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ON THE CONTINUOUS DEPENDENCE OF TRAJECTORIES OF BILINEAR SYSTEMS ON CONTROLS AND ITS APPLICATIONS

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Special representation of dependence of trajectories of a bilinear time dependent system on controls is obtained. On the basis of this representation an estimate for continuous dependence of trajectories of a bilinear time dependent system with single input on controls is developed. In the last section a numerical method for determination of optimal control in problems with constant parameters and fixed time interval is suggested. This method is a modification of the well-known gradient projection method and employs the developed estimate. Illustrative examples are given.

1. INTRODUCTION

Let us consider the following control system:

$$\dot{x} = A(t)\,x + B(t)\,u(t)\,x + c(t)\,u(t) + f(t)\,, \qquad x(t_0) = x_0\,, \quad t \in [t_0, t_1] \subset R\,. \tag{1}$$

In the sequel, (1) will be called the bilinear time dependent system with single input (BTDSSI). Here $A(t)$, $B(t)$ are $(n \times n)$-dimensional matrix-valued functions, $c(t)$, $f(t)$ are functions with values in R^n. The scalar control (or input) $u(t)$ is assumed to be a measurabe function on every finite time interval $[t_0, t_1]$ such that for given real numbers $u_{\min}$, $u_{\max}$ $u(t) \in [u_{\min}, u_{\max}]$ almost everywhere (a.e.) on $[t_0, t_1]$. Such a control will be called admissible. Finally, $x \in R^n$ is the vector of state variables and $x_0 \in R^n$ is the given initial state of the system.

The aim of this contribution is twofold. First, to generalize the results of [3]. Second, to give some more elaborated examples of possible applications of this kind of results. In [3] a time independent bilinear system was considered, namely, the system (1) with $A(t) \equiv A$, $B(t) \equiv B$, $c(t) \equiv c$ and $f(t) \equiv 0$. For this case a representation of dependence of trajectories of system (1) on controls was obtained. On the basis of this representation the following estimate for continuous dependence of trajectories of system (1) on controls in certain norms was derived:

$$\max_{t \in [t_0, t_1]} \|x_1(t) - x_2(t)\|_{R^n} \leqq K \max_{t \in [t_0, t_1]} \left|\int_{t_0}^{t_1} (u_1(s) - u_2(s))\,\mathrm{d}s\right|\,. \tag{2}$$

Here $x_1(t)$ and $x_2(t)$ are solutions of the system (1) for controls $u_1(t)$ and $u_2(t)$, respectively, K is a constant depending only on the parameters of system (1).

In this paper validity of the above facts is extended to the case of BTDSSI under some additional assumptions. Note that estimate (2) in fact establishes a Lipschitzian dependence of the trajectories of BTDSSI on controls in norms $\max_{t\in[t_0,t_1]} \|x(t)\|_{R^n}$ and $\max_{t\in[t_0,t_1]} |\int_{t_0}^t u(\alpha)\, d\alpha|$. Estimates of type (2) are useful in order to study bilinear control systems as it was shown in [3].

In this contribution also a numerical method for searching the optimal control of time independent bilinear systems is described, which is a modification of the well-known gradient projection method. This modification is based on estimate (2) and allows to obtain such an approximation of optimal control which has only two values: $u_{\min}$ and $u_{\max}$ (although a very high number of switchings may occur).

2. ANALYTICAL REPRESENTATION OF DEPENDENCE OF TRAJECTORIES OF BTDSSI ON CONTROLS

Let us introduce the following notation

$$w(t) = \int_{t_0}^t u(s)\, ds\,. \tag{3}$$

Let us remind that $\exp(F)$ (or e^F) denotes matrix-valued functions of an $(n \times n)$-matrix argument F defined by:

$$e^F = \exp(F) = \sum_{k=0}^{\infty} \frac{F^k}{k!}\,. \tag{4}$$

Our aim in this section is to find a special representation of $x(t)$, the solution of (1) for a given $u(t)$, which allows us to obtain the estimate (2). First we construct a special representation for the fundamental matrix $\Phi(t)$ of system (1) and its inverse $\Phi^{-1}(t)$. Let us recall that the fundamental matrix $\Phi(t)$ of system (1) is the solution of the following matrix differential equation

$$\frac{d}{dt} X(t) = (A(t) + B(t)\, u(t))\, X(t)\,, \quad X(t_0) = I \tag{5}$$

and its inverse matrix $\Phi^{-1}(t)$ is the solution of

$$\frac{d}{dt} Y(t) = -Y(t)\,(A(t) + B(t)\, u(t))\,, \quad Y(t_0) = I\,, \tag{6}$$

where I denotes the $(n \times n)$-dimensional identity matrix.

We say that a matrix-valued function is integrable (absolutely continuous) if each of its elements exhibits the appropriate property.

