mathbf{Q}$ and if $f_{\boldsymbol{a}}, f_{\boldsymbol{b}}$ are their membership functions then the membership function $f_{\boldsymbol{a}+\boldsymbol{b}}$ of the fuzzy quantity $\boldsymbol{a} + \boldsymbol{b}$ is defined by the relation

$$(2) \qquad f_{\boldsymbol{a}+\boldsymbol{b}}(k) = \sum_i f_{\boldsymbol{a}}(i) \,.\, f_{\boldsymbol{b}}(k - i)\,, \quad k \in \mathbb{Q}\,,$$

where the summation is considered over the set $\mathbb{Q}$.

Remark 1. Substituting $i = k - j$ we immediately obtain

$$\sum_i f_{\boldsymbol{a}}(i) \,.\, f_{\boldsymbol{b}}(k - i) = \sum_j f_{\boldsymbol{a}}(k - j) \,.\, f_{\boldsymbol{b}}(j)\,, \quad k \in \mathbb{Q}\,.$$

Some of the algebraical properties of the convolutionary operation (2) were shown in [4] already. Namely:

Lemma 1. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ are fuzzy quantities then $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$.

Proof. $f_{\boldsymbol{a}+\boldsymbol{b}}(k) = \sum_i f_{\boldsymbol{a}}(i) \,.\, f_{\boldsymbol{b}}(k - i) = \sum_j f_{\boldsymbol{b}}(j) \,.\, f_{\boldsymbol{a}}(k - j) = f_{\boldsymbol{b}+\boldsymbol{a}}(k)$. □

Lemma 2. If $\boldsymbol{a} \in \mathbf{Q}$, $\boldsymbol{b} \in \mathbf{Q}$, $\boldsymbol{c} \in \mathbf{Q}$ are fuzzy quantities then $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$.

Proof.

$$f_{(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}}(m) = \sum_k f_{\boldsymbol{a}+\boldsymbol{b}}(k) \,.\, f_{\boldsymbol{c}}(m - k) =$$
$$= \sum_k \sum_i f_{\boldsymbol{a}}(i) \,.\, f_{\boldsymbol{b}}(k - i) \,.\, f_{\boldsymbol{c}}(m - k) = \sum_i f_{\boldsymbol{a}}(i) \left(\sum_k f_{\boldsymbol{b}}(k - i) \,.\, f_{\boldsymbol{c}}(m - k)\right) =$$
$$= \sum_i f_{\boldsymbol{a}}(i) \left(\sum_n f_{\boldsymbol{b}}(n) \,.\, f_{\boldsymbol{c}}(m - i - n)\right) = \sum_i f_{\boldsymbol{a}}(i) \,.\, f_{\boldsymbol{b}+\boldsymbol{c}}(m - i) = f_{\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})}(m)\,,$$

where the substitution $n = k - i$ was used, and where all summations are considered over the whole set $\mathbb{Q}$. $\square$

Theorem 1. The set $\mathbf{Q}$ of fuzzy quantities with rational values and the addition operation defined by (2) forms an additive semigroup which is also commutative.

Proof. The theorem follows immediately from Lemmas 1 and 2. $\square$

2. THE GROUP PROPERTIES

The statement of Theorem 1 follows immediately from the general results presented in [4]. However, the set of rational numbers $\mathbb{Q}$ is an additive group, and it is desirable to check the validity of the group axioms also for $\mathbf{Q}$. First we introduce the concepts of the opposite element and zero element.

Definition 2. If $\boldsymbol{a} \in \mathbf{Q}$ and $f_{\boldsymbol{a}}$ is its membership function then we say that the fuzzy quantity $-\boldsymbol{a} \in \mathbf{Q}$ is an *opposite element* to $\boldsymbol{a}$ iff $f_{-\boldsymbol{a}}(i) = f_{\boldsymbol{a}}(-i)$ for all $i \in \mathbb{Q}$.

Lemma 3. If $\boldsymbol{a} \in \mathbf{Q}$, $\boldsymbol{b} \in \mathbf{Q}$ then for all $k \in \mathbb{Q}$

$$f_{\boldsymbol{a}+(-\boldsymbol{b})}(k) = \sum_j f_{\boldsymbol{a}}(k + j) \cdot f_{\boldsymbol{b}}(j) = \sum_i f_{\boldsymbol{a}}(i) \cdot f_{\boldsymbol{b}}(i - k) .$$

Proof.

$$f_{\boldsymbol{a}+(-\boldsymbol{b})}(k) = \sum_i f_{\boldsymbol{a}}(i) \cdot f_{-\boldsymbol{b}}(k - i) = \sum_i f_{\boldsymbol{a}}(i) \cdot f_{\boldsymbol{b}}(i - k) = \sum_j f_{\boldsymbol{a}}(k + j) \cdot f_{\boldsymbol{b}}(j) ,$$

where $j = i - k$ was substituted, and all summations are considered over the set $\mathbb{Q}$. $\square$

Lemma 4. If $\boldsymbol{a} \in \mathbf{Q}$, $\boldsymbol{b} \in \mathbf{Q}$ then $-(\boldsymbol{a} + \boldsymbol{b}) = (-\boldsymbol{a}) + (-\boldsymbol{b})$ and $-(-\boldsymbol{a}) = \boldsymbol{a}$.

