Kybernetika

THE MULTIDIMENSIONAL z-TRANSFORM AND ITS USE IN SOLUTION OF PARTIAL DIFFERENCE EQUATIONS

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1. INTRODUCTION

Functional transforms in linear spaces are often introduced to serve as a tool in solution of linear functional equations. The type of the transform depends on the type of equation and on the set of its solutions. Commonly, the use of transform methods assumes an established theory of the equation itself, i.e., existence and uniqueness theorems and further properties of solutions are supposed to be known.

On this basis we want to give a survey of (generalized) multidimensional z-transform (n-D-z-transform) method and its use in solution of linear partial difference equations with constant coefficients, as well as systems of such equations, whose solutions will be called sequences. This theory is aimed at forming a basis of multidimensional digital system theory, which attracted wide and increasing attention in the last decades.

This effort might seem to be superfluous. In some respects we can only repeat known results; however, some relevant formulations in basic monographs seem to be open to misinterpretations, the theory of digital n-D systems itself is stumbling over some unexpected difficulties when generalizing some "obvious" results, and a large part of the n-D theory, although emphasizing the fundamental distinctions between the classical one-dimensional theory and the n-D case, is modelled as the transform of one sided and one-dimensional sequences.

1.1. Notations

The following notation will be used: $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{C}$ will stand for all integers, nonnegative integers, real and complex numbers, respectively; $\mathbb{Z}^n$ is the set of n-tuples of integers and similarly $\mathbb{Z}^n_+$, $\mathbb{R}^n$, $\mathbb{C}^n$ are sets of n-tuples; $\alpha, \beta, \ldots$ are elements of $\mathbb{Z}^n$ $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{Z}$; A, B are subsets of $\mathbb{Z}^n$ (or $\mathbb{C}^n$). For $A \subset \mathbb{Z}^n$, $\beta \in \mathbb{Z}^n$, $A + \beta$ will be the set of all γ such that $\gamma = \alpha + \beta$, $\alpha \in A$ and similarly $A + B = \{\gamma : \gamma = \alpha + \beta, \alpha \in A, \beta \in B\}$; so $A + \beta$ is the "shifted" set A, while $A \cup \{\beta\}$ is the union of sets A and $\{\beta\}$. We use $A \setminus B$ for the set of elements of A which do not belong to B. For $\lambda \in \mathbb{C}$ we shall use $\boldsymbol{\lambda} = (\lambda, \lambda, \ldots, \lambda) \in \mathbb{C}^n$ and $e_k \in \mathbb{Z}^n$ will be the n-tuple $e_k = (e_{k1}, \ldots, e_{kn})$, where $e_{kj} = 0$ for $k \neq j$, $e_{kj} = 1$ for $k = j$. Capitals $F, G, \ldots$ will also denote mappings $F: D \to \mathbb{C}$, $D \subset \mathbb{C}^n$, while f, g, x, y will be used for mappings as in $f: A \to \mathbb{C}$, $A \subset \mathbb{Z}^n$ with one common exception: $\delta^\beta: \mathbb{Z}^n \to \mathbb{C}$ such that its values $\delta^\beta(\alpha) = \begin{cases} 0 \text{ for } \alpha \neq \beta \\ 1 \text{ for } \alpha = \beta \end{cases}$. These mappings from $\mathbb{Z}^n \supset A \to \mathbb{C}$ will also be called sequences. Beside the delta sequence δ^β the sequence $f: \mathbb{Z}^n \to \mathbb{C}$ with $f(\alpha) = \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \ldots \lambda_n^{\alpha_n}$, shortly denoted as λ^α with $\lambda \in \mathbb{C}^n$ being a fixed n-tuple of complex numbers, is often used. This shortened notation will further be generalized as follows: If $\lambda \in \mathbb{C}^n$ and U is an $n \times n$ matrix of integers with its rows denoted by U_i, $i = 1, 2, \ldots, n$, we shall denote the n-tuple λ to the matrix power $U = [u_{ik}]$ as

$$\lambda^U = (\lambda^{U_1}, \lambda^{U_2}, \ldots, \lambda^{U_n}) \in \mathbb{C}^n$$

Hence, $(\lambda^U)^\alpha = \lambda^a$ with $a = (a_1, a_2, \ldots, a_n)$, $a_k = \sum_{i=1}^{n} u_{ik}\alpha_i$

$$(\lambda^U)^\alpha = \lambda^{\alpha U} .$$

This formalism preserves the common rules of operations with powers as far as they are meaningfull, e.g., for $\lambda \in {}^\bullet\mathbb{C}^n = \{\lambda \in \mathbb{C}^n, \lambda_i \neq 0, i = 1, 2, \ldots, n\}$ we obtain

$$(\lambda^U)^{U^{-1}} = \lambda^I = (\lambda_1, \lambda_2, \ldots, \lambda_n) .$$

Sequences $f\colon A \to \mathbb{C}$, $A \subset \mathbb{Z}^n$, form a linear space S_A. Normed linear subspaces of S_A can be considered, e.g. the space l_2 with a norm $\|\cdot\|$ defined by

$$\|f\|^2 = \sum_{\alpha \in A} |f(\alpha)|^2 .$$

In the sequel we shall often deal with various linear operators defined on subsets of S_A and with values in S_A. In the set S of sequences $f\colon \mathbb{Z}^n \to \mathbb{C}$ convolution and shifted sequence are commonly defined as follows:

If f, $g \in S$, then $h = f * g$, defined by $h(\alpha) = \sum_\beta f(\alpha - \beta)\, g(\beta)$ where $\beta \in \mathbb{Z}^n$, is called their convolution, provided the sum defining $h(\alpha)$ converges for every $\alpha \in \mathbb{Z}^n$. The convolution so defined obeys the following rules

$$f * g = g * f , \tag{1.1}$$
$$(\lambda f) * g = f * (\lambda g) = \lambda(f * g) , \quad \lambda \in \mathbb{C} ,$$
$$f * (g + h) = f * g + f * h .$$

The sequence $f_\beta = \delta^{-\beta} * f$ is often called "shifted"; there is $f_\beta(\alpha) = f(\alpha + \beta)$ and evidently

$$f = \delta^\beta * (\delta^{-\beta} * f) = \delta^{-\beta} * (\delta^\beta * f) .$$

We are tempted to introduce a linear operator $L = \sum_{\beta \in B} a_\beta \delta^{-\beta} *$ [1]) with B a finite set so as to consider the equation $Lf = x$, i.e.

$$\sum_{\beta \in B} a_\beta f(\alpha + \beta) = x(\alpha) , \quad \alpha \in A \in \mathbb{Z}^n \tag{1.2}$$

as an input/output relation of a digital system.

A Mikuszinski type axiomatic approach to functional transform, used in solving equation (1.2), could result in the transform relation $\delta^\beta \rightleftharpoons z^\beta$ with $z \in \mathbb{C}^n$ and, since for every $f \in S$ there is

$$f(\cdot) = \sum_\alpha \delta^\alpha(\cdot) f(\alpha) , \quad \alpha \in \mathbb{Z}^n ,$$

we could obtain

$$f \rightleftharpoons \sum_\alpha f(\alpha)\, z^\alpha , \quad \alpha \in \mathbb{Z}^n . \tag{1.3}$$

Although this often is the way how *n-D* *z*-transform is introduced in handling

[1]) [13] calls L a partial difference operator with constant coefficients in the special case $n = 2$.

input/output relations (1.2) it has some crucial drawbacks. Equation (1.2) evidently has an infinite number of solutions. Indeed, with every solution φ defined on the set $A + B = \{\gamma = \alpha + \beta, \alpha \in A, \beta \in B\}$ the function φ_λ with $\varphi_\lambda(\alpha) = \varphi(\alpha) + \lambda^\alpha$ is also its solution, provided λ is any solution of equation $\sum a_\beta \lambda^\beta = 0$, $\lambda = (\lambda_{,1}\, \lambda_2, \ldots \ldots, \lambda_n)$, $\lambda_i \in {}^\bullet\mathbb{C}^n$. Applying the theorem as in (1.3) we could perhaps obtain one of these solutions, but certainly no additional conditions (initial, boundary, e.t.c.) can this way be respected. "Operationally speaking" the algebra of operators defined by $\delta^\beta \rightleftharpoons z^\beta$ has nontrivial divisors of zero. This is in most cases (e.g. in system theory) unacceptable.

Alternative definitions of functional transforms aimed at solution of equation(1.3), if such exist, must be able to respect uniqueness conditions (e.g. initial conditions) for the solution of (1.2). It remains to be clarified, under which conditions such functional transform, commonly called z-transform, can be defined so as its use could be justified in multidimensional cases as it has been for some special one-dimensional problems.

1.2. Reinhardt domains

To define the n-D z-transform we need some preliminary results from function theory and algebra.

Definition 1.2.1. A nonempty open set $D \subset \mathbb{C}^n$ is called a Reinhardt domain if $z = (z_1, z_2, \ldots, z_n) \in D$ implies $(z_1\, e^{j\varphi_1},\ z_2\, e^{j\varphi_2} \ldots, z_n\, e^{j\varphi_n}) \in D$ for every n-tuple $(\varphi_1, \varphi_2, \ldots, \varphi_n) \in \mathbb{R}^n$.

A Reinhardt domain D is called complete if $z^0 = (z_1^0, z_2^0, \ldots, z_n^0) \in D$ implies $z = (z_1, \ldots, z_n) \in D$ for all $|z_k| < |z_k^0|$.

A Reinhardt domain $D \subset \mathbb{C}^n$ is called relatively complete (RCRD) if it is either complete, or all its nonempty intersections with ${}^\circ\mathbb{C}_k^n$, $k = 1, 2, \ldots, n$, are relatively complete (in $\mathbb{C}^{n-1}$). Here, ${}^\circ\mathbb{C}_k^n = \{z\colon z \in \mathbb{C}^n, z_k = 0\}$ and for $n = 1$, D is called relatively complete either if it is complete or if $D = \{z\colon z \in \mathbb{C},\ 0 \leqq b < |z| < a \leqq \leqq +\infty\}$.

Any Reinhardt domain can be wholely characterized by the mapping $\lambda\colon \mathbb{C}^n \to \mathbb{R}^n$ defined as follows

$$\lambda(z) = (|z_1), |z_2|, \ldots, |z_n|) \in \mathbb{R}^n$$

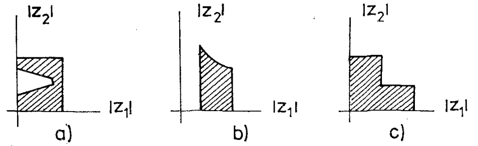

Fig. 1.

The image of an open, connected set $D \subset \mathbb{Z}^n$ in mapping λ is an open and connected set in $\mathbb{R}^n_+$. This set is called the Reinhardt diagram.

In Fig. 1. a), b), c) the Reinhardt diagrams of a noncomplete, relatively complete and complete Reinhardt domain are shown for $n = 2$.

The domain in Fig. 1 b) is not complete and that of 1 a) is not relatively complete. It is easy and useful to recall the Reinhardt diagrams for $n = 1$.

Reinhardt domains are important as convergence domains of power series in $\mathbb{C}^n$.

Lemma 1.2.2. Let P be a nontrivial polynomial in n-variables. Then there exists a relatively complete Reinhardt domain D such that $P(z) \neq 0$ for all $z \in D$.

Proof. For $n = 1$ the statement evidently holds true, In general, let $P(z) = z_n^k \sum_{i=0}^{m} P_i(z')\, z_n^i$, where $z' = (z_1, z_2, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}$, P_i are polynomials of $(n-1)$-variables, we can state: there exists an RCRD, say $B \subset \mathbb{C}^{n-1}$ such that $\varepsilon_1 < |P_0(z')| < \varepsilon_2$ for all $z' \in B$. To any ε_1 and all z_n with $|z_n| < \varepsilon$ there is

$$\Big|\sum_{i=0}^{m} P_i(z')\, z_n^i\Big| < \varepsilon_1 < |P_0(z')| \quad \text{for all} \quad z' \in B$$

and therefore $\sum_{i=0}^{m} P_i(z')\, z_n^i \neq 0$ on the set $B \times \{|z_n| < \varepsilon\}$. Finally we obtain $P(z) \neq 0$ for all $z \in B \times \{0 < |z_n| < \varepsilon\}$, which is a RCRD in $\mathbb{C}^n$. □

1.3. Semigroups in $\mathbb{Z}^n$

The set $\mathbb{Z}^n$ is an Abelian group with addition as its group operation and with $\mathbf{0}$ as its neutral element. In what follows we shall often use some special Abelian subsemigroups of $\mathbb{Z}^n$.

A semigroup A will be called ordered, when it is endowed with a reflexive, transitive and antisymmetric binary relation $\leqq$, called order, so that for every $\gamma \in A$ the following implication holds

$$\alpha, \beta \in A\,, \quad \alpha \leqq \beta \Rightarrow \alpha + \gamma \leqq \beta + \gamma\,.$$

A subsemigroup K of $\mathbb{Z}^n$ containing $\mathbf{0}$ (its neutral element) is a monoid.