Throughout this paper it is assumed that $B(t)$ is absolutely continuous and the norm of its derivative is almost everywhere on $[t_0, t_1]$ bounded by a constant

B^{DM}, $0 \leqq B^{DM} < \infty$. Moreover, it is assumed that for every $t', t'' \in [t_0, t_1]$ the matrices $B(t')$ and $B(t'')$ commute, that is, $B(t')\,B(t'') = B(t'')\,B(t')$. This second requirement is necessary in order to have possibility to represent the solution of the system $\dot{y} = B(t)\,u(t)\,y$, $y(t_0) = y_0$, as $y(t) = \exp\left(\int_{t_0}^{t} B(s)\,u(s)\,\mathrm{d}s\right) y_0$. It is fulfilled, e.g. when $B(t) = g(t)\,B$, where B is a certain matrix and $g(t)$ is a scalar function.

Theorem 1. Let us consider system (1), where $B(t)$ satisfies the above conditions and $A(t)$ is a function integrable on certain time interval $[t_0, t_1]$. Then we can represents the fundamental matrix of this system and its inverse on $[t_0, t_1]$ as the sums of the following infinite series:

$$(7) \qquad \Phi(t) = \exp\left(\int_{t_0}^{t} B(s)\,u(s)\,\mathrm{d}s\right)\left(I + \sum_{k=1}^{\infty} \int_{t_0}^{t=\tau_{k+1}} \ldots \int_{t_0}^{\tau_2} \prod_{j=1}^{k} \{\exp\left(-\int_{t_0}^{\tau_{k+1-j}} B(s)\,.\right.\right.$$

$$\left.\left. .\, u(s)\,\mathrm{d}s\right) A(\tau_{k+1-j}) \exp\left(\int_{t_0}^{\tau_{k+1-j}} B(s)\,u(s)\,\mathrm{d}s\right)\}\,\mathrm{d}\tau_1 \ldots \mathrm{d}\tau_k\right)$$

and

$$(8) \qquad \Phi^{-1}(t) = \left(I + \sum_{k=1}^{\infty} \int_{t_0}^{t=\tau_{k+1}} \ldots \int_{t_0}^{\tau_2} \prod_{j=1}^{k} \left(\exp\left(-\int_{t_0}^{\tau_j} B(s)\,u(s)\,\mathrm{d}s\right)\left(-A(\tau_j)\right)\,.\right.\right.$$

$$\left.\left. .\exp\left(\int_{t_0}^{\tau_j} B(s)\,u(s)\,\mathrm{d}s\right)\right\}\,.\,\mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n\right) \exp\left(-\int_{t_0}^{t} B(s)\,u(s)\,\mathrm{d}s\right).$$

Proof. Let us consider the following sequence of matrix-valued functions of real variable t $\{\Phi^i(t)\}_{i=0}^{\infty}$:

$$\Phi^0(t) = \exp\left(\int_{t_0}^{t} B(s)\,u(s)\,\mathrm{d}s\right),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\Phi^{i+1}(t) = B(t)\,u(t)\,\Phi^{i+1}(t) + A(t)\,\Phi^i(t)\,, \quad \Phi^{i+1}(t_0) = I\,,$$

that is

$$\Phi^{i+1}(t) = \exp\left(\int_{t_0}^{t} B(s)\,u(s)\,\mathrm{d}s\right)\left(I + \int_{t_0}^{t} \exp\left(-\int_{t_0}^{s} B(\alpha)\,u(\alpha)\,\mathrm{d}\alpha\right) A(s)\,\Phi^i(s)\,\mathrm{d}s\right),$$

or

$$\Phi^{i+1}(t) = \exp\left(\int_{t_0}^{t} B(s)\,u(s)\,\mathrm{d}s\right)\left(I + \sum_{k=1}^{i+1} \int_{t_0}^{t=\tau_{k+1}} \ldots \int_{t_0}^{\tau_2} \left(\prod_{j=1}^{k} \exp\left(-\int_{t_0}^{\tau_{k+1-j}} B(s)\,u(s)\,\mathrm{d}s\right)\,.\right.\right.$$

$$\left.\left. .\, A(\tau_{k+1-j}) \exp\left(\int_{t_0}^{\tau_{k+1-j}} B(s)\,u(s)\,\mathrm{d}s\right)\right)\mathrm{d}\tau_1 \ldots \mathrm{d}\tau_k\right).$$

For the solution of (5) it holds clearly:

$$\Phi(t) = \exp\left(\int_{t_0}^{t} B(s)\,u(s)\,\mathrm{d}s\right) + \int_{t_0}^{t} \exp\left(\int_{s}^{t} B(s)\,u(s)\,\mathrm{d}s\right) A(s)\,\Phi(s)\,\mathrm{d}s\,.$$

Hence

$$\|\Phi^{i+1}(t) - \Phi(t)\|_S \leqq \int_{t_0}^{t} \mathrm{e}^{B^M u_p (t-s)} A^M \|\Phi(s) - \Phi^i(s)\|_s\,\mathrm{d}s \leqq$$

$$\leqq M \int_{t_0}^{t} \|\Phi(s) - \Phi_i(s)\|_S\,\mathrm{d}s \leqq \frac{(M(t-t_0))^{i+1}}{(i+1)!} \max_{t \in [t_0, t_1]} \|\Phi(t) - \Phi^0(t)\|_S\,.$$

Here $\|\cdot\|_S$ denotes the spectral matrix norm,

$$M = \mathrm{e}^{B^M u_p (t_1 - t_0)} A^M\,, \quad B^M = \max_{t \in [t_0, t_1]} \|B(t)\|_S\,, \quad A^M = \max_{t \in [t_0, t_1]} \|A(t)\|_S\,.$$

So we can see that in the spectral matrix norm the sequence $\{\Phi^i\}_{i=0}^{\infty}$ converges to the solution of (5) as i tends to infinity. On the other hand, it is obvious that the series on the right hand side of (7) is just the limit of the sequence $\{\Phi^i\}_{i=0}^{\infty}$. Hence representation (7) has been proved.