Proof.

$$f_{-(\boldsymbol{a}+\boldsymbol{b})}(k) = f_{\boldsymbol{a}+\boldsymbol{b}}(-k) = \sum_i f_{\boldsymbol{a}}(i) \cdot f_{\boldsymbol{b}}(-k - i) =$$

$$= \sum_j f_{\boldsymbol{a}}(-j) \cdot f_{\boldsymbol{b}}(-k + j) = \sum_j f_{-\boldsymbol{a}}(j) \cdot f_{-\boldsymbol{b}}(k - j) = f_{(-\boldsymbol{a})+(-\boldsymbol{b})}(k)$$

for all $k \in \mathbb{Q}$. The second equality $-(-\boldsymbol{a}) = \boldsymbol{a}$ follows from Definition 2 immediately. $\square$

Definition 3. The fuzzy quantity $\boldsymbol{o} \in \mathbf{Q}$ such that

$$f_{\boldsymbol{o}}(0) = 1 , \quad f_{\boldsymbol{o}}(i) = 0 \quad \text{for} \quad i \in \mathbb{Q} , \quad i \neq 0 ,$$

is called the *zero element* of $\mathbf{Q}$.

Lemma 5. Let $\boldsymbol{a} \in \mathbf{Q}$, $\boldsymbol{b} \in \mathbf{Q}$ be fuzzy quantities. Then $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{a}$ if and only if $\boldsymbol{b} = \boldsymbol{o}$.

Proof. If $\boldsymbol{b} = \boldsymbol{o}$ then

$$f_{\boldsymbol{a}+\boldsymbol{o}}(k) = \sum_i f_{\boldsymbol{a}}(k - i) \cdot f_{\boldsymbol{o}}(i) = f_{\boldsymbol{a}}(k) \quad \text{for} \quad k \in \mathbb{Q} .$$

Let $\boldsymbol{b} \neq \boldsymbol{o}$. Then there exists $j \neq 0$ such that $f_{\boldsymbol{b}}(j) > 0$. If $\boldsymbol{a} = \boldsymbol{a} + \boldsymbol{b}$ then for all $k \in \mathbb{Q}$

$$f_{\boldsymbol{a}}(k) = f_{\boldsymbol{a}+\boldsymbol{b}}(k) = \sum_i f_{\boldsymbol{a}}(k - i) \cdot f_{\boldsymbol{b}}(i) \,.$$

As the set $\{i \in \mathbb{Q}: f_{\boldsymbol{a}}(i) > 0\}$ is bounded, there exists $k_0 \in \mathbb{Q}$ such that $f_{\boldsymbol{a}}(k_0) = 0$ but $f_{\boldsymbol{a}}(k_0 - j) > 0$ and by assumption also $f_{\boldsymbol{b}}(j) > 0$. It means that, for this k_0, $f_{\boldsymbol{a}}(k_0) \neq f_{\boldsymbol{a}+\boldsymbol{b}}(k_0)$. □

It has been already shown in Lemma 5 that the zero element $\boldsymbol{o} \in \mathbf{Q}$ fulfils the demands imposed on a zero element in the group axiomatics. Unfortunately, the opposite element $-\boldsymbol{a}$ to $\boldsymbol{a}$ does not do so. It can be easily seen that, except for some very special and atypical fuzzy quantities, the equation $\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{o}$ is not generally fulfilled.

In the next section we suggest a possibility how to overcome this discrepancy, and to find a way in which all the group axioms can be fulfilled for the addition operation over $\mathbf{Q}$ defined by (1).

3. EQUIVALENCE OF FUZZY QUANTITIES

It is possible to find certain similarity between some fuzzy quantities with rational values. This similarity can be used to partition of the set $\mathbf{Q}$ into equivalence classes having some useful properties.

Definition 4. A fuzzy quantity $\boldsymbol{s} \in \mathbf{Q}$ is called *symmetric* iff $-\boldsymbol{s} = \boldsymbol{s}$. The class of all symmetric fuzzy quantities from $\mathbf{Q}$ will be denoted by $\boldsymbol{S} \subset \mathbf{Q}$.

Remark 2. It is evident that the zero element $\boldsymbol{o}$ is symmetric.

Lemma 6. If $\boldsymbol{s} \in \boldsymbol{S}$ and $\boldsymbol{t} \in \boldsymbol{S}$ then $\boldsymbol{s} + \boldsymbol{t} \in \boldsymbol{S}$.