Let Q be an $n \times n$ nonsingular matrix of reals. The set $K_Q = \{\alpha \in \mathbb{Z}^n, \alpha Q = (\beta_1, \beta_2, \ldots, \beta_n), \beta_i \geqq 0\}$ is a monoid and, since Q is nonsingular, $K_Q \cap (-K_Q) = \{\mathbf{0}\}$. Moreover, for any positive integer λ and $\alpha \in K_Q$ there is $\lambda\alpha \in K_Q$; therefore K_Q can be called a cone. If the matrix Q consists of integers only, the set K_Q will be called a rational cone. Rational cones stem from special linear transforms of the first n-tant of $\mathbb{Z}^n$.

Linear transforms of $\mathbb{Z}^n$ onto itself will be considered. Any of these transforms can be characterized by a transform matrix U with integer elements and with det $U = 1$. The set $A \subset \mathbb{Z}^n$ is transformed onto a set A' by such matrix U, when $A' = \{\alpha' = \alpha U$

for every $\alpha \in A\}$. By such transform the order relation in A can be carried over to the set A' in an obvious manner. The set of matrices U has a canonical structure: it consists of a finite number of some special matrices. Let E_{kl}, $k \neq l$, be a square matrix of order n with elements $e_{ij} = \begin{cases} 0 & \text{for } i \neq k \text{ or } j \neq l \\ 1 & \text{for } i = k, j = l \end{cases}$. Then $U_{kl} = I + E_{kl}$ is a transform matrix and $I - E_{kl} = U_{kl}^{-1}$. The following theorem has been proved in [9].

A square matrix U of order n has integer elements and $\det U = 1$ if and only if U is a product of matrices U_{kl} and matrices U_{st}^{-1} with $1 \leqq k, l, s, t \leqq n$.

These transforms defined by a matrix U "preserve cones" in the following sense: Let K be a cone, let UK be the set of αU, $\alpha \in K$ and U the above described transform. Then UK is a cone; similarly, if K is a rational cone, UK is also a rational cone.

2. THE n-D z-TRANSFORM

2.1. Basic theorems

Definition 2.1.1. Let f be a sequence with values in $\mathbb{C}$, defined on the set $A \subset \mathbb{Z}^n$. Then the function F: $D \to \mathbb{C}$, $D \subset \mathbb{C}^n$ defined by

$$F(z) = \sum_{\alpha \in A} f(\alpha)\, z^\alpha\,, \quad z \in D \tag{2.1}$$

is called its z_A-transform, provided the series converges on the relatively complete Reinhardt domain D.

We shall write this correspondence as

$$f_A \rightleftharpoons F_D\; (\text{or } f_A(\alpha) \rightleftharpoons F_D(z))\,.$$

This definition has some obvious corollaries.

Theorem 2.1.2 (analyticity). The z-transform F_D of a sequence, if it exists, is analytic on the set D. If D is a CRD, then F_D is analytic on the logarithmically convex hull of D.

This theorem follows from the general theory of power series in $\mathbb{C}^n$; its proof can be found in textbooks on function theory, e.g. [10] p. 45.

Evidently, if the z_A-transform of a sequence f_A exists, then the $z_{A'}$-transform of its restriction $f_{A'}$ to the set $A' \subset A$ also exists. Further,

Theorem 2.1.3 (linearity). If $f_{A'} \rightleftharpoons F_{D'}$, $g_{A''} \rightleftharpoons G_{D''}$ and the sets $A = A' \cap A''$, $D = D' \cap D''$ are both nonempty, then for any constants $\lambda, \mu \in \mathbb{C}$ there is

$$(\lambda f + \mu g)_A \rightleftharpoons (\lambda F + \mu G)_D\,.$$

The proof is obvious.

Since the set D is indirectly determined by the set A, we can also make use of the notation: $F_D = Z(f_A)$ and formulate the following

Theorem 2.1.4 (additivity). If the domains of convergence D and D' of $Z(f_A)$ and $Z(f_B)$, respectively, have nonempty intersection, then (as far as the left-hand side exists)

$$Z(f_{A\cup B}) = Z(f_A) + Z(f_B) - Z(f_{A\cap B}) .$$

The proof is again almost self-evident; only the fact that the intersection of two relatively complete Reinhardt domains is either empty or again a relatively complete Reinhardt domain is to be taken into account.

Theorem 2.1.5 (inverse transform). Let be $f_A \rightleftharpoons F_D$. Then for all $\alpha \in A$

$$f_A(\alpha) = \frac{1}{(2\pi \mathrm{j})^n} \int_\Gamma F(z)\, z^{-\alpha-\mathbf{1}}\, \mathrm{d}z ,$$

where Γ is an n-tuple of positively oriented circles, all points of which are contained in the set $D \subset \mathbb{Z}^n$.

Remark. Here and further, the integral over Γ is to be understood as follows

$$\frac{1}{(2\pi \mathrm{j})^n} \int_\Gamma X(z)\, \mathrm{d}z =$$

$$= \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} X(r_1\, \mathrm{e}^{\mathrm{j}\varphi_1}, \dots r_n\, \mathrm{e}^{\mathrm{j}\varphi_n})\, r_1, r_2 \dots r_n\, \mathrm{e}^{\mathrm{j}(\varphi_1 + \dots + \varphi_n)}\, \mathrm{d}\varphi_1 \dots \mathrm{d}\varphi_n ,$$

where r_i are the radii of the circles, Γ consists of.

The proof is similar to that of the corresponding theorem of Laurent series in one dimensional function theory. Its details can be found e.g. in [10].

Strictly speaking, the z-transform as given in Definition 2.1.1 and its inverse is a one-to-one correspondence between pairs (A, f_A) and (D, F_D). While the set A uniquely determines D by Definition 2.1.1, Theorem 2.1.5 yields $\mathbb{Z}^n$ instead of A for the first member of the pair (A, f_A), since the corresponding formula can be applied for all $\alpha \in \mathbb{Z}^n$. This is quite common in inverse formulae of functional transforms: we obtain from them classes of equivalence between f_A and F_D. Here A has to be understood as the intersection of all sets A_k with $(A_k, f_{A_k}) \rightleftharpoons (D, F_D)$; this ends up with the set A consisting of all α such that the integral in Theorem 2.1.5 is nonzero.

To derive a theorem on z-transform of transformed sequences, we shall use the extended notation convention from Section 1.1. For any integer square matrix U we may denote $z = w^U$ and we obtain $w = (z^U)^{-1}$. Further, $z^\alpha = w^{\alpha U}$ supposing as before that α is a row-vector.

Theorem 2.1.6. (On linear mapping in the space domain). Let be $F_D(z) = Z(f_A)$ with A the first n-tant, i.e. $A = \{(\alpha_1, \dots, \alpha_n): \alpha_i \geqq 0, i = 1, 2, \dots, n\}$. Let further U be an $n \times n$ square matrix of integers such that $\det U = 1$. Then the function G with $G(w) = F(w^U)$ is the z-transform of the sequence g defined as $g(\alpha') = f(\alpha' U^{-1})$ for all $\alpha' \in K$, where $K = AU$. The convergence region of this transform is contained in $D^{U^{-1}}$.

Proof. Denote $\alpha' = \alpha U$ and in formula $F(z) = \sum_{\alpha \in A} f(\alpha) z^{\alpha}$ perform the substitution $\alpha = \alpha' U^{-1}$ as shown above. The rest becomes obvious, when using the rules on z^U as described in Section 1.1. (In this theorem and proof some almost selfevident notational conventions have been used.) □

2.2. Some examples

We want to show here by examples that our Definition 2.1.1 is a meaningfull generalization of the commonly accepted 1-D z-transform and that this generalization is not trivial.

Example 2.2.1. Take a constant $\lambda \in {}^{\circ}\mathbb{C}^n$, ${}^{\circ}\mathbb{C}^n = \{z \in \mathbb{C}^n : z_k \neq 0,\ k = 1, 2, \ldots, n\}$. Then $f(\alpha) = \lambda^{\alpha}$ is a sequence defined everywhere in $\mathbb{Z}^n$. Let its z_A-transform be considered for various sets $A \subset \mathbb{Z}^n$.

i) If $n = 1$, without loss of generality we may assume e.g. $A_0 = \{k : k \geqq 0\}$, $A_1 = \{k : k < 0\}$. Further, $Z_{A_0}(f) = 1/(1 - \lambda z)$ for all $|z| < 1/|\lambda|$, $Z_{A_1}(f) = 1/(\lambda z - 1)$ for all $|z| > 1/|\lambda|$. The corresponding convergence domains D_0, D_1 are disjoint, the z-transform of the "whole" sequence does not exist.

ii) For $n > 1$ the situation is similar only for the case $A_0 = \{\alpha : \alpha_i \geqq 0,\ i = 1, 2, \ldots, n\}$:

$$Z_{A_0}(f) = \prod_{i=1}^{n} \frac{1}{(1 - \lambda_i z_i)}$$

with $D = \{z \in \mathbb{C}^n : |z_i| < 1/|\lambda_i|\}$.

iii) For $n = 2$ and $A' = \{(i, k),\ i \geqq 0,\ |k| \leqq i\}$ after some rather tedious calculations we obtain

$$F_1(z) = \frac{(1 + \lambda_1 z_1)\, \lambda_2 z_2}{(1 - \lambda_1 \lambda_2 z_1 z_2)(\lambda_2 z_2 - \lambda_1 z_1)} \quad \text{for} \quad |\lambda_1 \lambda_2 z_1 z_2| < 1$$

and $|\lambda_1 z_1| < |\lambda_2 z_2|$. From the Reinhardt diagram of this region we reveal that the convergence region is a relatively complete Reinhardt domain.

Similarly, for $A'' = \{(i, k) : i \geqq 0,\ k \geqq -i\}$ we obtain

$$F_2(z) = \frac{\lambda_2 z_2}{(1 - \lambda_2 z_2)(\lambda_2 z_2 - \lambda_1 z_1)}$$

for $|\lambda_1 z_1| < 1$, $|\lambda_2 z_2| < 1$, $|\lambda_1 z_1| < |\lambda_2 z_2|$. The Reinhardt diagrams are open domains, they are given in Fig. 2a), b). Note that $A' \subset A''$ and for the corresponding convergence regions $D'' \subset D'$.

Taking now $A''' = A'' \setminus A'$, i.e. $A''' \cup A' = A''$ we obtain

$$F_3(z) = \sum_{\alpha \in A'''} \lambda^{\alpha} z^{\alpha} = \frac{\lambda_2 z_2}{(1 - \lambda_2 z_2)(1 - \lambda_1 \lambda_2 z_1 z_2)}$$

with a corresponding convergence domain D''' as in Fig. 2c). For all $z \in \mathbb{C}^n$ with $(|z_1|, |z_2|) \in D' \cap D'' \cap D'''$ we have in accordance with Theorem 2.1.4

$$F_3 + F_1 = F_2 , \quad D'' = D' \cap D''' .$$

which can readily be checked.

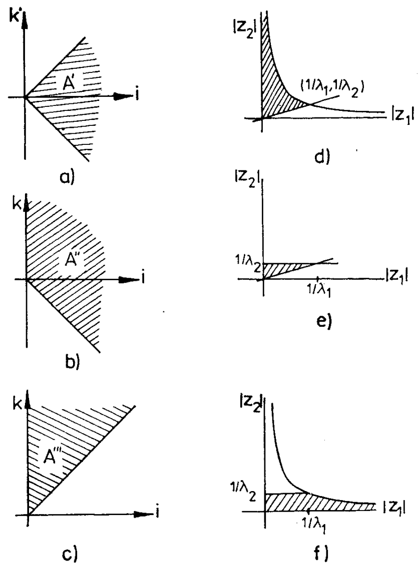

Fig. 2.

The sets A', A'', A''' are sketched in Fig. 2 a), b), c). The Reinhardt diagrams of the corresponding convergence domains are the shaded regions in Fig. 2 d), e), f).