In the same way we can prove representation (8), the only difference is that the appropriate sequence $\{\Phi_i^{-1}\}_{i=0}^{\infty}$ is of the form:

$$\Phi_0^{-1}(t) = \exp\left(-\int_{t_0}^t B(s)\, u(s)\, \mathrm{d}s\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \Phi_{i+1}^{-1} = \Phi_{i+1}^{-1}\left(-B(t)\, u(t)\right) - \Phi_i^{-1}\, A(t)\,, \quad \Phi_{i+1}^{-1}(t_0) = I\,. \qquad \square$$

The following theorem gives a representation of the solution $x(t)$ of system (1) in which $x(t)$ depends only on $w(t)$, $t \in [t_0, t_1]$, where $w(t)$ is given by (3).

Theorem 2. Let us consider system (1) with a given initial state $x(t_0) = x_0$ for the control $u(t)$. Let us assume in addition to the conditions of Theorem 1 that $c(t)$ is absolutely continuous and $f(t)$ is an integrable function on $[t_0, t_1]$. Then for the solution $x(t)$ of system (1) the following formula holds:

$$\begin{aligned} (9) \qquad x(t) = {} & \Phi(t)\left(x_0 + \int_{t_0}^t \Phi^{-1}(s)\, A(s)\, E_w(s) \exp\left(B(s)\, w(s)\right) c(s)\, \mathrm{d}s - \right. \\ & \left. - \int_{t_0}^t \Phi^{-1}(s)\, E_w(s) \exp\left(B(s)\, w(s)\right)\left(B'(s)\, w(s)\, c(s) + c'(s)\right) \mathrm{d}s\right) + \\ & + E_w(t) \exp\left(B(t)\, w(t)\right) c(t) + \Phi(t) \int_{t_0}^t \Phi^{-1}(s) f(s)\, \mathrm{d}s\,, \end{aligned}$$

where $E_w(s)$ is the following matrix-valued function of a real variable s (depending on $w(\alpha)$, $\alpha \in [t_0, t_1]$ given by (3)):

$$(10) \qquad E_w(s) = \sum_{i=0}^{\infty} \frac{B^i(s)\, w(s)^{i+1}}{(i+1)!}\, (-1)^i + \sum_{i=1}^{\infty} i \int_{t_0}^{s} \left(-B(\alpha)\right)^{i-1} B'(\alpha)\, \frac{w(\alpha)^{i+1}}{(i+1)!}\, \mathrm{d}\alpha$$

and $\Phi(t)$, $\Phi^{-1}(t)$ are given by (7) and (8). Moreover, as we can easily see, it is possible to replace everywhere in (7) and (8):

$$(11) \qquad \int_{t_0}^{\tau_i} B(s)\, u(s)\, \mathrm{d}s = B(\tau_i)\, w(\tau_i) - \int_{t_0}^{\tau_i} B'(\alpha)\, w(\alpha)\, \mathrm{d}\alpha\,.$$

Proof. Let us first remark that the infinite series on the right hand side of (10) evidently converge for any real matrix $B(\alpha)$, $\alpha \in [t_0, t_1]$, real number s and function $w(\alpha)$, $\alpha \in [t_0, t_1]$. Furthermore, the following relations hold:

$$(12) \qquad \frac{\mathrm{d}}{\mathrm{d}s}\, E_w(s) = u(s) \exp\left(-B(s)\, w(s)\right), \quad E_w(t_0) = 0\,.$$

As it is known from the theory of ordinary differential equations (see e.g. [11], p. 135) the solution $x(t)$ of system (1) has the following form:

$$(13) \qquad x(t) = \Phi(t)\left(x_0 + \int_{t_0}^t \Phi^{-1}(s)\left(c(s)\, u(s) + f(s)\right) \mathrm{d}s\,.\right.$$

Here $\Phi(t)$ is again the fundamental matrix of (1). From (8) it follows that we can write

$$(14) \qquad \Phi^{-1}(t) = \Phi_A(t) \exp\left(-\int_{t_0}^t B(\alpha)\, w(\alpha)\, \mathrm{d}\alpha\right),$$

where

$$(15) \qquad \Phi_A(t) = I + \sum_{k=1}^{\infty} \int_{t_0}^{t=\tau_{k+1}} \dots \int_{t_0}^{\tau_2} \prod_{i=1}^{k} \{\exp\left(-\int_{t_0}^{\tau_i} B(\alpha)\, u(\alpha)\, \mathrm{d}\alpha\right)\left(-A(\tau_i)\right) . \\ . \exp\left(\int_{t_0}^{\tau_i} B(\alpha)\, u(\alpha)\, \mathrm{d}\alpha\right)\}\, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_k .$$