Proof.

$$f_{\boldsymbol{s}+\boldsymbol{t}}(k) = \sum_i f_{\boldsymbol{s}}(i) \cdot f_{\boldsymbol{t}}(k - i) = \sum_i f_{\boldsymbol{s}}(-i) \cdot f_{\boldsymbol{t}}(-k + i) =$$
$$= \sum_j f_{\boldsymbol{s}}(j) \cdot f_{\boldsymbol{t}}(-k - j) = f_{\boldsymbol{s}+\boldsymbol{t}}(-k) \,,$$

where $j = -i$ was substituted, and all sums are considered over the set $\mathbb{Q}$. □

Lemma 7. If $\boldsymbol{a} \in \mathbf{Q}$ then $\boldsymbol{a} + (-\boldsymbol{a}) \in \boldsymbol{S}$.

Proof. For any $k \in \mathbb{Q}$

$$f_{\boldsymbol{a}+(-\boldsymbol{a})}(k) = \sum_i f_{\boldsymbol{a}}(i) \cdot f_{-\boldsymbol{a}}(k - i) = \sum_i f_{-\boldsymbol{a}}(-i) \cdot f_{\boldsymbol{a}}(-k + i) =$$
$$= \sum_j f_{-\boldsymbol{a}}(j) \cdot f_{\boldsymbol{a}}(-k - j) = f_{\boldsymbol{a}+(-\boldsymbol{a})}(-k) \,.$$

□

Definition 5. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$. We say that $\boldsymbol{a}$ and $\boldsymbol{b}$ are *equivalent* and write $\boldsymbol{a} \sim \boldsymbol{b}$

iff there exist symmetric fuzzy quantities $\boldsymbol{s}_1 \in \mathbf{S}$, $\boldsymbol{s}_2 \in \mathbf{S}$ such that

$$\boldsymbol{a} + \boldsymbol{s}_1 = \boldsymbol{b} + \boldsymbol{s}_2 \,. \tag{3}$$

Theorem 2. The equivalence relation $\boldsymbol{a} \sim \boldsymbol{b}$ defined by (3) is reflexive, symmetric and transitive.

Proof. If $\boldsymbol{a} \in \mathbf{Q}$ then $\boldsymbol{a} \sim \boldsymbol{a}$ as $\boldsymbol{a} + \boldsymbol{o} = \boldsymbol{a} + \boldsymbol{o}$, where $\boldsymbol{o}$ is the zero fuzzy quantity. If $\boldsymbol{a} \sim \boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ then obviously $\boldsymbol{b} \sim \boldsymbol{a}$ as follows from Definition 5. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbf{Q}$ and $\boldsymbol{a} \sim \boldsymbol{b}$, $\boldsymbol{b} \sim \boldsymbol{c}$ where

$$\boldsymbol{a} + \boldsymbol{s}_1 = \boldsymbol{b} + \boldsymbol{s}_2 \,, \quad \boldsymbol{b} + \boldsymbol{s}_3 = \boldsymbol{c} + \boldsymbol{s}_4 \,, \quad \boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4 \in \mathbf{S} \,,$$

then also

$$\boldsymbol{a} + \boldsymbol{s}_1 + \boldsymbol{s}_3 = \boldsymbol{b} + \boldsymbol{s}_2 + \boldsymbol{s}_3 \quad \text{and} \quad \boldsymbol{b} + \boldsymbol{s}_2 + \boldsymbol{s}_3 = \boldsymbol{c} + \boldsymbol{s}_2 + \boldsymbol{s}_4 \,.$$

It means that $\boldsymbol{a} + \boldsymbol{s}_1 + \boldsymbol{s}_3 = \boldsymbol{c} + \boldsymbol{s}_2 + \boldsymbol{s}_4$. By Lemma 6 $\boldsymbol{s}_1 + \boldsymbol{s}_3 = \boldsymbol{s}_5 \in \mathbf{S}$, $\boldsymbol{s}_2 + \boldsymbol{s}_4 = \boldsymbol{s}_6 \in \mathbf{S}$ and $\boldsymbol{a} + \boldsymbol{s}_5 = \boldsymbol{c} + \boldsymbol{s}_6$. Hence $\boldsymbol{a} \sim \boldsymbol{c}$. □

Remark 3. If $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbf{S}$ then $\boldsymbol{s}_1 \sim \boldsymbol{s}_2$ as $\boldsymbol{s}_1 + \boldsymbol{s}_2 = \boldsymbol{s}_2 + \boldsymbol{s}_1$.

Lemma 8. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ then $\boldsymbol{a} \sim \boldsymbol{b}$ iff $(-\boldsymbol{a}) \sim (-\boldsymbol{b})$.

Proof. If $\boldsymbol{a} + \boldsymbol{s}_1 = \boldsymbol{b} + \boldsymbol{s}_2$ for some $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbf{S}$ then $-(\boldsymbol{a} + \boldsymbol{s}_1) = -(\boldsymbol{b} + \boldsymbol{s}_2)$ as follows from Definition 2. □

Theorem 3. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ and if $\boldsymbol{a} + (-\boldsymbol{b}) \in \mathbf{S}$ then $\boldsymbol{a} \sim \boldsymbol{b}$.