Example 2.2.2. For

$$f(\alpha) = \frac{1}{(1 + \alpha_1 + \alpha_2)!}, \quad \alpha_i \geqq 0 , \quad i = 1, 2 ,$$

there is

$$Z\{f\} = \frac{e^{z_1} - e^{z_2}}{z_1 - z_2} \quad \text{for} \quad z \in \mathbb{C}^2 .$$

Indeed,

$$\frac{1}{z_1 - z_2}\left(1 + \frac{z_1}{1!} + \frac{z_1^2}{2!} + \ldots - \left(1 + \frac{z_2}{1!} + \frac{z_2^2}{2!} + \ldots\right)\right) =$$
$$= 1 + \frac{1}{2!}(z_1 + z_2) + \frac{1}{3!}(z_1^2 + z_1 z_2 + z_2^2) + \ldots$$

and the result follows by inspection. Many more transforms can be derived in a similar manner from the Taylor series of one-dimensional analytic functions, e.g. for $z \in \mathbb{C}^2$ we have:

$$Z^{-1}\left\{\frac{\sinh z_1 + \sinh z_2}{z_1 + z_2}\right\} = f(i, k) = \begin{cases} 0 \text{ for } i + k \text{ odd} \\ \dfrac{(-1)^i}{(1 + i + k)!} \text{ for } i + k \text{ even } i \geqq 0,\ k \geqq 0 . \end{cases}$$

In a similar way we obtain for $i \geqq 0$, $k \geqq 0$ with $\ln 1 = 0$:

$$\frac{1}{1 + i + k} \rightleftharpoons \frac{1}{z_1 - z_2} \ln \frac{1 + z_1}{1 + z_2}, \quad |z_1| < 1 , \quad |z_2| < 1 ,$$

and

$$\frac{(-1)^{i+k}}{1 + i + k} \rightleftharpoons \frac{1}{z_1 - z_2} \ln \frac{1 - z_1}{1 - z_2}, \quad |z_1| < 1 , \quad |z_2| < 1 .$$

We may conclude:
If for $n = 1$ there is $F(z) = f_0 + f_1 z + f_2 z^2 + \ldots, |z| < \varrho$, then for $i \geqq 0$, $k \geqq 0$ and $n = 2$

$$f_{i+k} \rightleftharpoons \frac{z_1 F(z_1) - z_2 F(z_2)}{z_1 - z_2}, \quad |z_1| < \varrho , \quad |z_2| < \varrho .$$

Example 2.2.3. Let $f_A \rightleftharpoons F_D$ and let $\xi \in {}^{\bullet}\mathbb{C}$ be a complex constant such that the point $(z', \xi) \in D$, $z' \in \mathbb{C}^{n-1}$. Define the sequence $g: A' \to \mathbb{C}$, $A' \subset \mathbb{Z}^{n-1}$ as folows: $g('\alpha) = '\sum f(\alpha)\, \zeta^{\alpha_n}$, where $'$ denotes summation over all the values of α_n such that $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n) \in A$, i e A' is the set of $(n - 1)$ tuples α' such that there exists α_n with $(\alpha', \alpha_n) \in A$. Then, as it can readily be seen,

$$Z(g_A) = F_D(z_1, z_2, \ldots, z_{n-1}, \zeta) .$$

This result could be called theorem on partial summation. It is formulated for the nth coordinate only to simplify the notation. The result can be generalized: more than one of the coordinates of $z \in \mathbb{C}^n$ could be fixed and accordingly, the sum defining g should be modified.

Example 2.2.4. Since the partial derivative of a power series is again a power series with the same region of convergence, we may state: If $f_A \rightleftharpoons F_D$, then $z_k(\partial F/\partial z_k) \rightleftharpoons \alpha_k f(\alpha)$ with $\alpha \in A \subset \mathbb{Z}^n$, $z \in D \subset \mathbb{C}^n$.

Example 2.2.5. Let again be $f_A \rightleftharpoons F_D$. Choose a constant $\lambda \in {}^{\bullet}\mathbb{C}^n$ such that the set

$\lambda D = \{w = (\lambda_1 z_1, \lambda_2 z_2, \ldots, \lambda_n z_n), z \in D\}$ is again a relatively complete Reinhardt domain. Then $f_A(\alpha)\, \lambda^\alpha \rightleftharpoons F_{D'}(\lambda_1 z_1, \lambda_2 z_2, \ldots, \lambda_n z_n)$ with $\alpha \in A$, $D' = \lambda^{-1} D$.

Example 2.2.6. The computation of the inverse z-transform of rational functions $F: \mathbb{C}^n \to \mathbb{C}$, e.g. $F(z) = P(z)/Q(z)$, where P, Q are polynomials, $Q(\mathbf{0}) \neq 0$, can easily be performed. This inverse z-transform consists of the coefficients of the Taylor series, which in this case can be expressed as values of the corresponding partial derivatives. Thus, from $P = FQ$ we obtain e.g. for $n = 2$: If $P(z, w)/Q(z, w) = \sum_{\substack{i=0\\j=0}} f(i, j)\, z^i w^j$, then

$$\frac{\partial^{k+q} P}{\partial z^k\, \partial w^q} = \sum_{i=0}^{k} \sum_{j=0}^{q} \frac{k!\, q!}{(k-i)!\, (q-j)!} \frac{{}^{k+q-i-j}Q}{\partial z^{k-i}\, \partial w^{q-j}} f(i, j)\,,$$

where substitution of $z = 0$ is understood. Denoting $P(z, w) = \sum b_{ik} z^i w^k$, $Q(z, w) = \sum a_{ik} z^i w^k$, after simplification we obtain

$$b_{kq} = \sum_{\substack{i=0\\j=0}}^{\substack{i=k\\j=q}} a_{k-i,q-j} f(i, j)\,.$$

It would be much easier to derive this result with the use of convolution of sequences (see Section 3.1).

Example 2.2.7. To illustrate the notational convention as used in Theorem 2.1.6 and the transform matrix U as introduced in Section 1.3 take the sequence λ^α, $\alpha \in A_0$, $\lambda \in {}^*\mathbb{C}^2$, λ fixed. Its z_{A_0}-transform equals

$$F_D(z) = \frac{1}{(1 - \lambda_1 z_1)(1 - \lambda_2 z_2)}\,, \quad D = \{z \in \mathbb{C}^2 : |z_1| < |\lambda_1|^{-1}\,, \quad |z_2| < |\lambda_2|^{-1}\}.$$

With $U = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ the first quadrant A_0 becomes transformed onto $A_0 U = \{\alpha' : \alpha'_1 + \alpha'_2 \geqq 0, \alpha'_2 \geqq 0\}$. Since for $w \in \mathbb{C}^2$ there is $w^U = (w_1, w_2/w_1)$, we obtain

$$F(w^U) = G(w) = \frac{1}{(1 - \lambda_1 w_1)\left(1 - \dfrac{\lambda_2 w_2}{w_1}\right)}$$

as the z-transform of the sequence $g(\alpha') = \lambda^{\alpha' U^{-1}} = \lambda^{(\alpha_1' + \alpha_2', \alpha_2')} = \lambda_1^{\alpha_1'} (\lambda_1 \lambda_2)^{\alpha_2'}$. The correspondence

$$G(z) \rightleftharpoons \lambda_1^{\alpha_1} (\lambda_1 \lambda_2)^{\alpha_2}\,, \quad z \in D^U\,, \quad (\alpha_1, \alpha_2) \in A_0 U^{-1}$$

may easily be verified. Here, $D^U = \{z \in \mathbb{C}^2 : z^U \in D\}$; since $z^U = (z^{(1,0)}, z^{(-1,1)}) = (z_1, z_1^{-1} z_2)$, there is

$$D^U = \{z \in \mathbb{C}^2 : |z_1| < |\lambda_1|^{-1}, |z_2| < |\lambda_2^{-1} z_1|\}\,.$$

2.3 THE CHARACTERISTIC TRANSFORMS

The result of Example 2.2.4 shows that the following definition might be useful.

Definition 2.3.1. Let $A \subset \mathbb{Z}^n$; the z-transform W_D^A of its characteristic function χ_A is called the characteristic transform of the set A; the region of convergence of W is called the characteristic domain of A.

It follows that

$$W_D^A(z) = \sum_{\alpha \in A} z^\alpha .$$

Some examples have been already given. From ii) in Example 2.2.1 we conclude that

$$W_D^{A_0}(z) = \prod_{i=1}^{n} \frac{1}{(1 - z_i)};$$

in iii) of the same example the following relations are given

$$W_D^{A'}(z) = \frac{z_2(1 + z_1)}{(1 - z_1 z_2)(z_2 - z_1)}$$

$$W_D^{A''}(z) = \frac{z_2}{(1 - z_2)(z_2 - z_1)}$$

$$W_D^{A'''}(z) = \frac{z_2}{(1 - z_2)(1 - z_1 z_2)}$$

The characteristic domain depends on the set A only and it is uniquely determined. We may therefore speak of a mapping $m(A) = D$. Theorem 2.1.4 can be taken as a subadditive property of this mapping, or more precisely

$$m(A \cup B) \subset m(A) \cap m(B)$$

provided the left-hand side is nonempty.

The characteristic domain of a set $A \subset \mathbb{Z}^n$ is a relatively complete Reinhardt domain, if nonempty.

Theorem 2.3.2. If the characteristic domain of a set $A \subset \mathbb{Z}^n$ is nonempty, then for any two elements $\xi, \eta \in A$ there exists an integer t such that $\xi + t\eta \notin A$.

Proof. Indeed, $\xi + t\eta \in A$ for all $t \in \mathbb{Z}$ would imply that the series $\sum_{t=-\infty}^{\infty} (z^\eta)^t$ has an open set of convergence, which is impossible. □

In n-D digital systems theory so called asymmetric half-planes are considered e.g. as $H = A_0 \cup \{(\alpha_1, \alpha_2): \alpha_1 > 0, \alpha_2 < 0\}$. The set H is a cone with $H \cap (-H) = \{\mathbf{0}\}$, but $m(H) = \emptyset$.

Theorem 2.3.3. The set $A \subset \mathbb{Z}^n$ has a nonempty characteristic domain if it belongs to a rational cone K.

Proof. It is sufficient to prove that any rational cone has a nonempty characteristic domain, i.e. that the series $\sum_{\alpha \in K} z^\alpha$ converges in an RCRD. Let the rational cone K be defined by a nonsingular integer matrix V (see Section 1.2) as follows: $\alpha \in K$ iff the vector $V\alpha = \mu$ has only nonnegative coordinates. Denote $z = w^V$ and write $\sum_{\alpha \in K} z^\alpha = \sum_{\alpha \in K} (w^V)^\alpha$. All terms of this series are contained among the terms of the series $\sum_{\mu \geqq 0} w^\mu$, which is convergent for all $|w| = |w_1 w_2 \dots w_n| < 1$. Since $d = |\det V|$ is a positive integer, the last condition is equivalent to $|w_2 w_2 \dots w_n|^d < 1$. From $w = z^{V^{-1}}$ we obtain $|w|^d = |z^Y|$, where Y is an integer matrix. The sum $\sum_{\alpha \in K} z^\alpha$ will be convergent for all $z \in \mathbb{C}^n$ satisfying $|z^Y| < 1$ and we are left with a set of n inte-qualities $|z^{\eta^{(i)}}| < 1$. If the matrix Y consists of nonnegative integers only, then the series $\sum_{\alpha \in K} z^\alpha$ converges in a complete Reinhardt domain (see Definition 1.2.1). If Y contains negative elements, then the domain of convergence is relatively complete and the statement is proven. □

It can be proved that any cone is contained in a rational cone. (For $n = 2$ this statement and proof is essentially contained in the paper by R. Eising [5], but the method can hardly be generalized for $n > 2$.) It means that for the characteristic domain of a monoid $M \subset \mathbb{Z}^n$ to be nonempty it is sufficient if M is a cone. Theorem 2.3.2 shows that a corresponding necessary condition must include requirements similar to that of $M \cap (-M) = \{\mathbf{0}\}$. An appropriate weaker condiition could be of importance in n-D systems theory.

The existence of the characteristic transform is closely related to the existence of n-D z-transforms of some classes of sequences.

A sequence f will be called λ-bounded on the set $A \subset \mathbb{Z}^n$ if there exists a vector $\lambda \in {}^{*}\mathbb{C}^n$ such that $|f(\alpha)\, \lambda^\alpha| \leqq M$ for some $M > 0$ and all $\alpha \in A$.

Theorem 2.3.4. If the set $A \subset \mathbb{Z}^n$ has a nonempty characteristic domain, $m(A) \neq 0$, then all λ-bounded sequences have convergent z_A-transforms; if $m(A) = 0$, then there exist λ-bounded sequences, which are not z_A-transformable.

Proof. If f is λ-bounded, then

$$|Z_A\{f\}| \leqq M \sum_{\alpha \in A} \left|\frac{z}{\lambda}\right|^\alpha .$$

Here and further

$$\frac{z}{\lambda} = \frac{z_1}{\lambda_1} \cdot \frac{z_2}{\lambda_2} \dots \frac{z_n}{\lambda_n} .$$

Since $\sum_{\alpha \in A} z^\alpha$ converges in an RCRD, say $D \subset \mathbb{C}^n$, then this inequality implies that there exists an RCRD $D' = \lambda D$, where $Z_A\{f\}$ is holomorphic (here, $\lambda D = \{w : w = \lambda z, z \in D\}$). On the other hand, no constant sequence is z_A-transformable if $m(A) = \emptyset$. □

Example 2.3.5. The sequence $f(\alpha) = \lambda^{|\alpha|}$, $\alpha \in \mathbb{Z}$, $\lambda \in {}^{*}\mathbb{C}$ has its z-transform

$$Z(f) = \frac{(1 - \lambda^2)z}{(z - \lambda)(1 - \lambda z)}, \quad \lambda < |z| < 1/\lambda,$$

provided $|\lambda| < 1$ although $m(A) = \emptyset$. For $f(a) = \alpha!$, $\alpha \in \mathbb{Z}$, $\alpha \geqq 0$, its z-transform does not exist, although $m(A) = \{z \in \mathbb{C}, |z| < 1\}$.

A further lemma can easily be proved.

Lemma 2.3.6. Every sequence f, which is λ-bounded on a rational cone $K \subset \mathbb{Z}^n$ has a convergent z_K-transform.

This lemma follows from Theorem 2.3.3. Further applications of the concept of characteristic transform will be shown in subsequent sections.

Some characteristic transforms can easily be found using Theorem 2.1.6.