By direct evaluation we obtain that

$$(16) \qquad \frac{\mathrm{d}}{\mathrm{d}t} \Phi_A(t) = -\Phi_A(t) \exp\left(-\int_{t_0}^{t} B(\alpha)\, u(\alpha)\, \mathrm{d}\alpha\right) A(t) \exp\left(\int_{t_0}^{t} B(\alpha)\, u(\alpha)\, \mathrm{d}\alpha\right),$$

$$(17) \qquad \Phi_A(t_0) = I .$$

Using relations (11)–(12) and (14)–(17) we have

$$\int_{t_0}^{t} \Phi^{-1}(s)\, c(s)\, u(s)\, \mathrm{d}s = \int_{t_0}^{t} \Phi_A(s)\, u(s) \exp\left(-\int_{t_0}^{s} B(\alpha)\, u(\alpha)\, \mathrm{d}\alpha\right) c(s)\, \mathrm{d}s =$$
$$= \int_{t_0}^{t} \Phi_A(s)\, u(s) \exp\left(-B(s)\, w(s)\right) \exp\left(\int_{t_0}^{s} B'(\alpha)\, w(\alpha)\, \mathrm{d}\alpha\right) c(s)\, \mathrm{d}s =$$
$$= \int_{t_0}^{t} \Phi_A(s) \left(\frac{\mathrm{d}}{\mathrm{d}s} E_w(s)\right) \exp\left(\int_{t_0}^{s} B'(\alpha)\, w(\alpha)\, \mathrm{d}\alpha\right) c(s)\, \mathrm{d}s =$$
$$= \Phi_A(t)\, E_w(t) \exp\left(\int_{t_0}^{t} B'(\alpha)\, w(\alpha)\, \mathrm{d}\alpha\right) c(t) - \int_{t_0}^{t} \left(-\Phi_A(s)\right) \exp\left(-\int_{t_0}^{s} B(\alpha)\, u(\alpha)\, \mathrm{d}\alpha\right) .$$
$$. A(s) \exp\left(\int_{t_0}^{s} B(\alpha)\, u(\alpha)\, \mathrm{d}\alpha\right) E_w(s) \exp\left(\int_{t_0}^{s} B'(\alpha)\, w(\alpha)\, \mathrm{d}\alpha\right) c(s)\, \mathrm{d}s -$$
$$- \int_{t_0}^{t} \Phi_A(s)\, E_w(s)\, B'(s)\, w(s) \exp\left(\int_{t_0}^{s} B'(\alpha)\, w(\alpha)\, \mathrm{d}\alpha\right) c(s)\, \mathrm{d}s -$$
$$- \int_{t_0}^{t} \Phi_A(s)\, E_w(s) \exp\left(\int_{t_0}^{s} B'(\alpha)\, w(\alpha)\, \mathrm{d}\alpha\right) c'(s)\, \mathrm{d}s .$$

The last equality is integration by parts. Let us remark that under the conditions of Theorem 1

$$\exp\left(\int_{t_0}^{s} B'(\alpha)\, w(\alpha)\, \mathrm{d}\alpha\right); \quad \exp\left(B(s)\, w(s)\right), \quad B'(s), \quad E_w(s)$$

commute evidently with each other. So we can write

$$(18) \qquad \int_{t_0}^{t} \Phi^{-1}(s)\, c(s)\, u(s)\, \mathrm{d}s = \Phi_A(t)\, E_w(t) \exp\left(\int_{t_0}^{t} B'(\alpha)\, w(\alpha)\, \mathrm{d}\alpha\right) c(t) +$$
$$+ \int_{t_0}^{t} \Phi^{-1}(s)\, A(s)\, E_w(s) \exp\left(B(s)\, w(s)\right) c(s)\, \mathrm{d}s - \int_{t_0}^{t} \Phi^{-1}(s)\, E_w(s) .$$
$$. \exp\left(B(s)\, w(s)\right)\left(B'(s)\, w(s)\, c(s) + c'(s)\right) \mathrm{d}s .$$

When the substitution from (18) into the right hand side of (13) is performed taking into account (14) and (11) we obtain representation (9). Theorem 2 is proved. □

Remark 1. Analyzing relations (7)–(11) we can find that $x(t)$ depends on $u(\alpha)$, $\alpha \in [t_0, t_1]$, explicitly only through the function $w(s) = \int_{t_0}^{s} u(\alpha)\, \mathrm{d}\alpha$, $s \in [t_0, t_1]$. This fact will be employed in Theorem 3 which establishes estimate (2).

Remark 2. Note that there are no requirements on the commutativity between $A(t)$ and $B(t)$, so the assumptions of Theorem 2 may be considered to be quite general.

3. ESTIMATE FOR CONTINUOUS DEPENDENCE OF TRAJECTORIES OF BTDSSI ON CONTROLS

In this section estimate (2) will be derived.