Proof. If there exists $\boldsymbol{s}_1 \in \mathbf{S}$ such that $\boldsymbol{a} + (-\boldsymbol{b}) = \boldsymbol{s}_1$ then

$$\boldsymbol{s}_1 + \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b}) + \boldsymbol{b} = \boldsymbol{a} + \boldsymbol{s}_2 \,, \quad \boldsymbol{s}_2 \in \mathbf{S} \,,$$

according to Lemma 7. □

Theorem 4. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ and if there exists $\boldsymbol{s} \in \mathbf{S}$ such that $\boldsymbol{a} + \boldsymbol{s} = \boldsymbol{b}$ then $\boldsymbol{a} \sim \boldsymbol{b}$.

Proof. The statement follows immediately from Definition 5 if we put $\boldsymbol{s}_1 = \boldsymbol{s}$, $\boldsymbol{s}_2 = \boldsymbol{o} \in \mathbf{S}$. □

Theorem 5. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbf{Q}$ are fuzzy quantities, then

$$\boldsymbol{a} + \boldsymbol{c} \sim \boldsymbol{b} + \boldsymbol{c} \quad \text{if and only if} \quad \boldsymbol{a} \sim \boldsymbol{b} \,.$$

Proof. If $\boldsymbol{a} + \boldsymbol{c} \sim \boldsymbol{b} + \boldsymbol{c}$ then $\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{s}_1 = \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{s}_2$ for some $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbf{S}$, and $\boldsymbol{a} + \boldsymbol{c} + (-\boldsymbol{c}) + \boldsymbol{s}_1 = \boldsymbol{b} + \boldsymbol{c} + (-\boldsymbol{c}) + \boldsymbol{s}_2$. It means that $\boldsymbol{a} + \boldsymbol{s}_3 = \boldsymbol{b} + \boldsymbol{s}_4$ for $\boldsymbol{s}_3 = \boldsymbol{c} + (-\boldsymbol{c}) + \boldsymbol{s}_1 \in \mathbf{S}$, $\boldsymbol{s}_4 = \boldsymbol{c} + (-\boldsymbol{c}) + \boldsymbol{s}_2 \in \mathbf{S}$ as follows from Lemma 6 and Lemma 7. Hence, $\boldsymbol{a} \sim \boldsymbol{b}$. On the other hand, if $\boldsymbol{a} \sim \boldsymbol{b}$ and $\boldsymbol{c} \in \mathbf{Q}$ then $\boldsymbol{a} + \boldsymbol{s}_1 = \boldsymbol{b} + \boldsymbol{s}_2$ and $\boldsymbol{a} + \boldsymbol{s}_1 + \boldsymbol{c} = \boldsymbol{b} + \boldsymbol{s}_2 + \boldsymbol{c}$ for some $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbf{S}$. It means that $\boldsymbol{a} + \boldsymbol{c} \sim \boldsymbol{b} + \boldsymbol{c}$. □

The equivalence relation introduced above offers a new look of the quality of the group properties concerning the convolutive addition of fuzzy quantities from $\mathbf{Q}$. The group properties can be successfully guaranteed up to the equivalence relation defined here. This conclusion can be formulated in the following statement.

Theorem 6. The addition operation defined by (2) is a group operation over the set $\mathbf{Q}$ of fuzzy quantities up to the equivalence relation (3). It means that

$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} &\sim \boldsymbol{b} + \boldsymbol{a}, \\ (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} &\sim \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}), \\ \boldsymbol{a} + \boldsymbol{t} \sim \boldsymbol{a} &\quad \text{iff} \quad \boldsymbol{t} \sim \boldsymbol{o}, \\ \boldsymbol{a} + \boldsymbol{b} \sim \boldsymbol{o} &\quad \text{iff} \quad \boldsymbol{b} \sim (-\boldsymbol{a}), \end{aligned}$$

for any $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c} \in \mathbf{Q}$, $(-\boldsymbol{a})$ being the opposite element to $\boldsymbol{a}$ and $\boldsymbol{o}$ being the zero element from $\mathbf{Q}$).