Example 2.3.7. Let U be a nonsingular matrix of integers, $\det U = 1$. Then the characteristic transform W^B of the region $B = A_0 U$, where $A_0 = \{\alpha \in \mathbb{Z}^n, \alpha_i \geqq 0\}$ is the first n-tant, equals

$$W^B(z) = \frac{1}{1 - z^U}$$

and its characteristic domain $m(B) = D^{U^{-1}}$. E.g. for

$$U = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{we obtain} \quad z^U = (z_1, z_2/z_1)$$

and therefore

$$W^B(z) = \frac{z_1}{(1 - z_1)(z_1 - z_2)} \quad \text{for all} \quad z \in \mathbb{C}^2 \quad \text{such that} \quad |z^U| < 1,$$

i.e. $|z_1| < 1$, $|z_2| < |z_1|$. For $U = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ we obtain the result as for $W^{A''}$ in this section.

Together with the theorem in Section 1.3 on the canonical structure of the set of transformation matrices U, $\det U = 1$, we have now a method of construction of characteristic transforms for regions in $\mathbb{Z}^n$, for which such one-to-one mapping to the first n-tant does exist. This method can also be used for integer nonsingular transform matrices V with $\det V = -1$. E.g. for

$$V = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

we obtain $z^V = (z_1 z_3, z_1 z_2, 1/z_3)$. Here $B = A_0 V$ consists of points α' satisfying $\alpha'_1 + \alpha'_3 \geqq 0$, $-\alpha'_1 + \alpha'_2 - \alpha'_3 \geqq 0$, $-\alpha'_3 \geqq 0$ and from Theorem 2.1.6 follows for the characteristic transform of the set B

$$W^B(z) = \frac{1}{(1 - z_1 z_3)(1 - z_1 z_2)\left(1 - \dfrac{1}{z_3}\right)}.$$

3. THE CONVOLUTION

3.1. Convolution and Product of sequences

In the preceding section the algebraic structure of the set A was mostly irrelevant. From now on, it has to be assumed that A is a commutative semigroup with addition as its semigroup operation. Additional requirements (e.g. such as the existence of the neutral element) will be formulated whenever necessary. For sequences f, g defined on the sets A, B, respectively we shall introduce their convolution as follows:

Definition 3.1.1. If $f: A \to \mathbb{C}$, $g: B \to \mathbb{C}$, $A, B \subset \mathbb{Z}^n$, then the sequence $h: A + B \to \mathbb{C}$, defined by

(3.1.1)
$$h(\gamma) = \sum_{\substack{\alpha\in A\\ \beta\in B\\ \gamma=\alpha+\beta}} f(\alpha)\, g(\beta)$$

is called convolution of f with g (and denoted by $h = f * g$) provided the sum exists for all $\alpha + \beta \in A + B$.

To obtain the common form of formula (3.1.1) we may rewrite it as follows

$$h(\gamma) = \sum_{\alpha\in A}' f(\alpha)\, g(\gamma - \alpha), \quad \gamma \in A + B,$$

where $\sum'$ denotes that for every $\gamma \in A + B$ the summation is taken over all such $\alpha \in A$ for which $\gamma - \alpha \in B$. Evidently

(3.1.2)
$$f * g = g * f.$$

Also

(3.1.3)
$$(\lambda f) * g = f * (\lambda g) = \lambda(f * g),$$
$$(f + g) * h = (f * h) + (g * h)$$

provided the right-hand sides exist.

The domain of convolution is equal to that of the constituents only in exceptional cases. Some cases are of special interest. If e.g. $A = B = \mathbb{Z}^n$, we arrive at the case given in Section 1.1. If only one of the sets, say $A = \mathbb{Z}^n$, then $A + B = \mathbb{Z}^n$ for all possible choices of B. Here the most important example is as follows

Example 3.1.2. Let $f: A \to \mathbb{C}$, then $f * \delta^\beta: \mathbb{Z}^n \to \mathbb{C}$ and

$$(f * \delta^\beta)(\gamma) = \sum_{\alpha} f(\gamma - \alpha)\, \delta^\beta(\alpha) =$$

$$= \begin{cases} f(\gamma - \beta) & \text{for} \quad \gamma - \beta \in A \\ 0 & \text{for} \quad \gamma - \beta \notin A \,, \end{cases} \quad \text{where} \quad \alpha \in \mathbb{Z}^n \,.$$

For $\beta = 0$ this operation simply extends the domain of f, putting its value equal to zero for $\alpha \notin A$.

Definition 3.1.3. Let $f\colon A \to \mathbb{C}$; then the sequence $f_\beta\colon \mathbb{Z}^n \to \mathbb{C}$ is called "f shifted by $\beta \in \mathbb{Z}^n$" if

$$f_\beta = f * \delta^{-\beta} \,.$$

According to the previous example, the shifted sequence

$$f_\beta(\gamma) = \begin{cases} f(\gamma + \beta) & \text{for} \quad \gamma + \beta \in A \\ 0 & \text{for} \quad \gamma + \beta \notin A \end{cases}. \tag{3.1.4}$$

This definition is not always satisfactory. The main reason is that $\delta^\beta * (\delta^{-\beta} * f) \neq f$ if $A \neq \mathbb{Z}^n$. We shall deal with this problem in the next section. Here an important theorem on *n-D* *z*-transform has to be formulated.

Theorem 3.1.4. Let $f_A \rightleftharpoons F_D$, $g_B \rightleftharpoons G_H$ and let $D \cap H = K \neq \emptyset$. Then, provided the left-hand side exists, there is

$$f_A * g_B \rightleftharpoons (F \,.\, G)_K \,.$$

Proof. Take the right-hand side of the correspondence for $z \in K$. Sinde the intersection of relatively complete Reihardt domains is again an RCR domain we have to consider the product

$$\sum_{\alpha \in A} f(\alpha)\, z^\alpha \sum_{\beta \in B} g(\beta)\, z^\beta = \sum_{\substack{\alpha \in A \\ \beta \in B}} f(\alpha)\, g(\beta)\, z^{\alpha+\beta} =$$

$$= \sum_{\gamma \in A+B} \Big(\sum_{\alpha \in A}{}' f(\alpha)\, g(\gamma - \alpha) \Big) z^\gamma = \sum_{\gamma \in A+B} (f * g)(\gamma)\, z^\gamma \,,$$

which completes the proof. □

A dual theorem can also be proved

Theorem 3.1.5. Let be

$$F(z) = \sum_{\alpha \in A} f(\alpha)\, z^\alpha \quad \text{for all} \quad z \in D \,,$$

$$G(z) = \sum_{\alpha \in A} g(\alpha)\, z^\alpha \quad \text{for all} \quad z \in H$$

and $D \cap H = K \neq \emptyset$. (Here D, H, K are relatively complete Reinhardt domains.) Then

$$\frac{1}{(2\pi \mathrm{j})^n} \int_\Gamma F\left(\frac{z}{\zeta}\right) G(\zeta)\, \frac{\mathrm{d}\xi}{\xi} = \sum_{\alpha \in A} f(\alpha)\, g(\alpha)\, z^\alpha$$

for all $z \in K$, provided Γ is a suitable "closed path of integration encircling the point $\mathbf{0}$", entirely contained in K.

Proof. Consider the following evident fact

$$\frac{1}{(2\pi j)^n} \int_\Gamma \frac{d\xi}{\xi^\alpha} = \begin{cases} 1 & \text{for} \quad \alpha = \mathbf{1}\,, \\ 0 & \text{for} \quad \alpha \neq \mathbf{1}\,, \end{cases} \quad \alpha \in \mathbb{Z}^n\,. \tag{3.1.5}$$

After using the series expansions of F and G in the integral (3.1.5), the interchange of summations and integration is justified due to the absolute and uniform convergence of the series. Hence,

$$\frac{1}{(2\pi j)^n} \int_\Gamma \left(\sum f(\alpha) \left(\frac{z}{\xi}\right)^\alpha \right) \left(\sum g(\beta)\, \xi^\beta \right) \frac{d\xi}{\xi} = \frac{1}{(2\pi j)^n} \int_\Gamma \sum_{\substack{\alpha \in A \\ \beta \in A}} f(\alpha)\, g(\beta)\, z^\alpha \xi^{\beta - \alpha - \mathbf{1}}\, d\xi =$$

$$= \frac{1}{(2\pi j)^n} \sum_{\substack{\alpha \in A \\ \beta \in A}} z^\alpha \int_\Gamma f(\alpha)\, g(\beta)\, \xi^{\beta - \alpha - \mathbf{1}}\, d\xi = \sum_{\alpha \in A} f(\alpha)\, g(\alpha)\, z^\alpha\,,$$

with the use of formula (3.1.5). □

This theorem has interesting applications in the following special case: Let $f_A \rightleftharpoons F_D(z)$ and let $B \subset A$. Then the restriction of f_A to f_B has a convergent z-transform and it can be expressed as

$$F_R(z) = \frac{1}{(2\pi j)^n} \int_\Gamma F_D(\xi)\, W_{D'}^B \left(\frac{z}{\xi}\right) \frac{d\xi}{\xi}\,, \tag{3.1.6}$$

where $W_{D'}^B$ is the characteristic transform of the set B, $R = D \cap D'$. (The reader may not be surprised discovering, that the special case $B = A$ in (3.1.6) gives the well-known Cauchy's formula).

Some examples could be in order here.

Example 3.1.6. Let the sequence f

$$f_A(\alpha) = \frac{1}{(1 + \alpha_1 + \alpha_2)!}\,, \quad A = \{\alpha_1 \geqq 0,\ \alpha_2 + \alpha_1 \geqq 0\}$$

be considered. Its z-transform can be obtained similarly as in Example 2.2.2.

$$Z_A\{f\} = \frac{e^{z_2} - 1}{z_2 - z_1} \quad \text{for} \quad |z_1| < |z_2|\,.$$

The set A contains the first quadrant A_0. With the characteristic transform of A_0 and with (3.1.6) we can obtain $Z\{f_{A_0}\}$ as follows

$$Z\{f_{A_0}\}\,(\xi) = \frac{1}{(2\pi j)^2} \int_\Gamma \frac{e^{z_2} - 1}{z_2 - z_1} \frac{dz}{(z_1 - \xi_1)(z_2 - \xi_2)}\,,$$

where $|z_1| < |z_2|$ and therefore $\Gamma = \{z_1 = e^{j\varphi};\ z_2 = r\, e^{j\psi},\ r > 1\}$, hence

$$Z\{f_{A_0}\}\,(\xi) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \frac{r\, e^{j\psi}(e^{r e^{j\psi}} - 1)}{r\, e^{j\psi} - \xi_2}\, I(\xi_1, \psi)\, d\psi\,,$$

with

$$I(\xi_1, \psi) = \int_0^{2\pi} \frac{e^{j\varphi}\, d\varphi}{(r\, e^{j\psi} - e^{j\varphi})(e^{j\varphi} - \xi_1)}.$$

After some further calculations we obtain

$$Z\{f_{A_0}\}(\xi) = \frac{e^{\xi_1} - e^{\xi_2}}{\xi_1 - \xi_2}$$

in accordance with Example 2.2.2.

3.2. The shift operator

Although the set S_A of sequences defined on $A \subset \mathbb{Z}^n$ forms a linear space, sequences "shifted by β", as they are characterized by Definition 3.1.3 do not belong to this space. For reasons given in Section 1.1 we have to construct a different "shift operator" so that these operators could be treated by the z-transform.

Taking S_A as a starting point, the "shifted sequence" would be defined on a "shifted region", say $A + \beta$, which may in various ways differ from the originally given region A. This can be illustrated by an example as in Fig. 3.

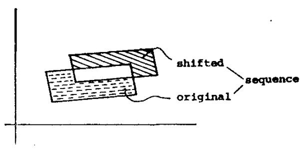

Fig. 3.

The shift creates certain "abandoned" regions (dotted area) and some "newly occupied" regions. In general, the "abandoned" regions are formed by the set $A \setminus (A + \beta)$, the "newly occupied" ones by the set $(A + \beta) \setminus A$ (either one or both of these may be void). To create a linear space of shifted sequences demands to somehow take care of the "abandoned" and "newly occupied" regions. This is the basic motivation of the following seemingly overcomplicated

Definition 3.2.1. For a given set $A \subset \mathbb{Z}^n$ and a given point $\beta \in \mathbb{Z}^n$ the operator T^β: $S_A \to S_A$ is called a complete shift operator if

1) for any given $f \in S_A$ the values $f(\gamma)$ are specified for all $\gamma \in (A + \beta) \setminus A$
2) the values of $g = T^\beta f$ for any given sequence $f \in S_A$ are given by

$$g(\alpha) = f(\alpha + \beta) \quad \text{for all} \quad \alpha \in A\,.$$

The complete shift operator maps the linear space S_A into the linear space S_A, but the operator T^β is not linear. Nevertheless, for given numbers λ, μ and given vectors α, β the sum $\lambda T^\alpha + \mu T^\beta$ is again an operator T: $S_A \to S_A$ provided T^α, T^β

are complete shift operators. More generally: for any finite set $B \subset \mathbb{Z}^n$ and any mapping $a\colon B \to \mathbb{C}$ the following operator $T\colon S_A \to S_A$ can be defined

$$T = \sum_{\beta \in B} a_\beta T^\beta ; \tag{3.2.1}$$

here, T^β are complete shift operators.