Theorem 3. Let us consider the BTDSSI (1) defined on the time interval $[t_0, t_1]$. We impose the following assumptions:
1) $B(t)$, $c(t)$ are absolutely continuous and almost everywhere on $[t_0, t_1]$

$$\|B(t)\|_S \leqq B^M < \infty,$$
$$\|B'(t)\|_S \leqq B^{DM} < \infty,$$
$$\|c(t)\|_{R^n} \leqq c^M < \infty,$$
$$\|c'(t)\|_{R^n} \leqq c^{DM} < \infty.$$

2) $A(t)$, $f(t)$ are essentially bounded measurable functions and almost everywhere on $[t_0, t_1]$

$$\|A(t)\|_S \leqq A^M < \infty, \quad \|f(t)\|_{R^n} \leqq f^M < \infty.$$

Let $x(t_0) = x_0$ be the initial state of system (1) and let $x_1(t)$ and $x_2(t)$ be trajectories of this system for admissible controls $u_1(t)$ and $u_2(t)$, respectively. Then estimate (2) is valid, where

$$(19) \qquad K = K_1K_2\|x_0\|_{R^n} + 2K_1K_2^2(K_3 + K_4K_5u_p(t_1 - t_0))\,K_4K_6(t_1 - t_0) + \\ + K_2^2K_4(1 + K_4K_5(1 + B^Mu_p(t_1 - t_0)))\,K_6(t_1 - t_0) + \\ + K_2^2K_4(K_3 + K_4K_5u_p(t_1 - t_0))\,B^{DM}c^M(t_1 - t_0) + \\ + K_4(1 + K_4K_5(1 + B^Mu_p(t_1 - t_0)))\,c^M + 2K_1K_2^2(t_1 - t_0)\,f^M.$$

Hete we used the notation

$$(20) \qquad K_1 = (B^M + (t_1 - t_0)\,B^{DM})(1 + A^M(t_1 - t_0)) + B^MA^M(t_1 - t_0)$$
$$(21) \qquad K_2 = e^{((A^M + B^Mu_p)(t_1 - t_0))}$$
$$(22) \qquad K_3 = \{e^{(B^Mu_p(t_1 - t_0))} - 1\}/B^M$$
$$(23) \qquad K_4 = e^{(B^Mu_p(t_1 - t_0))}$$
$$(24) \qquad K_5 = u_p(t_1 - t_0)^2\,B^{DM}$$
$$(25) \qquad K_6 = A^Mc^M + B^{DM}c^Mu_p(t_1 - t_0) + c^{DM}$$
$$(26) \qquad u_p = \max\{|u_{\min}|, |u_{\max}|\}.$$

Finally, $\|\cdot\|_S$ stands for the spectral matrix norm and $\|\cdot\|_{R^n}$ for the Euclidean vector norm in R^n.

Proof. Let us denote the fundamental matrices of system (1) for control $u_1(t)$ and $u_2(t)$ by $\Phi_1(t)$ and $\Phi_2(t)$, respectively. First we establish the estimates for

$$\max_{t_0 \leqq t \leqq t_1} \|\Phi_1(t) - \Phi_2(t)\|_S \quad \text{and} \quad \max_{t_0 \leqq t \leqq t_1} \|\Phi_1^{-1}(t) - \Phi_2^{-1}(t)\|_S.$$

It is easily verified that for any square matrices $X_1, X_2, \ldots, X_k, Y_1, Y_2, \ldots, Y_k$ the following identity holds

(27) $$\prod_{i=1}^{k} X_i - \prod_{i=1}^{k} Y_i = \sum_{i=1}^{k} \left(\prod_{j=1}^{i-1} Y_j\right)(X_i - Y_i)\left(\prod_{j=i+1}^{k} X_j\right).$$

(We define that $\prod_{i=p}^{q} D_i = 1$ for $q < p$.)

Using this identity, formulas (7) and (11) we obtain:

(28) $$\Phi_1(t) - \Phi_2(t) = F(t) + \sum_{k=1}^{\infty} \int_{t_0}^{t=\tau_{k+1}} \ldots \int_{t_0}^{\tau_2} \sum_{i=1}^{k} \left(\prod_{j=0}^{i-2} D_2^{k-j} A(\tau_{k-j})\right) .$$
$$. \left(D_1^{k-i+1} - D_2^{k-i+1}\right) A(\tau_{k-i+1}) \left(\prod_{j=i}^{k-1} D_1^{k-j} A(\tau_{k-j})\right) \exp\left(B(\tau_1)\, w(\tau_1) -\right.$$
$$\left. - \int_{t_0}^{\tau_1} B(\alpha)\, w(\alpha)\, \mathrm{d}\alpha\right) \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_k +$$
$$+ \sum_{k=1}^{\infty} \int_{t_0}^{t=\tau_{k+1}} \ldots \int_{t_0}^{\tau_2} \left(\prod_{i=0}^{k-1} D_2^{k-i} A(\tau_{k-i})\right) F(\tau_1)\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_k ,$$