Proof. The associativity and commutativity follow from Lemma 1 and Lemma 2 immediately. Let $\boldsymbol{a} \in \mathbf{Q}$, $\boldsymbol{t} \sim \boldsymbol{o}$, $\boldsymbol{t} \in \mathbf{Q}$. Then there exist $\boldsymbol{s}_1 \in \mathbf{S}$, $\boldsymbol{s}_2 \in \mathbf{S}$ such that $\boldsymbol{t} + \boldsymbol{s}_1 = \boldsymbol{s}_2$. It means that $\boldsymbol{a} + \boldsymbol{t} + \boldsymbol{s}_1 = \boldsymbol{a} + \boldsymbol{s}_2$ and consequently $\boldsymbol{a} + \boldsymbol{t} \sim \boldsymbol{a}$. On the other hand, if $\boldsymbol{a} + \boldsymbol{t} \sim \boldsymbol{a} = \boldsymbol{a} + \boldsymbol{o}$ then by Theorem 5 $\boldsymbol{t} \sim \boldsymbol{o}$. The last property follows from Lemma 7, Remark 3 and Remark 2; namely if $\boldsymbol{b} \sim -\boldsymbol{a}$ then there exist $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4 \in \mathbf{S}$ such that $(-\boldsymbol{a}) + \boldsymbol{s}_1 = \boldsymbol{b} + \boldsymbol{s}_2$ and further $\boldsymbol{a} + (-\boldsymbol{a}) + \boldsymbol{s}_3 = \boldsymbol{o} + \boldsymbol{s}_4$ by Lemma 7 and Remark 3. Then

$$\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{s}_2 + \boldsymbol{s}_3 = \boldsymbol{a} + (-\boldsymbol{a}) + \boldsymbol{s}_1 + \boldsymbol{s}_3 = \boldsymbol{o} + \boldsymbol{s}_4 + \boldsymbol{s}_1$$

and consequently $\boldsymbol{a} + \boldsymbol{b} \sim \boldsymbol{o}$. If, on the other hand, $\boldsymbol{a} + \boldsymbol{b} \sim \boldsymbol{o}$ then there exist $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbf{S}$ such that $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{s}_1 = \boldsymbol{s}_2$ and $\boldsymbol{a} + \boldsymbol{b} + (-\boldsymbol{b}) + \boldsymbol{s}_1 = -\boldsymbol{b} + \boldsymbol{s}_2$. It means that $\boldsymbol{a} + \boldsymbol{s} + \boldsymbol{s}_1 = -\boldsymbol{b} + \boldsymbol{s}_2$ and $\boldsymbol{a} \sim -\boldsymbol{b}$ as $\boldsymbol{s} + \boldsymbol{s}_1 \in \mathbf{S}$. Lemma 8 and Definition 2 imply that $-\boldsymbol{a} \sim \boldsymbol{b}$. □

Remark 4. It was shown by Lemma 1, Lemma 2 and Lemma 5 that the first two group axioms are fulfilled also in the form of equations, and that $\boldsymbol{a} + \boldsymbol{o} = \boldsymbol{a}$ and $\boldsymbol{a} + (-\boldsymbol{a}) \sim \boldsymbol{o}$.

4. FUZZY QUANTITIES ISOTONIC WITH NATURAL NUMBERS

This section is subjected to a special but in practical applications frequent type of fuzzy quantities with sparse sets of possible values. Namely the fuzzy quantities are considered here whose sets of possible values are isotonic with some subset of the set of natural numbers. It means that the sets of values with positive membership function are sparse and there exist their minimal elements.

Definition 6. Let $\boldsymbol{a} \in \mathbf{Q}$ be a fuzzy quantity. We say that $\boldsymbol{a}$ is *isotonic with natural numbers* iff there exists a sequence

$$\{r_n\}_{n \in M}, \quad M \subset \mathbb{N}, \quad r_n \in \mathbf{Q} \quad \text{for} \quad n \in M, \quad r_n < r_m \quad \text{iff} \quad n < m,$$

and

$$\{i \in \mathbb{Q} : f_a(i) > 0\} = \{r_n\}_{n \in M}.$$

Remark 5. If the set $\{i \in \mathbb{Q} : f_a(i) > 0\}$ is finite then $\boldsymbol{a}$ and $-\boldsymbol{a}$ are isotonic with natural numbers.

Remark 6. If $s \in \mathbf{S}$ is isotonic with natural numbers then the set $\{i \in \mathbb{Q}: f_s(i) > 0\}$ is finite and there exist its minimal and maximal elements.

Lemma 9. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$ are isotonic with natural numbers then

$$\{i \in \mathbb{Q}: f_{a+b}(i) > 0\} = \{k \in \mathbb{Q}: \exists j \in \mathbb{Q}, f_a(j) > 0 \wedge f_b(k-j) > 0\}$$

and this set is well ordered.

Proof. Let us denote

$$\{i \in \mathbb{Q}: f_a(i) > 0\} = \{r_k\}_{k \in K}, \quad K \subset \mathbb{N},$$
$$\{i \in \mathbb{Q}: f_b(i) > 0\} = \{s_l\}_{l \in L}, \quad L \subset \mathbb{N}.$$

If $i \in \mathbb{Q}$ is such that $i = r_k + s_l$ for some $k \in K$, $l \in L$ then

$$f_{a+b}(i) = \sum_{n \in K} f_a(r_n) \cdot f_b(i - r_n) \geqq f_a(r_k) \cdot f_b(s_l) > 0\,.$$