Consider now the inverse of T^β, formally denoted by $(T^\beta)^{-1}$. Note, that for T^β to be defined, the vector β and the values $f(\gamma)$, $\gamma \in A + \beta \setminus A$ must be uniquely specified. Therefore, $(T^\beta)^{-1}$ will exist if and only if there exists a unique extension of $g = T^\beta f \in S_A$ to the set $A - \beta \setminus A$ such that $g(\alpha - \beta) = f(\alpha)$ for all $\alpha \in A$. More generally:

Theorem 3.2.2. The operator $T\colon S_A \to S_A$, defined by formula (3.2.1) has an inverse operator T^{-1} if and only if equation (1.2) has a unique solution f for any sequence $x \in S_A$.

In our next step we want to investigate the impact of the z-transform on the operator T^β, i.e. to derive $Z_A(T^\beta f)$ from a known $Z_A(f)$.

Theorem 3.2.3. Let be $f_A \rightleftharpoons F_D$ and let T^β be a complete shift operator with $A \cap \cap (A + \beta) \neq \emptyset$. Then

$$Z_A(T^\beta f) = z^{-\beta}\big(Z_A(f) + \sum_{\alpha \in R_\beta} f(\alpha)\, z^\alpha - \sum_{\alpha \in S_\beta} f(\alpha)\, z^\alpha\big),$$

where $R_\beta = (A + \beta) \setminus A$, $S_\beta = A \setminus (A + \beta)$.

Proof. The proof is straightforward:

$$Z_A(T^\beta f) = \sum_{\alpha \in A} f(\alpha + \beta)\, z^\alpha = z^{-\beta} \sum_{\gamma \in A + \beta} f(\gamma)\, z^\gamma =$$

$$= z^{-\beta}\Big(\sum_{\gamma \in (A+\beta)\setminus A} + \sum_{\gamma \in A \cap (A+\beta)} \Big) = z^{-\beta}\big(\sum_{\gamma \in R_\beta} f(\gamma)\, z^\gamma + F(z) - \sum_{\gamma \in S_\beta} f(\gamma)\, z^\gamma \big)$$

and the proof is completed. □

To illustrate the situation, take $n = 1$ and $\beta_1 = 1$, $\beta_2 = -1$. With the sequence

$$f_{-1} = 3\,, \quad f_0 = 2\,, \quad f_1 = f_2 = \dots f_n = 1, \dots,$$

we have $A = \{\alpha \geqq -1\}$. For $\beta_1 = 1$ there is

$$R_{\beta_1} = \{\alpha \geqq 0\} \setminus \{\alpha \geqq -1\} = \emptyset\,, \quad S_{\beta_1} = \{\alpha \geqq -1\} \setminus \{\alpha \geqq 0\} = \{-1\}\,.$$

Since

$$Z_A\{f\} = \frac{3 - z - z^2}{z(1 - z)}\,,$$

we obtain

$$Z_A\{T^{\beta_1} f\} = \frac{1}{z}\left(\frac{3 - z - z^2}{z(1 - z)} - \frac{3}{z}\right) = \frac{2}{z} - \frac{1}{z - 1}$$

and the sequence $T^{\beta_1} f = f'$ is therefore

$$f'_{-1} = 2\ ; \quad f'_0 = f'_1 = \ldots = f'_n = 1\ , \ldots .$$

Similarly for β_2 we obtain $R_{\beta_2} = \{-2\}$; $S_{\beta_2} = \emptyset$ and therefore

$$Z_A(T^{\beta_2} f) = z\left(\frac{3 - z - z^2}{z(1 - z)} + \frac{f_{-2}}{z^2}\right),$$

with the value f_{-2} of the original sequence undefined.

The last two theorems make possible the use of z-transform in solving partial difference equations (1.2). Other applications are also possible.

Example 3.2.4. z-transform techniques are often used in deriving combinatorial identities. Starting with the correspondence

$$\binom{m + n}{n} \rightleftharpoons \frac{1}{1 - z_1 - z_2}\,, \qquad m, n \geqq 0\ ; \quad |z_1 + z_2| < 1\ ,$$

which can easily be obtained, we may construct the "convolutional square" $\binom{m + n}{n} * \binom{m + n}{n}$ as follows

$$g(m, n) = \sum_{\substack{i=0\\ j=0}}' \binom{i + j}{j} \binom{m + n - i - j}{n - j}$$

(see Definition 3.1.1). With Theorem 3.1.4 we obtain for the z-transform of the sequence

$$G(z) = \frac{1}{(1 - z_1 - z_2)^2}\,.$$

On the other hand, Example 2.2.4 yields

$$\frac{z_1}{(1 - z_1 - z_2)^2} \rightleftharpoons m\binom{m + n}{n} = f(m, n)$$

and from Theorem 3.2.2 we obtain

$$\frac{1}{(1 - z_1 - z_2)^2} \rightleftharpoons T^{(1,0)} f = (m + 1)\binom{m + n + 1}{n}.$$

Comparing these results, we obtain

$$\sum_{\substack{i=0\\ j=0}} \binom{i + j}{j} \binom{m + n - i - j}{n - j} = (m + 1)\binom{m + n + 1}{n},$$

where, as usual, $\binom{k}{q} = 0$ if $q > k$, or $k < 0$, or $q < 0$.

4. SOLUTION OF n-D DIFFERENCE EQUATIONS BY z-TRANSFORM

4.1. The existence and uniqueness of solution

We want to use z-transform to find the solution of equation (1.2). Besides linearity (Theorem 2.1.3) mainly the shift operator (replacing in a certain sense the partial difference operator) has to be treated; here Theorem 3.2.3 will be used. We shall show that in the applicability of z-transform the sets S_β, R_β play a crucial role.

When solving linear functional equations of any kind by functional transform method some preliminary investigations are necessary. These must include the one-to-one correspondence of the transform and its inverse, which is a necessary condition of its application. Often such correspondence can be ensured only on certain classes of solutions of the functional equation originally given. This is why functional transform methods can hardly be used for obtaining existence and uniqueness results for functional equations. As shown by the correspondence of region A of the sequences f and domain D of its z-transform (see Definition 2.1.1), a similar situation arises in solving partial difference equations. Application of z-transform to the solution of equation (1.2) can be reasonable only after the existence and uniqueness of its solution has been guaranteed. However, this cannot be done without some additional information. Indeed, supposing f^* is a solution of equation (1.2), we realize that any sequence f with $f(\alpha) = f^*(\alpha) + \lambda^\alpha$, where $\lambda \in {}^\bullet\mathbb{C}^n$ is a constant vector satisfying $\sum_{\beta \in B} a_\beta \lambda^\beta = 0$, is also its solution. Therefore if $n > 1$, equation (1.2) either has no solution at all or it has an infinite (non-countable) set of linearly independent solutions. To ensure uniqueness some additional requirements have to be imposed on the solution of (1.2). These may have e.g. the form of boundary conditions or "initial" conditions. Here rather complicated situations may arise, which are not attainable by z-transform methods. Therefore we shall restrict ourselves to special type of solutions, called recursively computable solutions (see [1] and also [2], [4]), which have been mostly dealt with in n-D digital systems theory.

The definition of a recursively computable (RC-) solution has to be carefully formalized. The basic idea is that of successive computations of the values $f(\alpha)$ from earlier computated values $f(\alpha')$, values $x(\alpha)$ of the input and some given "initial" values. The existence, uniqueness and construction of an RC solution of equation (1.2) essentially depends on an order relation $\leqq$ in the set, where this solution is supposed to satisfy the equation. Supposing in equation (1.2) that $\alpha \in A \subset \mathbb{Z}^n$, the solution f has to be defined on all the "sets A shifted by $\beta \in B$", i.e. $f: A + B \to \mathbb{C}$, where $A + B = \{\gamma \in \mathbb{Z}^n: \gamma = \alpha + \beta,\ \alpha \in A,\ \beta \in B\}$. As mentioned above, RC solutions imply that the set $A + B$ is an ordered semigroup, its ordering relation will be henceforth denoted by $\leqq$. To guarantee the existence and uniqueness of the RC-solution for arbitrary initial values, a certain set G, say "initial set" has to be suitably chosen. It has been shown [1], that even in more general cases of linear partial difference equations such choice is always possible. Its actual construction can be derived from the order $\leqq$ and from the sets A, B; it does not depend on the values of the coefficients a_β as far as they are not equal to zero.

The most simple situation arises with a well-ordered set $A + B$. Since any cone A can always be endowed with an order $\leqq$ such that $A + B$ is a well-ordered set, recalling Theorems 2.3.3, 2.3.4, we can qualify this special case as the most important one in applications of the z-transform. The following Theorem contains also the construction of the "initial set" G.

Theorem 4.1.1.a Let the equation

$$\sum_{\beta \in B} a_\beta f(\alpha + \beta) = x(\alpha)\,, \quad \alpha \in A \subset \mathbb{Z}^n \tag{4.1.1}$$

be considered with A, B nonempty sets, $B \subset \mathbb{Z}^n$ finite and containing at least two elements, $a_\beta \neq 0$ for all $\beta \in B$. Let further an order relation $\leqq$ in $\mathbb{Z}^n$ be given such that A is a well-ordered cone with respect to $\leqq$. Denote $\max_{\leqq} B = \beta^0$ and

$$G = (A + B) \setminus (A + \beta^\circ)\,. \tag{4.1.2}$$

Then for any mapping $g: G \to \mathbb{C}$ there exists one and only one sequence $f: A + B \to \mathbb{C}$ such that f satisfies equation (4.1.1) and $f(\gamma) = g(\gamma)$ for all $\gamma \in G$. Moreover, all the values $f(\gamma)$, $\gamma \in A + B \setminus G$ are recursively computable from the "initial" values $g(\gamma)$.

To give a nontrivial illustration, we shall reconsider the difference equation from Example 2 (see [7], p. 93), which is claimed not to have a "well-defined recursive solution" under "zero initial conditions". The equation reads as follows (original notation of Huang's paper is retained): $h(m, n) = \delta(m, n) + eh(m - 1, n) + fh(m, n - 1) + gh(m + 1, n + 1)$, $m, n \geqq 0$ and with "zero initial conditions". To use Theorem 4.1.1a, let $\leqq$ be the lexicographic order, i.e. $(m, n) \leqq (i, k)$ iff $[m < i$ or $(m = i, n < k)$ or $m = i, n = k]$. Since $B = \{(0, 0), (-1, 0), (0, -1), (1, 1)\}$, we have $\beta^0 = (1, 1)$. With $A = \{(m, n), m \geqq 0, n \geqq 0\}$ we obtain $A + B = \{(m, n): m \geqq -1, n \geqq -1\} \setminus \{(-1, -1)\}$, $A + \beta^0 = \{(m, n), m \geqq 1, n \geqq 1$.

Therefore

$$G = \{(m, n): m = 0, m = -1, n \geqq 0\} \cup \{(m, n): n = 0, n = -1, m \geqq 0\}\,,$$

and "zero initial conditions" say that $g(m, n) = 0$ for all $(m, n) \in G$. A simple figure (which the reader is asked to sketch) shows by inspection that the solution can readily be computed by recursion. The mistake in Huang's paper also becomes evident: the value $h(-1, -1)$ must not be chosen arbitrarily and therefore it cannot be put equal to zero beforehand.

The last theorem is an overspecialized version of a more general result, which has been proved in [1]. In what follows we need a slightly more general result: the assumption of A to be a well-ordered set is too resctrictive since it is not included in the problem as it is formulated by equation (1.2). Let therefore the last theorem be reformulated without this assumption.

Theorem 4.1.1. Let the equation

$$\sum_{\beta\in B} a_\beta f(\alpha + \beta) = x(\alpha)\,, \quad \alpha \in A \subset \mathbb{Z}^n \tag{4.1.1}$$

be considered, where A, B are nonempty sets, $A, B \subset \mathbb{Z}^n$, $a_\beta \neq 0$, B finite and with at least two elements. Then there exist an order relation $\leqq$ in the set A and a set $G \subset A + B$ such that for any function $g: G \to \mathbb{C}$ the equation has exactly one recursively computable solution $f: A + B \to \mathbb{C}$ satisfying the (initial) conditions $f(\gamma) = g(\gamma)$ for all $\gamma \in G$.

The proof of this theorem can be found in [1], where also the construction of the initial set G is described in case when the set A is not well-ordered. The basic idea for this construction lies in construction of the order relation $\leqq$ in the set A and a mapping $\beta^*: A \to B$, which satisfy the following implication

$$\alpha' + \beta^*(\alpha') \in \alpha + B \Rightarrow \alpha' \leqq \alpha \quad \text{for all} \quad \alpha, \alpha' \in A\,. \tag{4.1.3}$$

This construction is proved to be always possible, and moreover, the initial set G can be expressed as follows

$$G = (A + B) \setminus \bigcup_{\alpha\in A} (\alpha + \beta^*(\alpha))\,. \tag{4.1.4}$$

Some examples are shown in Section 4.2. The above theorem together with Theorem 2.1.3, Theorem 3.2.3 and the basic Definition 2.1.1 enables to find the classes of n-D difference equations which are directly solvable by z-transform. (It cannot be anything surprising in the fact that not all equations (1.2) can be directly solved by z-transform: in our rather general formulation even in one dimensional cases some equations of the considered type cannot be solved by z-transform.) Since the main difficulty lies in discrepancies between the initial set G (see Theorem 4.1.1) and the sets S_β, R_β (see Theorem 3.2.3), their properties will be investigated now.