where

$$D_p^q = \exp\left(\int_{\tau_q}^{\tau_{q+1}} B(s)\, u_p(s)\, \mathrm{d}s\right) = \exp\left(B(\tau_{q+1})\, w_p(\tau_{q+1}) - B(\tau_q)\, w_p(\tau_q) -\right.$$
$$\left. - \int_{\tau_q}^{\tau_{q+1}} B'(s)\, w_p(s)\, \mathrm{d}s\right),$$
$$F(s) = \exp\left(B(s)\, w_1(s) - \int_{t_0}^{s} B'(\alpha)\, w_1(\alpha)\, \mathrm{d}\alpha\right) -$$
$$- \exp\left(B(s)\, w_2(s) - \int_{t_0}^{s} B'(\alpha)\, w_2(\alpha)\, \mathrm{d}\alpha\right).$$

Further

$$\left\|D_1^q - D_2^q\right\|_S \leqq \max_{0 \leqq \Theta \leqq 1} \left\{\left\|\exp\left(W^*\right)\right\|_S\right\} B(\tau_{q+1})\, w_1(\tau_{q+1}) - B(\tau_q)\, w_1(\tau_q) +$$
$$+ \int_{\tau_{q+1}}^{\tau_q} B'(s)\, w_1(s)\, \mathrm{d}s - B(\tau_{q+1})\, w_2(\tau_{q+1}) + B(\tau_q)\, w_2(\tau_q) - \int_{\tau_{q+1}}^{\tau_q} B'(s) w_2(s)\, \mathrm{d}s \big| ,$$

where

$$W^* = (1 - \Theta) \int_{\tau_q}^{\tau_{q+1}} B(s)\, u_1(s)\, \mathrm{d}s + \Theta \int_{\tau_q}^{\tau_{q+1}} B(s)\, u_2(s)\, \mathrm{d}s .$$

So we can write

$$\left\|W^*\right\|_S \leqq (1 - \Theta)\, B^M u_p \left|\tau_{q+1} - \tau_q\right| + \Theta B^M u_p \left|\tau_{q+1} - \tau_q\right| = B^M u_p \left|\tau_{q+1} - \tau_q\right| .$$

Hence

$$\left\|D_1^q - D_2^q\right\|_S \leqq \mathrm{e}^{B^M u_p |\tau_{q+1} - \tau_q|} . \left\{B^M\left(\left|w_1(\tau_{q+1}) - w_2(\tau_{q+1})\right| + \left|w_1(\tau_q) - w_2(\tau_q)\right|\right) +\right.$$
$$\left. + B^{DM} \left|\tau_{q+1} - \tau_q\right| \max_{s \in [\tau_q, \tau_{q+1}]} \left|w_1(s) - w_2(s)\right|\right\} .$$

Now we can write the estimate

(29) $$\left\|D_1^q - D_2^q\right\|_S \leqq \mathrm{e}^{B^M u_p |\tau_{q+1} - \tau_q|} \left(2B^M + B^{DM} \left|\tau_{q+1} - \tau_q\right|\right) .$$
$$\max_{t \in [t_0, t_1]} \left|w_1(t) - w_2(t)\right| .$$

Moreover,

(30) $$\left\|D_q^p\right\|_S \leqq \mathrm{e}^{B^M |\tau_{q+1} - \tau_q| u_p} .$$

In the same way as for $\|D_1^q - D_2^q\|_S$ we can obtain the estimate for $\|F(s)\|_S$

(31) $$\|F(s)\|_S \leqq e^{B^M u_p |s - t_0|}(B^M + (s - t_0) B^{DM}) \max_{t \in [t_0, t_1]} |w_1(t) - w_2(t)| .$$

Using relations (28)–(31) we obtain

$$\|\Phi_1(t) - \Phi_2(t)\|_S \leqq \{e^{B^M u_p(t_1 - t_0)}(B^M + (t_1 - t_0) B^{DM}) +$$
$$+ (2B^M + B^{DM}(t_1 - t_0)) \sum_{k=1}^{\infty} k \int_{t_0}^{\tau_{k+1}} \dots \int_{t_0}^{\tau_2} e^{B^M u_p \sum_{q=1}^{k} |\tau_{q+1} - \tau_q|} (A^M)^k e^{B^M u_p(\tau_1 - t_0)} d\tau_1 \dots d\tau_k$$
$$+ (B^M + (t_1 - t_0) B^{DM}) \sum_{k=1}^{\infty} \int_{t_0}^{\tau_{k+1}} \dots \int_{t_0}^{\tau_2} e^{B^M u_p \sum_{q=1}^{k} |\tau_{q+1} - \tau_q|} (A^M)^k .$$
$$. e^{B^M u_p(\tau_1 - t_0)} d\tau_1 \dots d\tau_k\} \max_{t \in [t_0, t_1]} |w_1(t) - w_2(t)| .$$