If $i \in \mathbb{Q}$, $i \neq r_k + s_l$ for any $k \in K$, $l \in L$ then for every product $f_a(r_n) \cdot f_b(i - r_n)$, $n \in K$, $f_b(i - r_n) = 0$ and consequently $f_{a+b}(i) = 0$. As r_1 and s_1 are minimal elements of $\{r_k\}_{k \in K}$ and $\{s_l\}_{l \in L}$, respectively, then $k_0 = r_1 + s_1$ is the minimal element of $\{i \in \mathbb{Q}: f_{a+b}(i) > 0\}$. □

Lemma 10. Let $\boldsymbol{a} \in \mathbf{Q}$ and $\boldsymbol{s} \in \mathbf{S}$ be fuzzy quantities isotonic with natural numbers. Let $\boldsymbol{a} + \boldsymbol{s}$ be symmetric. Then $\boldsymbol{a}$ is symmetric, too; $\boldsymbol{a} \in \mathbf{S}$.

Proof. Let $\boldsymbol{a} \in \mathbf{Q}$, $\boldsymbol{s} \in \mathbf{S}$, and let $\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{s} \in \mathbf{S}$. Let us denote

$$\{i \in \mathbb{Q}: f_a(i) > 0\} = \{r_k\}_{k \in K}, \quad K \subset \mathbb{N},$$
$$\{i \in \mathbb{Q}: f_s(i) > 0\} = \{s_l\}_{l \in L}, \quad L \subset \mathbb{N}.$$

Then

$$f_r(m) = \sum_i f_a(i) \cdot f_s(m - i)\,,$$
$$f_r(-m) = \sum_j f_a(j) \cdot f_s(-m - j) = \sum_j f_a(j) \cdot f_s(m + j) =$$
$$= \sum_i f_a(-i) \cdot f_s(m - i)\,,$$

where all summations are considered over the set $\mathbb{Q}$ and where $i = -j$ was substituted. As $\boldsymbol{r} \in \mathbf{S}$, $f_r(m) = f_r(-m)$ and

$$0 = f_r(m) - f_r(-m) = \sum_i f_s(m - i) \cdot [f_a(i) - f_a(-i)] =$$
$$= \sum_{k \in K} f_s(m - r_k) \cdot [f_a(r_k) - f_a(-r_k)]\,,$$

for all $m \in \mathbb{Q}$. Let us choose $m = r_1 + s_1 \in \mathbb{Q}$. Then

$$0 = \sum_{k \in K} f_s(m - r_k) \cdot [f_a(r_k) - f_a(-r_k)] = f_s(s_1) \cdot [f_a(r_1) - f_a(-r_1)]\,.$$

As $f_s(s_1) > 0$ then $f_a(r_1) = f_a(-r_1)$. Let us suppose that $f_a(r_k) = f_a(-r_k)$ for $k = 1, 2, \ldots, n$, and let $m = r_{n+1} + s_1$. Then

$$0 = \sum_{k \in K} f_s(m - r_k) \cdot [f_a(r_k) - f_a(-r_k)] =$$

$$= f_s(s_1) \cdot [f_a(r_{n+1}) - f_a(-r_{n+1})] +$$
$$+ \sum_{k=1}^{n} f_s(s_1 + r_{n+1} - r_k) \cdot [f_a(r_k) - f_a(-r_k)] =$$
$$= f_s(s_1) \cdot [f_a(r_{n+1}) - f_a(-r_{n+1})] ,$$
and consequently $f_a(r_{n+1}) = f_a(-r_{n+1})$. It means that $f_a(r_k) = f_a(-r_k)$ for all $k \in K$, and $\boldsymbol{a} \in \mathbf{S}$. □

Corollary. If we denote by $\mathbf{Q}^* \subset \mathbf{Q}$ the set of all fuzzy quantities isotonic with natural numbers then any $\boldsymbol{a} \in \mathbf{Q}^*$ is equivalent to the zero element, $\boldsymbol{a} \sim \boldsymbol{o}$, if and only if $\boldsymbol{a}$ is symmetric. The set $\mathbf{S} \cap \mathbf{Q}^*$ is then one of the equivalence classes from the partition of $\mathbf{Q}^*$ created by the equivalence relation $\sim$.

Remark 7. If the assumptions of Lemma 10 are fulfilled then the set $\{i \in \mathbb{Q} : f_a(i) > 0\}$ is finite as follows from Lemma 10 and Remark 6.

Theorem 7. If $\boldsymbol{a} \in \mathbf{Q}$ and $\boldsymbol{b} \in \mathbf{Q}$ are fuzzy quantities isotonic with natural numbers then $\boldsymbol{a} \sim \boldsymbol{b}$ if and only if $\boldsymbol{a} + (-\boldsymbol{b}) \in \mathbf{S}$.