Lemma 4.1.2. Let be $R_B = \bigcup_{\beta\in B} R_\beta$, $S_B = \bigcup_{\beta\in B} S_\beta$. Then

$$\begin{aligned} R_B &= (A + B) \setminus A \\ S_B &= A \setminus \bigcap_{\beta\in B} (A + \beta)\,, \end{aligned} \tag{4.1.5}$$

and the following equivalences hold true:

$$\begin{aligned} R_B &= \emptyset \Leftrightarrow A + B \subset A \\ S_B &= \emptyset \Leftrightarrow A \subset \bigcap_{\beta\in B} (A + \beta) \Rightarrow A \subset A + B\,. \end{aligned} \tag{4.1.6}$$

The proof follows from well known identities of set theory. □

Definition 4.1.3. Equation (4.1.1) with a given initial set G ensuring the existence and uniqueness of its solution is said to be z-complete if

$$G = S_B \cup R_B . \tag{4.1.7}$$

This definition is motivated by comparison of Theorems 3.2.3 and 4.1.1. Evidently, only for equations satisfying condition (4.1.7) the z-transforms of their solutions can be obtained by direct application of Theorem 3.2.3 and 2.1.3.

Theorem 4.1.4. The equation (4.1.1) with $\mathbf{0} \in B$ and $A \neq \mathbb{Z}^n$ is z-complete if and only if

$$\bigcap_{\beta \in B} (A + \beta) = \bigcup_{\alpha \in A} (\alpha + \beta^*(\alpha)) , \tag{4.1.8}$$

where the mapping β^*: $A \to B$ satisfies condition (4.1.3).

Proof. If $\mathbf{0} \in B$, then $\bigcap (A + \beta) \subset A \subset A + B$ and the rest follows from Lemma 4.1.2 and formula (4.1.4). □

Example 4.1.5. Let in equation (4.1.1) be $n = 1$. With the assumptions of the previous theorem this equation has the common form

$$\sum_{k=0}^{p} a_k f(m + k) = x(m) ,$$

where m is a positive integer, p is the order of the difference equation. Here $\bigcap_{\beta \in B} (A + \beta)$ is the set of integers $\geqq p$ and $\beta^*(\alpha) = p$. Therefore $\bigcup_{\alpha \in A} (\alpha + \beta^*(\alpha))$ also equals the set of integers $\geqq p$. Hence every initial value problem for a one-dimensional difference equation on a half line is z-complete. We shall see that here the assumption $n = 1$ together with that of A being a half-line is essential.

4.2. Difference equations on well-ordered sets

Since conditions ensuring the existence of a unique, recursively computable solution of equation (4.1.1) have their simplest form in case when the set A is a semigroup, which is well-ordered with respect to order $\leqq$ (compare (4.1.4) and (4.1.2)) we start our discussion with the special case. It includes, among others, the mostly investigated QP filters in multidimensional systems theory, together with all the classical results in one-dimensional digital filtering and, at the same time, it is closely related to a certain "one-sidedness" or, casually speaking, to causality of n-D systems.

From Theorem 3.2.3 the following classification scheme can be derived:

Definition 4.2.1. The equation (4.1.1) is said to be of
delayed type if $A \subset \bigcap_{\beta \in B} (A + \beta)$,
semidelayed type if $A \subset A + B$
advanced type if $A + B \subset A$

mixed type if it is neither of delayed nor of advanced type
neutral type if it is both of delayed and of advanced type.

Some examples could be in order.

Example 4.2.2. (i) In 2-D digital system theory mostly the following example is considered

$$(4.2.1) \qquad \sum_{\substack{i=0\\k=0}}^{i=N,k=M} a_{ik} f(\alpha_1 - i, \alpha_2 - k) = x(\alpha_1, \alpha_2),$$

$$\text{with} \quad \alpha = (\alpha_1, \alpha_2) \in A_0, a_{00} \neq 0, N, M \quad \text{positive}.$$

This is an equation of delayed type. Moreover, with lexicographic ordering of $\mathbb{Z}^2$, it is also z-complete and therefore its solution by z-transform methods runs without difficulties. Similar is the situation with such equations for $n > 2$.

(ii) Equation (4.1.1) considered for $A = \mathbb{Z}^n$ is an equation of neutral type. Its direct solution by z-transform is impossible, e.g., since initial conditions cannot be respected or for reasons described in Section 1.1.

(iii) Equation

$$(4.2.2) \qquad \sum_{\substack{i=0\\k=0}}^{i=N,k=M} a_{ik} f(\alpha_1 + i, \alpha_2 + k) = x(\alpha_1, \alpha_2), \quad \alpha \in A_0, \quad N, M > 0$$

is an example of equations of advanced type. It is also z-complete if $a_{NM} \neq 0$.

(iv) Equation (4.2.1) with $A = \{(\alpha_1, \alpha_2): \alpha_1 \geqq 0, \ \alpha_2 \geqq 0 \text{ or } \alpha_1 > 0, \alpha_2 < 0\}$ describes the so-called ASHP (asymmetric half-plane) filters (see [7]). It is again an equation of delayed type, ($S_B = 0$), but now $R_B \nsubseteq G$ (more explicitly $G = R_B \cup \{(\alpha_1, \alpha_2): \alpha_1 \in \mathbb{Z}, \alpha_2 = -1\}$ when lexicographic ordering is considered) and therefore (4.2.1) cannot be z complete in this case.

(v) Equation

$$(4.2.3) \qquad f(\alpha_1, \alpha_2) + a f(\alpha_1 + 1, \alpha_2 - 1) + b f(\alpha_1 - 1, \alpha_2 + 1) = x(\alpha), \quad \alpha \in A_0$$

is of semidelayed type, but it is not of delayed type. As of its solution, the situation is somewhat similar to that of Example 4.2.4. This equation is not z-complete.

(vi) Equation

$$(4.2.4) \qquad 4 f(\alpha_1, \alpha_2) = f(\alpha_1 + 1, \alpha_2) + f(\alpha_1 - 1, \alpha_2) + f(\alpha_1, \alpha_2 + 1) + f(\alpha_1, \alpha_2 - 1) \quad \text{for} \quad A = \{(\alpha_1, \alpha_2): 0 \leqq \alpha_1 \leqq M, 0 \leqq \alpha_2 \leqq N\}$$

is obtained by discretization of the Laplace partial differential equation. In our classification scheme it is an equation of semidelayed type, which is not z-complete.

(vii) Equation

(4.2.5) $$f(\alpha_1 + 1, \alpha_2) + f(\alpha_1, \alpha_2 + 1) = x(\alpha_1, \alpha_2),$$

with $A = \{(\alpha_1, \alpha_2): 0 \leqq \alpha_1 \leqq M, 0 \leqq \alpha_2 \leqq N\}$ is an example of mixed type equation which is not z-complete.

It has to be emphasized that all the equations in this example with properly chosen initial conditions have a unique and recursively computable solution. Now the following theorem on the solution of partial difference equations by z-transform can be formulated:

Theorem 4.2.3. Let the sets A, $B \subset \mathbb{Z}^n$ be as in equation (4.1.1). Let the set A be an ordered semigroup such that (4.1.1) is z-complete, i.e. condition (4.1.7) is satisfied. Let further initial values $g(\alpha)$ be given for all α belonging to the corresponding initial set G. Then the z_A-transform F of the solution f of equation (4.1.1), satisfying conditions $f(\alpha) = g(\alpha)$ for all $\alpha \in G$, can be expressed by

(4.2.6) $$F(z) = \frac{X(z) - N(z) + M(z)}{\sum_{\beta \in B} a_\beta z^{-\beta}},$$

where $X(z) \rightleftarrows x(\alpha)$ and

$$N(z) = \sum_{\beta \in B} a_\beta z^{-\beta} \sum_{\alpha \in R_\beta} g(\alpha)\, z^\alpha,$$ [1]

$$M(z) = \sum_{\beta \in B} a_\beta z^{-\beta} \sum_{\alpha \in S_\beta} g(\alpha)\, z^\alpha$$ [1]

are determined by the given initial conditions.

If (4.1.1) is of delayed or semidelayed type, then

$$M(z) \equiv 0.$$

If (4.1.1) is of advanced type, then

$$N(z) \equiv 0.$$

If $g(\alpha) = 0$ for all $\alpha \in G$, then

$$N(z) \equiv M(z) \equiv 0.$$

The proof of this theorem consists of application of Theorems 3.2.3, 4.1.1, 4.1.3.

Although the last theorem did not assume that the set A is well-ordered, we shall investigate now this case in more detail. Our attention will be focused on equations which are not z-complete.

Example 4.2.4. Find the solution equation of

(4.2.7) $$f(\alpha_1 - 1, \alpha_2) + f(\alpha_1, \alpha_2 - 1) = x(\alpha_1, \alpha_2), \quad \alpha \in A_0$$

satisfying initial conditions $f(-1, \alpha_2) = 0$ for all $\alpha_2 \geqq 0$, if $x(\alpha) = 1$ for all $\alpha \in A_0$.

[1]) The sum is considered equal to zero if the corresponding set R_β or S_β is empty.

Since $B = \{(-1, 0),\ (0, -1)\}$, we find that $A_0 = \bigcap_{\beta \in B} (A_0 + \beta)$ and therefore (4.2.7) is of delayed type. From Theorem 4.1.1 we conclude that (4.2.7) has a unique solution. To apply Theorem 3.2.2, we find $R_{(-1,0)} = G$, $R_{(0,-1)} = \{(\alpha_1, -1),\ \alpha_1 \geqq 0\}$. Hence the equation is not z-complete. For zero initial conditions and given function $x(\alpha)$ we obtain

$$F(z)\,(z_1 + z_2) = \frac{1}{(1 - z_1)(1 - z_2)} - \varphi(z_1)\,,$$

where $\varphi(z_1) = \sum_{\alpha \in R_{(0,-1)}} f(\alpha)\, z_1^{\alpha_1}$ is a one-variable function. This function cannot be obtained from given data; it is by no means arbitrary. Direct application of z_A-transform fails.

Trying to improve the situation, we might attempt to substitute "new variables" γ by $\alpha_1 = \gamma_1$, $\alpha_2 = \gamma_2 + 1$, to obtain equation

$$f(\gamma_1 - 1, \gamma_2 + 1) + f(\gamma_1, \gamma_2) = x(\gamma_1, \gamma_2 + 1)\,, \quad \gamma_1 \geqq 0\,, \quad \gamma_2 \geqq -1\,.$$

This equation is no longer of delayed type, since $S_{(-1,1)} \neq \emptyset$. With corresponding initial conditions again a unique solution exists, nevertheless it still remains to be not z-complete.

Let now in (4.2.7) the set $A^* = \{(\alpha_1, \alpha_2),\ \alpha_1 \geqq 0,\ \alpha_2 \in \mathbb{Z}\}$ be considered instead of A_0. Evidently $A_0 \subset A^*$. With $G^* = \{(-1, \alpha_2), \alpha_2 \in \mathbb{Z}\}$ we have also $G \subset G^*$ and equation (4.2.7) for $\alpha \in A_0$ could have a unique solution provided initial conditions on the set G^* and values of the input x for $\alpha \in A^*$ are known. On the set $G^* \setminus G$ and $A^* \setminus A_0$, respectively, the initial and input values can be chosen arbitrarily: let us put them all equal to zero. This "new" equations is henceforth characterized by the triple (A^*, B, G^*). Since $A^* + B \setminus A^* = G^* = (A^* + B) \setminus (A^* + (0, -1))$, and $\bigcap_{\beta} (A^* + \beta) = A^*$, the equation is of delayed type. Furthermore $R_{(-1,0)} = G^*$, $R_{(0,-1)} = \emptyset$, and the equation becomes z-complete. With zero initial conditions and with the above accepted definition of x as

$$x(\alpha) = \begin{cases} 1 & \text{for} \quad \alpha \in A_0 \\ 0 & \text{for} \quad \alpha \in A^* \setminus A_0 \end{cases}$$

we have from Theorem 4.2.3

$$F_{A^*}(z)\,(z_1 + z_2) = \frac{1}{(1 - z_1)(1 - z_2)}\,.$$

Note that F is indeed holomorphic in an RCRD (e.g. $|z_1| < |z_2|$, $|z_1| < 1$, $|z_2| < 1$.) To obtain the z_A-transform of the solution of the originally formulated problem, we have to use Theorem 3.1.5 as characterized by formula (3.1.6); this means

$$F(z) = -\frac{1}{4\pi^2} \int_{\Gamma} \frac{\mathrm{d}w_1\, \mathrm{d}w_2}{(w_1 - z_1)(w_2 - z_2)(w_1 + w_2)(1 - w_1)(1 - w_2)}$$

with $\Gamma = \{w_1 = \varrho\, \mathrm{e}^{\mathrm{i}\varphi},\ w_2 = r\, \mathrm{e}^{\mathrm{j}\psi},\ 0 < \varrho,\ r \leqq 1,\ 0 \leqq \varphi, \psi \leqq 2\pi\}$.