Let us observe that $\tau_{k+1} \geqq \tau_k \geqq \dots \geqq \tau_2 \geqq \tau_1$, hence

$$\sum_{q=1}^{k} |\tau_{q+1} - \tau_q| = \tau_{k+1} - \tau_1 = t - \tau_1 .$$

Thus

$$\|\Phi_1(t) - \Phi_2(t)\|_S \leqq e^{B^M u_p(t_1 - t_0)} \Big\{ (B^M + (t_1 - t_0) B^{DM}) + (2B^M + B^{DM}(t_1 - t_0)) .$$
$$. \sum_{k=1}^{\infty} k (A^M)^k \frac{(t_1 - t_0)^k}{k!} + (B^M + (t_1 - t_0) B^{DM}) \sum_{k=1}^{\infty} (A^M)^k \frac{(t_1 - t_0)^k}{k!} \Big\} \max_{t \in [t_0, t_1]} |w_1(t) - w_2(t)| =$$
$$= e^{B^M u_p(t_1 - t_0)} \Big\{ e^{A^M (t_1 - t_0)}(B^M + (t_1 - t_0) B^{DM}) + (2B^M + (t_1 - t_0) B^{DM}) .$$
$$. (A^M(t_1 - t_0)) \sum_{k=1}^{\infty} (A^M)^{k-1} \frac{(t_1 - t_0)^{k-1}}{(k-1)!} \Big\} \max_{t \in [t_0, t_1]} |w_1(t) - w_2(t)| .$$

Finally, we can write the following estimate

(32) $$\max_{t \in [t_0, t_1]} \|\Phi_1(t) - \Phi_2(t)\|_S \leqq K_1 K_2 \max_{t \in [t_0, t_1]} |w_1(t) - w_2(t)| ,$$

where K_1, K_2 are given by (20) and (21).

Analogously as for $\Phi(t)$ we can obtain the estimate for $\Phi^{-1}(t)$:

(33) $$\max_{t \in [t_0, t_1]} \|\Phi_1^{-1}(t) - \Phi_2^{-1}(t)\|_S \leqq K_1 K_2 \max_{t \in [t_0, t_1]} |w_1(t) - w_2(t)| .$$

We can also see that

(34) $$\max_{t \in [t_0, t_1]} \|\Phi(t)\|_S \leqq K_2$$

and

(35) $$\max_{t \in [t_0, t_1]} \|\Phi^{-1}(t)\|_S \leqq K_2 .$$

By similar arguments it follows from (10):

$$\|E_w(s)\|_S \leqq (e^{B^M u_p(s - t_0)} - 1)/B^M + e^{B^M u_p(s - t_0)} u_p^2 (s - t_0)^3 B^{DM} ,$$

that is,

$$\|E_w(s)\|_S \leqq K_3 + K_4 K_5\, u_p(s - t_0)\,. \tag{36}$$

In the same way as (29) we obtain:

$$\|E_{w_1}(s) \exp(B(s)\, w_1(s)) - E_{w_2}(s) \exp(B(s)\, w_2(s))\|_S \leqq \tag{37}$$
$$\leqq K_4(1 + K_4 K_5(1 + B^M u_p(t_1 - t_0))) \max_{t \in [t_0, t_1]} |w_1(t) - w_2(t)|\,.$$

Now we can complete the proof of estimate (2). From (9) it follows that

$$x_1(t) - x_2(t) = (\Phi_1(t) - \Phi_2(t))\,(x_0 + \int_{t_0}^{t} \Phi_1^{-1}(s)\, A(s)\, E_{w_1}(s) \exp(B(s)\, w_1(s))\,.$$
$$.\, c(s)\, \mathrm{d}s - \int_{t_0}^{t} \Phi_1^{-1}(s)\, E_{w_1}(s) \exp(B(s)\, w_1(s))\,(B'(s)\, w_1(s)\, c(s) + c'(s))\, \mathrm{d}s) +$$
$$+ \Phi_2(t) \int_{t_0}^{t} (\Phi_1^{-1}(s) - \Phi_2^{-1}(s))\, A(s)\, E_{w_1}(s) \exp(B(s)\, w_1(s))\, c(s)\, \mathrm{d}s -$$
$$- \Phi_2(t) \int_{t_0}^{t} (\Phi_1^{-1}(s) - \Phi_2^{-1}(s))\, E_{w_1}(s) \exp(B(s)\, w_1(s))\,(B'(s)\, w_1(s)\, c(s) + c'(s))\,.$$
$$.\, \mathrm{d}s + \Phi_2(t) \int_{t_0}^{t} \Phi_2^{-1}(s)\, A(s)\,(E_{w_1}(s) \exp(B(s)\, w_1(s)) -$$
$$- E_{w_2}(s) \exp(B(s)\, w_2(s)))\, c(s)\, \mathrm{d}s - \Phi_2(t) \int_{t_0}^{t} \Phi_2^{-1}(s)\,(E_{w_1}(s) \exp(B(s)\, w_1(s)) -$$
$$- E_{w_2}(s) \exp(B(s)\, w_2(s)))\,(B'(s)\, w_1(s)\, c(s) + c'(s))\, \mathrm{d}s -$$
$$- \Phi_2(t) \int_{t_0}^{t} \Phi_2^{-1}(s)\, E_{w_2}(s) \exp(B(s)\, w_2(s))\,(B'(s)\,(w_1(s) - w_2(s))\, c(s))\, \mathrm{d}s +$$
$$+ (E_{w_1}(t) \exp(B(t)\, w_1(t)) - E_{w_2}(t) \exp(B(t)\, w_2(t)))\, c(t) +$$
$$+ (\Phi_1(t) - \Phi_2(t)) \int_{t_0}^{t} \Phi_2^{-1}(s) f(s)\, \mathrm{d}s + \Phi_2(t) \int_{t_0}^{t} (\Phi_1^{-1}(s) - \Phi_2^{-1}(s) f(s)\, \mathrm{d}s\,.$$