Proof. If $\boldsymbol{a} + (-\boldsymbol{b})$ is symmetric then $\boldsymbol{a} \sim \boldsymbol{b}$ as follows from Theorem 3. On the other hand, if $\boldsymbol{a} \sim \boldsymbol{b}$ then there exist $s_1, s_2 \in \mathbf{S}$ such that $\boldsymbol{a} + s_1 = \boldsymbol{b} + s_2$. It means that
$$\boldsymbol{a} + s_1 + (-\boldsymbol{b}) = \boldsymbol{b} + s_2 + (-\boldsymbol{b})$$
and consequently $\boldsymbol{a} + (-\boldsymbol{b}) + s_1 = s \in \mathbf{S}$ where $s = \boldsymbol{b} + (-\boldsymbol{b}) + s_2$. Lemma 9 implies that $\boldsymbol{a} + (-\boldsymbol{b}) \in \mathbf{S}$. □

5. DETERMINISTIC MULTIPLICATION

In this section we briefly mention the operation of multiplication of a fuzzy quantity from $\mathbf{Q}$ by a given rational number from $\mathbb{Q}$. This multiplication, together with the addition (1), offers sufficients tools for the introduction of some types of linear combinations of fuzzy quantities. It is useful in some applications of the fuzzy quantities theory.

Definition 7. Let $\boldsymbol{a} \in \mathbf{Q}$ be a fuzzy quantity with rational values, let $\boldsymbol{o} \in \mathbf{Q}$ be the zero element of $\mathbf{Q}$ and $r \in \mathbb{Q}$ be a rational number. Then the fuzzy quantity $r \,.\, \boldsymbol{a} \in \mathbf{Q}$ such that for all $i \in \mathbb{Q}$
$$\begin{aligned} f_{r.a}(i) &= f_a(i/r) \quad \text{if} \quad r \neq 0 \\ &= f_o(i) \quad\quad \text{if} \quad r = 0 \end{aligned}$$
is called the *product* by r of the fuzzy quantity $\boldsymbol{a}$.

Remark 8. If $\boldsymbol{a} \in \mathbf{Q}$, $s \in \mathbf{S}$ and $r, t \in \mathbb{Q}$ then evidently $r \,.\, \boldsymbol{o} = \boldsymbol{o}$, $r \,.\, s \in \mathbf{S}$, $r \,.\, (t \,.\, \boldsymbol{a}) = (r \,.\, t) \,.\, \boldsymbol{a}$, $r \,.\, (1/r) \,.\, \boldsymbol{a} = \boldsymbol{a}$, $r \,.\, (-\boldsymbol{a}) = (-r) \,.\, \boldsymbol{a} = -(r \,.\, \boldsymbol{a})$.

Theorem 8. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{Q}$, $r \in \mathbb{Q}$ then $r \,.\, (\boldsymbol{a} + \boldsymbol{b}) = r \,.\, \boldsymbol{a} + r \,.\, \boldsymbol{b}$.

Proof. If $r \neq 0$ then for any $k \in \mathbb{Q}$

$$f_{r.(a+b)}(k) = f_{a+b}(k/r) = \sum_j f_a(j) \cdot f_b((k/r) - j) =$$
$$= \sum_i f_a(i/r) \cdot f_b((k - i)/r) = \sum_i f_{r.a}(i) \cdot f_{r.b}(k - i) = f_{r.a+r.b}(k),$$

where $i = r, j$ was substituted, and the summations are considered over the set $\mathbb{Q}$. If $r = 0$ then $r . (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{o} = \boldsymbol{o} + \boldsymbol{o} = r . \boldsymbol{a} + r . \boldsymbol{b}$. □

Lemma 11. If $\boldsymbol{a} \in \mathbf{Q}$ is isotonic with natural numbers and $r \in \mathbb{Q}$, $r > 0$, then $r . \boldsymbol{a}$ is also isotonic with natural numbers.

Proof. If $\{i \in \mathbb{Q}: f_a(i) > 0\} = \{r_k\}_{k \in K}$, $K \subset \mathbb{N}$, then $f_{r.a}(i) > 0$ iff $i = (r_k/r)$ for some r_k, $k \in K$, and $f_{r.a}(i) = f_a(i/r)$ in such a case. As $r > 0$, $r_k/r < r_n/r$ iff $k < n$ and

$$\{i \in \mathbb{Q}: f_{r.a}(i) > 0\} = \{r_k/r\}_{k \in K} = \{s_k\}_{k \in K}, \quad K \subset \mathbb{N}. \qquad \square$$

It is easy to verify that the dual distributivity rule, i.e. $(r + s) . \boldsymbol{a} = r . \boldsymbol{a} + s . \boldsymbol{a}$ for $r, s \in \mathbb{Q}$, $\boldsymbol{a} \in \mathbf{Q}$, is not generally fulfilled as follows from the next example.