After some rather tedious calculation in which we consider $|z_1| < |w_1| < 1$, $|z_2| < |w_2| < 1$ and get

$$F(z) = -\frac{1}{4\pi^2}\int_{\Gamma_2}\frac{1}{(w_2 - z_2)(1 - w_2)}\left(\int_{\Gamma_1}\frac{dw_1}{(w_1 + w_2)(w_1 - z_1)(1 - w_1)}\right)dw_2 =$$

$$= -\frac{2\pi j}{4\pi^2}\int_{\Gamma_2}\frac{1}{(w_2 - z_2)(1 - w_2)}\,\frac{1}{(z_1 + w_2)(1 - z_1)}\,dw_2 =$$

$$= \frac{1}{1 - z_1}\left(\frac{1}{(1 - z_2)(z_1 + z_2)} - \frac{1}{(z_1 + z_2)(1 + z_1)}\right)$$

($w_1 + w_2 \neq 0$ has been assumed, e.g. as $|w_1| < |w_2|$) we obtain

$$F(z) = \frac{1}{(1 - z_1^2)(1 - z_2)},$$

which can easily be verified as the correct answer to the originally formulated problem.

The above described procedure can be generalized at least in cases when the initial set G is a proper subset of $R_B \cup S_B$. This procedure means "to enlarge" the set A to A^* in (4.1.1), so as this equation with A replaced by $A^* \supset A$ becomes z-complete. Henceforth Theorem 4.2.3 can be applied and the use of Theorem 3.1.4 gives the solution of the originally formulated problem.

4.3. Difference equations on semigroups of $\mathbb{Z}^n$

In the previous section we have seen how difference equations satisfying conditions $G \subsetneqq R_B \cup S_B$ can be handled by z-transform. It remains to investigate cases with $G \supset R_B \cup S_B$ and with the undispensable assumption that the set A is an ordered semigroup, $A \subset \mathbb{Z}^n$. A simple example is here equation (4.1.1) with $A = \mathbb{Z}^n$, since in this case both R_B and S_B are empty sets for any choice of B. With given initial values $g(\alpha)$, $\alpha \in G$ we have more information on the solution f than direct application of z-transform can absorb. This may be caused by the fact that z-transform of the unique solution does not exist (as it is the case of 1-D difference equations), or perhaps that some indirect method has to be applied similarly to Example 4.2.4.

In the process of proving Theorem 4.1.1 it has been shown that the set A can always be subdivided into subsets $A^{(i)}$ such that they are well-ordered. It means that after such decomposition, say $A = A^1 \cup A^2$, application of Theorem 4.2.3 might become possible. This may or need not result in the z-transform of the solution of (4.1.1). Let results of this type be shown via the following

Example 4.3.1. Solve the equation

$$f(\alpha_1, \alpha_2) + a f(\alpha_1 - 1, \alpha_2) + b f(\alpha_1, \alpha_2 - 1) +$$
$$+ c f(\alpha_1 - 1, \alpha_2 - 1) = 0$$

for $\alpha_1 \geqq 0$, $\alpha_2 \in \mathbb{Z}$ with $G = \{(-1, q), q \in \mathbb{Z}\} \cup \{(k, -1), k \geqq 0\}$ and $g(\alpha) = 1$ for all $\alpha \in G$.

This is a correctly formulated problem: it has a unique solution. The set $B = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ with $\beta_1 = (0, 0)$, $\beta_2 = (-1, 0)$, $\beta_3 = (0, -1)$, $\beta_4 = (-1, -1)$ and therefore $R_B \cup S_B = \{(-1, q), q \in \mathbb{Z}\}$. We have $S_B = \emptyset$, $G \neq R_B$, $G \supset R_B$; the equation is not z-complete, but now (contrary to the previous example) the initial conditions contain more information than application of z-transform can respect.

Consider $A = A_0 \cup A_1$ with $A_0 = \{(\alpha_1, \alpha_2),\ \alpha_1 \geqq 0,\ \alpha_2 \geqq 0\}$, $A_1 = \{(\alpha_1, \alpha_2),\ \alpha_1 \geqq 0,\ \alpha_2 \leqq -1\}$. For $\alpha \in A_0$ and for the given initial conditions g the equation is z-complete (disregarding, of course, the part of G which does not belong to $A_0 + B$). For its z-transform we obtain

$$F(z) + az_1\left(F(z) + \frac{1}{z_1}\sum_{k=0}^{\infty} z_2^k\right) + bz_2\left(F(z) + \frac{1}{z_2}\sum_{k=0}^{\infty} z_1^k\right) + $$
$$+ cz_1z_2\left(F(z) + \frac{1}{z_1}\sum_{k=0}^{\infty} z_2^k + \frac{1}{z_2}\sum_{k=0}^{\infty} z_1^k\right) + c = 0\,,$$

which yields

$$F_{A_0}(z) = \frac{az_1 + bz_2 - (a + b + c) + cz_1z_2}{(z_1 - 1)(z_2 - 1)(1 + az_1 + bz_2 + cz_1z_2)}\,, \quad |z_1| < 1\,, \quad |z_2| < 1\,.$$

Let now $\alpha \in A_1$. The sets R_β, S_β are as follows:

β	R_β	S_β
$(0,\ 0)$	$\emptyset$	$\emptyset$
$(-1,\ 0)$	$(-1, k);\ k \leqq -1$	$\emptyset$
$(\ 0, -1)$	$\emptyset$	$(k, -1),\ k \geqq 0$
$(-1, -1)$	$(-1, k),\ k \leqq -2$	$(k, -1),\ k \geqq 0$

The equation is again z-complete and for the z-transform F of its solution on the set A_1 we obtain

$$F(z) + az_1\left(F(z) + \frac{1}{z_1}\sum_{k=-1}^{-\infty} z_2^k\right) + bz_2\left(F(z) - \frac{1}{z_2}\sum_{k=0}^{\infty} z_1^k\right) + $$
$$+ cz_1z_2\left(F(z) + \frac{1}{z_1}\sum_{k=-2}^{-\infty} z_2^k - \frac{1}{z_2}\sum_{k=0}^{\infty} z_1^k\right) = 0\,,$$

from where

$$F_{A_1}(z) = \frac{cz_1z_2 + az_1 + bz_2 - (a + b + c)}{(1 - z_1)(z_2 - 1)(1 + az_1 + bz_2 + cz_1z_2)}\,, \quad |z_1| < 1\,, \quad |z_2| > 1\,.$$

Not surprisingly (see Example 2.2.1 and Theorem 2.3.2) the z-transforms of the solution have disjoint regions of convergence.

Since every semigroup $A \subsetneqq \mathbb{Z}^n$ can be subdivided into a disjoint union of well-ordered semigroups the procedure explained in the last example can always be applied.

We may conclude:

The direct applicability of the z-transform, as it is defined by Definition 2.1.1, to solution of correctly formulated initial value problems for n-D difference equations with constant coefficients depends on fulfilment of a certain condition called z-completeness (see Definition 4.1.3). If this condition is not met, i.e. $G \neq S_B \cup R_B$, various situation may arise. Examples show that in cases with $G \subset S_B \cup R_B$ the use of z-transform may become possible for an "amplified" region $A^* \subset \mathbb{Z}^n$ (with a corresponding "new" initial set G^*) instead of the original region A. Such extension could make the equation z-complete. On the other hand, if $G \supset S_B \cup R_B$, then subdivision of the region A into suitably choosen subsets may lead to a small number of z-complete problems, which are readily solvable. Here the most important example is that of $A = \mathbb{Z}^n$, since for every finite set B the corresponding sets S_B and R_B are both empty sets. For $n = 2$ the mentioned subdivision of $\mathbb{Z}^2$ often goes by inspection when the corresponding "initial set" G is taken into account. There exist examples which do not fall into one of these cases and, apparently, no general recommendation of how to apply the z-transform in such cases can be given.

The impact of these conclusions to basic system-theoretical concepts such as the concept of transfer function, impulse response, stability etc., deserves further investigations.

4.4. Systems of n-D difference equations

System of partial difference equations form a natural extension of the preceding section. Moreover, in multidimensional linear systems theory these systems are essential parts of state-space models. Mostly so-called 1-st order systems are considered, although there seems to be no unified and widely accepted definition of such systems. We aim at discussing these systems from the point of view of their solution by n-D z-transform, however, as it has been shown, existence and uniqueness properties of the solution are here essential perliminaries.

In this section mappings x, f, g will be (column-) vectors of dimension m, e.g. $x\colon A \to \mathbb{C}^m$, $A \subset \mathbb{Z}^n$; capitals besides of sets in $\mathbb{Z}^n$ will also denote matrices with real or complex elements. It is believed that these notational convention will not be confusing. We remind the reader that e_i will denote the point of $\mathbb{Z}^n$ with all coordinates equal zero except the ith, which is equal to one. With this notation the general systems of 1-st order linear system of partial difference equations with constant coefficients can be written in the following form

$$Qx(\alpha) = \sum_{i=1}^{n} P_i x(\alpha + \varepsilon_i e_i) + u(\alpha)\,, \quad \alpha \in A \subset \mathbb{Z}^n\,, \tag{4.4.1}$$

where Q, P_i are constant square matrices of order m, $\varepsilon_i = \pm 1$. To simplify what follows, we shall assume that Q and P_i are nonsingular. Therefore we may assume without loss of generality, that any one of the matrices Q, P_i is the unit matrix of corresponding order. Similarly as in Section 4.1, for a given set A a certain initial set C is to be found, so as (4.4.1) has a unique solution. Here it is much simpler to start with a given set G and ask for the corresponding region A, where the recursively computable solution is uniquely determined.

Choose a fixed variable, say the kth, and write (4.4.1) in the following form

$$x(\alpha + \varepsilon_k e_k) = Q^* x(\alpha) - \sum_{\substack{i=1 \\ i \neq k}}^{n} P_i x(\alpha + \varepsilon_i e_i) + P_k^{-1} u(\alpha) . \tag{4.4.2}$$

In the set $\mathbb{Z}^n$, the vector $\varepsilon_k e_k$ can be considered as the normal vector of a hyperplane H to which belong all the vectors $\varepsilon_i e_i$, $i \neq k$. Denote A the half-space induced by this hyperplane; it contains the vector $\varepsilon_k e_k$. Supposing now that values of x in (4.4.2) are given at all points of H, from (4.4.2) the values of $x(\alpha)$ for all $\alpha \in A$ can uniquely be calculated. We have proved the following

Theorem 4.4.1. Let in equation (4.4.1) k be a fixed integer, $1 \leqq k \leqq n$. Denote $G = \{(\alpha_1, \alpha_2, \ldots, \alpha_n)\colon \alpha_k = 0\}$, $A = \{(\alpha_1, \alpha_2, \ldots, \alpha_n)\colon \varepsilon_k \alpha_k \geqq 0\}$. If the values $x(\alpha)$ are given for all $\alpha \in G$, then there exists a unique solution s of (4.4.1) on the set A such that it assumes the given values on the set $G \subset A$.

The assumptions of this theorem are unnecessarily restrictive; in fact, it is sufficient to assume the nonsingularity of the matrix P_k only.

Applying now Theorem 3.2.3 to the situation described above, we obtain the following

Corollary 4.4.2. The z_A-transform of the vector $T^{\varepsilon_i e_i} x$, with the set A described in Theorem 4.4.1 and with $T^{\varepsilon_i e_i}$ being a complete shift operator, can be expressed as follows

$$Z_{A_{(k)}}(T^{\varepsilon_i e_i} x) = z^{-\varepsilon_i e_i}\big(X(z) - \delta_{ik} \sum_{\alpha \in G} x(\alpha)\, z^{\alpha}\big) .$$

Here, $\delta_{ik} = \begin{cases} 0 \text{ for } i \neq k \\ 1 \text{ for } i = k \end{cases}$ and G denotes the initial set as given in Theorem 4.4.1.

From these results we obtain the following equation from (4.4.2):

$$z^{-\varepsilon_k e_k}\big(X(z) - \sum_{\alpha \in G} x(\alpha)\, z^{\alpha}\big) = \big(Q^* - \sum_{\substack{i=1 \\ i \neq k}}^{n} z^{-\varepsilon_i e_i} P_i\big) X(z) + P_k^{-1}\, U(z)$$

or directly from (4.4.1)

$$\big(Q - \sum_{i=1}^{n} z^{-\varepsilon_i e_i} P_i\big) X(z) = U(z) - z^{-\varepsilon_k e_k}\big(\sum_{\alpha \in G} x(\alpha)\, z^{\alpha}\big) P_k . \tag{4.4.3}$$

For "zero initial conditions", i.e. if $x(\alpha) = 0$ for all $\alpha \in G$ we obtain

$$X(z) = \big(Q - \sum_{i=1}^{n} z^{-\varepsilon_i e_i} P_i\big)^{-1}\, U(z) \tag{4.4.4}$$

where there is apparently no connection with the integer value k previously considered. However, we have to keep in mind that $Z^{-1}\{X(z)\}$ has to be understood as a sequence defined on the set A induced by the considered initial set G. As a simple illustration here the last part of Example 4.2.4 could be given.

The above described procedure cannot be used if all the matrices P_i are singular, which occurs in the "difference equation" – parts of most of the state-space models of linear shift invariant systems [11]. With our notation, these systems can be written in the form of (4.4.1), where P_i are diagonal matrices of rank r_i, $r_1 + r_2 + \ldots + r_n = m$ such that $P_i \,.\, P_j = 0$ for all $1 \leqq i, j \leqq n$, $i \neq j$. If $n = 2$, (4.4.1) can be rewritten e.g. as

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} r(\alpha) \\ s(\alpha) \end{bmatrix} = \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r(\alpha + e_1) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} s(\alpha + e_2) \\ 0 \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix},$$

where P_{11} and P_{22} are diagonal and nonsingular matrices. Hence, Theorem 4.4.1 and Corollary 4.4.2 can be applied to obtain the z-transform to the solution for $A = \{\alpha_1 \geqq 0, \alpha_2 \geqq 0\}$ from given initial values $r(0, \alpha_2)$ and $s(\alpha_1, 0)$. Details can be omitted, since the system under consideration falls into the class of z-complete equations.