Example. Let $\boldsymbol{a} \in \mathbf{Q}$ and let $f_a(1) = 0{\cdot}5$, $f_a(2) = 0{\cdot}5$, $f_a(i) = 0$ for $i \in \mathbb{Q}$, $i \neq 1, 2$. Then $\boldsymbol{a} + \boldsymbol{a} \in \mathbf{Q}$ and

$$f_{a+a}(2) = 0{\cdot}25 = f_{a+a}(4), \quad f_{a+a}(3) = 0{\cdot}5, \quad f_{a+a}(i) = 0 \quad \text{for} \quad i \neq 2, 3, 4.$$

If we put $r = 1$, $s = 1$, $r + s = 2$ then

$$f_{2.a}(i) = 0 \quad \text{for} \quad k \neq 2, 4, \quad f_{2.a}(2) = 0{\cdot}5 = f_{2.a}(4).$$

It means that $\boldsymbol{a} + \boldsymbol{a} \neq 2 . \boldsymbol{a}$.

6. CONCLUSIVE REMARKS

The possibility to perform some algebraical operations over fuzzy quantities shown above offers more occasions to apply the concept of the fuzzy quantities to a wide scale of problems. Some of those applications were suggested in [6], [7] and [8], some others can be found in the branches in which vague numerical data are handled. The results presented above can be very easily generalized also to the summation and deterministic multiplication of more-dimensional fuzzy quantities with rational values, as well as to their applications.

However, there are still some open problems and topics for discussion connected with the theory presented above. In this conclusive section we briefly mention a few of them.

The first discrepancy of the suggested method is the possibility of the membership function to take values $f_a(i)$ greater than 1, and, moreover, practically unlimited. Namely the summs of fuzzy quantities with large support sets $\{i \in \mathbb{Q}: f_a(i) > 0\}$ and with sufficiently high values $f_a(i)$ can be described by unpleasantly unlimited membership functions. This values can be interpreted in the same way as $f_a(i) = 1$

and in this sense accepted, as it was done above, or they can be cut off by means of the minimum operation $\min(f_{\boldsymbol{a}}(i), 1)$ as done in [4]. None of these two approaches to the large values of the membership functions is satisfactory.

In fact, the possibility of the large values of membership functions follows from the convolutionary concept of the addition formulated in Definition 1. The definition does too much imitate the probabilistic approach to the uncertainty which is not always adequate to the concept of fuzziness. It is possible to define the addition over fuzzy quantities in another way, namely to put for every $k \in \mathbb{Q}$.

$$f_{\boldsymbol{a}+\boldsymbol{b}}(k) = \sup_i (\min (f_{\boldsymbol{a}}(i), f_{\boldsymbol{b}}(k - i))) , \tag{4}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are fuzzy quantities with membership functions $f_{\boldsymbol{a}}$ and $f_{\boldsymbol{b}}$, respectively, and where the supremum is considered over the whole set $\mathbb{Q}$. The algebraic properties of such addition are investigated in paper [11] submitted for publication. Addition defined by (4) better corresponds with the concept of fuzziness as a representation of uncertainty. On the other hand, its analytical properties are less convenient for the theoretical and practical handling such operation. More details are presented in[11].

Other difficulties appear if we try to combine the addition and multiplication of fuzzy quantities. Some of them were shown in [5], others can be easily deduced from the problems solved, for the operation of addition, in this paper and in [11]. Even the multiplication by deterministic rational values is not without difficulties. Example presented in Section 5 shows that the multiplication introduced by Definition 7 does not fulfil all linearity conditions. It is possible to define the multiplication of a fuzzy quantity $\boldsymbol{a} \in \mathbf{Q}$ by a rational number $r \in \mathbb{Q}$ in another way. Namely, if $r = p/q$ where p and q are integers, and $p > 0$ then for $k \in \mathbb{Q}$

$$f_{r.\boldsymbol{a}}(k) = f_{\underbrace{\boldsymbol{a}+\dots+\boldsymbol{a}}_{p\text{-times}}}(k \,.\, q) \,.$$

It is not difficult to prove that $r \,.\, (\boldsymbol{a} + \boldsymbol{b}) = r \,.\, \boldsymbol{a} + r \,.\, \boldsymbol{b}$ and that $(r + s) \,.\, \boldsymbol{a} = r \,.\, \boldsymbol{a} + s \,.\, \boldsymbol{a}$, where $r = m/q$, $s = n/q$ and m, n, q are integers. This conception of the deterministic product, besides its rather strange structure, has other discrepancies. For example, it is not difficult to verify that generally

$$(p/q) \,.\, \boldsymbol{a} \neq ((m \,.\, p)/(m \,.\, q)) \,.\, \boldsymbol{a}$$

for integers p, q, m and for $\boldsymbol{a} \in \mathbf{Q}$.

All the mentioned problems illustrate that some algebraical properties of fuzzy quantities are complex and their deduction is not easy. Nevertheless, even the results presented above and their analogies in [11] offer a few effective tools suitable for mathematical models of situations in which vague numerical data appear. Considering the fact that the sets of possible values of many fuzzy quantities in real applications are either finite or isotonic with the set of natural numbers, the results presented above can be applied to many interesting problems.

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