4.5. Another approach to *n-D* difference equations

While the notation of z-completness is indispensable in solution of partial difference equations, the complete shift operator violating the superposition principle could be eliminated from our consideration if general z-complete difference equations are reduced to those with zero initial conditions. To this end some results on difference equations are necessary, first of all the following

Theorem 4.5.1. Let $f^h \in S_{A+B}$ be the unique solution of

$$\sum_{\beta \in B} a_\beta f(\alpha + \beta) = 0 \tag{4.5.1}$$

satisfying the initial conditions $f^h(\alpha) = g(\alpha)$ for all $\alpha \in G$, where G is an initial set and g a given function of G. Define g^* by $g^*(\alpha) = \begin{cases} g(\alpha) & \text{for } \alpha \in G \\ 0 & \text{otherwise.} \end{cases}$
Let further f be the unique solution of equation

$$\sum_{\beta \in B} a_\beta f(\alpha + \beta) = - \sum_{\beta \in \beta} a_\beta \, g^*(\alpha + \beta)$$

satisfying zero initial conditions on the set G. Then

$$f^h(\alpha) = f(\alpha) \quad \text{for all} \quad \alpha \in (A + B) \setminus G\,.$$

Proof. Denote $h(\alpha) = f^h(\alpha) - f(\alpha) - g^*(\alpha)$. We obtain $h(\alpha) = 0$ for all $\alpha \in G$. Now, h satisfies the equation $\sum a_\beta \, h(\alpha + \beta) = 0$. Indeed, its left-hand side equals

$$\sum a_\beta f^h(\alpha + \beta) - \sum a_\beta f(\alpha + \beta) - \sum a_\beta \, g^*(\alpha + \beta)\,.$$

For $\alpha \in A$ such that $\alpha + \beta \in (A + B) \setminus G$ the first sum equals zero since f^h solves the homogeneous equation (4.5.1) and similarly the remaining two terms yield a zero due to the definition of f. Hence $h(\alpha) \equiv 0$ for all $\alpha \in A + B$, moreover $g^*(\alpha) = 0$ for $\alpha \in (A + B) \setminus G$. Therefore $f^h(\alpha) - f(\alpha) = 0$ for all $\alpha \in (A + B) \setminus G$ which has to be proven.

Corollary 4.5.2. The equation

$$\sum_{\beta \in B} a_\beta f(\alpha + \beta) = x(\alpha)\,, \quad \alpha \in A\,, \quad f(\alpha) = g(\alpha) \quad \text{for all} \quad \alpha \in G$$

and with given $g(\alpha)$, and the equation

$$\sum_{\beta \in B} a_\beta f^*(\alpha + \beta) = x(\alpha) - \sum_{\beta \in B} a_\beta\, g^*(\alpha + \beta)\,, \quad \alpha \in A$$

with zero initial conditions on the set G, have solutions identically equal on the set

$$(A + B) \setminus G\,, \quad \text{i.e.} \quad f(\alpha) = f^*(\alpha) \quad \text{for all} \quad \alpha \in (A + B) \setminus G\,.$$

The proof of this corollary can be given by reasoning similar to that of the previous theorem.

These results, both independent of functional transform consideration, show that under certain well described conditions we may restrict the methods of solution of partial difference equations to those with zero initial conditions. Nonzero initial conditions can always be respected by adequate changes of the input function. Perhaps, it is not superfluous to reformulate the corresponding results of Section 3.2 and Section 4.1.

Let be considered the class of uniquely solvable initial value problems for the equation (4.1.1), i.e. the initial set G for (4.1.1) is specified. If this equation is z-complete (see Definition 4.1.3) and if the solution is supposed to vanish on the set G then a shift operator V^β: $S_A \to S_A$ can be defined as follows: for $f \in S_A$ there is $g = V^\beta f$ given by

$$g(\alpha) = \begin{cases} f(\alpha + \beta) & \text{for} \quad \alpha + \beta \in A \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.5.2}$$

The corresponding Theorem 3.2.3 reads now

$$Z_A(V^\beta f) = z^{-\beta}\, Z_A(f) \tag{4.5.3}$$

and, accordingly, Theorem 4.2.3 can be simplified: instead of (4.2.6) we have

$$F(z) = \frac{X(z)}{\sum_{\beta \in B} \alpha_\beta\, z^{-\beta}} \tag{4.5.4}$$

Although in multidimensional system theory mostly formulae (4.5.3) and (4.5.4) are used, it can be seen that their prerequisites and restrictions are rather severe. On the other hand, Theorem 4.5.1 and its corrollaries throw some light on the origin of the well known problem of BIBO stability of multidimensional systems and its dependence on initial conditions.

5. CONCLUDING REMARKS

In this paper the attention has been focused on such development of the z-transform method, which could be immediately used in solution of initial value problems for partial difference equations. Special results on z-transform of sequences defined on $\mathbb{Z}^n$ were deliberately omitted, since they are sufficiently covered in the literature [4]. This orientation of our efforts made it necessary to formulate some results on partial difference equation which generalize to some extent also the 1-D (ordinary) difference equations and the common procedures of z-transform applications. Some basic formulae and rudiments of a table of transforms are summarized in the Appendix.

The conclusion, that a wide class of linear partial difference equations can be solved by z-transform is certainly not surprising. Here the surprise comes rather from the unexpected difficulties and not always evident methods to obtain established results. It seems that the class of linear shift invariant systems, for which partial difference equations are considered to be their I/O relations, contains much more than systems with transfer functions holomorphic in an open disc.

In this paper system-theoretical corollaries and inverse problems have been left aside. It seems that except for special cases of so-called quarter-plane filters, the way from an I/O relation to the transfer function is still not clear enough and therefore synthesis problems of n-D filters with desired properties encounter so many difficulties. It is hoped that topics here presented will promote solutions of the difficult problems of n-D digital systems synthesis.

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APPENDIX

In Table 1 below a summary of basic formulae can be found. References here are theorems or formulae as numbered in the paper. Table 2 and Table 3 list some examples of the mostly used case $n = 2$. Although the number of these examples could easily be increased, a systematic "dictionary" of transform relations can hardly be given. It is interesting to note that comparatively simple sequences have nonrational z-transforms. In the table of characteristic transforms the depicted regions have to be understood with the "bordering" lines included. It is hoped that this appendix may help the reader to follow the reasoning of the paper.

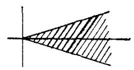
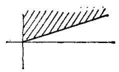
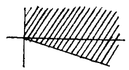
Table 1. Basic relations.

	Region of support	Sequence	Transform	Domain of convergence
(2.1.1)	$A \subset \mathbb{Z}^n$	f	$F(z) = \sum_{\alpha \in A} f(\alpha)\, z^\alpha$	D
(2.1.5)	$\mathbb{Z}^n$	$f(\alpha) = \frac{1}{(2\pi \mathrm{j})^n} \int_\Gamma F(z)\, z^{-\alpha-1}\, \mathrm{d}z$	$F(z)$	D
(2.2.4)	$A \subset \mathbb{Z}^n$	$\alpha_k f(\alpha)$	$z_k \frac{\partial F}{\partial z_k}$	D
(2.2.5)	$A \subset \mathbb{Z}^n$	$\lambda^\alpha f(\alpha),\ \lambda_i \neq 0$	$F(\lambda z)$	$\lambda^{-1} . D$
(3.2.3)	$A \subset \mathbb{Z}^n$	$T^\beta f$	$z^{-\beta}(F(z) + \sum_{\alpha \in R_\beta} f(\alpha)\, z^\alpha - \sum_{\alpha \in S_\beta} f(\alpha)\, z^\alpha)$	D
(2.1.6)	AU, $\det U = 1$	$g(\alpha) = f(\alpha U^{-1})$	$G(w) = F(\omega^U)$	DU^{-1}
(3.1.5)	$A \subset \mathbb{Z}^n$	$f(\alpha)\, g(\alpha)$	$\frac{1}{(2\pi \mathrm{j})^n} \int_\Gamma F\left(\frac{z}{\xi}\right) G(\xi) \frac{\mathrm{d}\xi}{\xi}$	$D \cap H$
(3.1.6)	$B \subset A$	$f(\alpha)$	$\frac{1}{(2\pi \mathrm{j})^n} \int_\Gamma F(\xi)\, W^B\left(\frac{z}{\xi}\right) \frac{\mathrm{d}\xi}{\xi}$	D'

Table 2. Examples of transforms in $\mathbb{Z}^2$.

Region of support	Sequence	Transform	Domain of convergence
$A_0 = \{\alpha \in \mathbb{Z}^2, \alpha_i \geqq 0\}$	λ^α	$\dfrac{1}{(1-\lambda_1 z_1)(1-\lambda_2 z_2)\dots}$	$\lvert z_i\rvert < \dfrac{1}{\lvert\lambda_i\rvert}$
A_0	$\dfrac{1}{1+\alpha_1+\alpha_2}$	$\dfrac{1}{z_1-z_2}\ln\dfrac{1+z_1}{1+z_2}$; $(\ln 1 = 0)$	$\lvert z_i\rvert < 1$
A_0	$\dfrac{1}{(1+\alpha_1+\alpha_2)!}$	$\dfrac{e^{z_1}-e^{z_2}}{z_1-z_2}$	$\mathbb{C}^2$
A_0	$\begin{cases} \dfrac{(-1)^\alpha}{(1+\alpha_1+\alpha_2)!} & \text{for } \alpha_1+\alpha_2 \text{ even} \\ 0 & \text{for } \alpha_1+\alpha_2 \text{ odd} \end{cases}$	$\dfrac{\sinh z_1 + \sinh z_2}{z_1+z_2}$	$\mathbb{C}^2$
A_0	$\dfrac{(-1)^{\alpha_1+\alpha_2}}{1+\alpha_1+\alpha_2}$	$\dfrac{1}{z_1-z_2}\ln\dfrac{1-z_1}{1-z_2}$; $(\ln 1 = 0)$	$\lvert z_i\rvert < 1$
$\{\alpha \in \mathbb{Z}^2, \alpha_1 \geqq 0, \alpha_1 \geqq \alpha_2 \geqq 0\}$	$\dbinom{\alpha_1}{\alpha_2}$	$\dfrac{1}{1-z_1-z_1 z_2}$	$D \subset \{\lvert z_1(1+z_2)\rvert < 1\}$
A_0	$\dfrac{1}{k^{\alpha_1+\alpha_2}}\dbinom{\alpha_1+\alpha_2}{\alpha_2}$; $(k \in \mathbb{Z}, k \neq 0)$	$\dfrac{k}{k-z_1-z_2}$	$D \subset \{\lvert z_1+z_2\rvert < \lvert k\rvert\}$
A_0	$\dfrac{\alpha_1+\alpha_2}{k^{\alpha_1+\alpha_2+1}}\dbinom{\alpha_1+\alpha_2}{\alpha_1}$; $(k \in \mathbb{Z}, k \neq 0)$	$\dfrac{z_1+z_2}{(k-z_1-z_2)^2}$	$D \subset \{\lvert z_1+z_2\rvert < \lvert k\rvert\}$
A_0	$\dfrac{\alpha_1\alpha_2}{k^{\alpha_1+\alpha_2+1}}\dbinom{\alpha_1+\alpha_2}{\alpha_1}$; $(k \in \mathbb{Z}, k \neq 0)$	$\dfrac{2z_1 z_2}{(k-z_1-z_2)^3}$	$D \subset \{\lvert z_1+z_2\rvert < \lvert k\rvert\}$

Table 3. Characteristic transforms of regions in $\mathbb{Z}^2$.

Region		Transform	Domain of convergence
$A_0 = \{\alpha \in \mathbb{Z}^n, \alpha_i \geqq 0\}$	1-st n-tant	$\dfrac{1}{(1-z_1)(1-z_2)\dots(1-z_n)}$	$D\colon \lvert z_i\rvert < 1;\ i = 1, 2, \dots, n$
$\{\alpha \in \mathbb{Z}^2, \alpha_1 \geqq 0, \lvert\alpha_2\rvert \leqq \alpha_1\}$		$\dfrac{z_2(1+z_1)}{(1-z_1z_2)(z_2-z_1)}$	$\lvert z_1z_2\rvert < 1;\ \lvert z_1\rvert < \lvert z_2\rvert$
$\{\alpha \in \mathbb{Z}^2, \alpha_1 \geqq 0, \alpha_2 > \alpha_1\}$		$\dfrac{z_2}{(1-z_2)(1-z_1z_2)}$	$\lvert z_2\rvert < 1;\ \lvert z_1z_2\rvert < 1$
$\{\alpha \in \mathbb{Z}^2, \alpha_1 \geqq 0, \alpha_2 \geqq -\alpha_1\}$		$\dfrac{z_2}{(1-z_2)(z_2-z_1)}$	$\lvert z_2\rvert < 1;\ \lvert z_1\rvert < \lvert z_2\rvert$
$B = A_0U, \det U = 1$		$\dfrac{1}{1-z^u}$	$z \in D^{u^{-1}}$