Using estimates (32)–(37), the triangle inequality and the relation between spectral matrix norm of an $(n \times n)$-dimensional matrix F and the Euclidean vector norm of some vector $y \in R^n$ (see e.g. [2]):

$$\|F\,.\,y\|_{R^n} \leqq \|F\|_S\,.\,\|y\|_{R^n}\,,$$

we obtain estimate (2) with K given by (19)–(26). The proof of Theorem 3 is completed. □

Remark 3. Theorem 3 is a direct generalization of Theorem 3 in [3]. If we take $B^{DM} = f^M = c^{DM} = 0$, we obtain the constant K, given by (19)–(26), which is exactly the same as in [3].

4. MODIFIED GRADIENT METHOD FOR NUMERICAL SOLUTION OF OPTIMAL CONTROL PROBLEMS

In this section we suggest some applications of estimate (2). In [3] some interesting properties of the so-called attainable set of bilinear systems were derived on the basis of the estimate (2). This properties can be evidently extended also to the case of time dependent systems.

In [3] an attempt was made to use estimate (2) for numerical computations, namely, an algorithm for determination of trajectories of a bilinear system with arbitrary control $u(t)$ was suggested. This algorithm was based on the following lemma.

Lemma 1. Let us consider any function $u(s)$ measurable on a closed interval $[t_0, t_1]$, such that $u(s) \in [u_{\min}, u_{\max}]$ a.e. on $[t_0, t_1]$. Let us divide the closed interval $[t_0, t_1]$ into k closed subintervals $[t_0 + (i - 1) h, t_0 + ih]$, $i = 1, 2, \ldots, k$, $h = (t_1 - t_0)/k$. Then there exists a function $u^*(s)$ with the following properties:

1) $u^*(s)$ is constant on each subinterval of the form $[t_0 + (i - 1) h, t_0 + ih]$
2) $u^*(s) \in \{u_{\min}, u_{\max}\}$ for all $s \in [t_0, t_1]$
3) for all $t \in [t_0, t_1]$

$$\left|\int_{t_0}^{t} u(s)\,\mathrm{d}s - \int_{t_0}^{t} u^*(s)\,\mathrm{d}s\right| \leqq \frac{u_{\max} - u_{\min}}{2} h . \tag{38}$$

The proof of this lemma is performed in [3] in detail. The proof is constructive, i.e., it gives a simple algorithm how to construct for any function $u(s)$ the appropriate function $u^*(s)$. Moreover, by combining Theorem 3 and Lemma 1 we can obtain:

Theorem 4. Let us consider an arbitrary admissible control $u(s)$ for the system (1) on time interval $[t_0, t_1]$ and let us denote by $x(t)$ the corresponding trajectory of system (1) with initial condition $x(t_0) = x_0$. Further, let $[t_0, t_1]$ be divided into k subintervals as in Lemma 1, let $u^*(s)$ be the control constructed to the control $u(s)$ by Lemma 1 and let $x^*(t)$ be the corresponding trajectory of system (1) with initial condition $x(t_0) = x_0$. Then

$$\max_{t \in [t_0, t_1]} \|x(t) - x^*(t)\|_{R^n} \leqq K \frac{u_{\max} - u_{\min}}{2} \frac{t_1 - t_0}{k} . \tag{39}$$

Here K is given by (19)–(26).

So we can see that $x^*(t)$ may be considered as a numerical approximation of $x(t)$ with the first-order accuracy. Furthermore, $x^*(t)$ may be computed in the following way. (For the sake of simplicity we consider further only time independent case, i.e. $A(t) \equiv A$, $B(t) \equiv B$, $c(t) \equiv c$, $f(t) \equiv 0$.)

Let us consider two operators L_h^+ and L_h^- which act from R^n to R^n:

$$L_h^+ x = \exp\left((A + Bu_{\max}) h\right) x + u_{\max} \int_0^h \exp\left((A + Bu_{\max})(h - s)\right) c\,\mathrm{d}s$$

$$L_h^- x = \exp\left((A + Bu_{\min}) h\right) x + u_{\min} \int_0^h \exp\left((A + Bu_{\min})(h - s)\right) c\,\mathrm{d}s .$$

Then

$$x^*(t_0 + ih) = L_1 L_2 \ldots L_i x(t_0) , \quad i = 1, 2, \ldots, k ,$$

where

$$L_j = L_h^+ , \quad \text{if} \quad u^*(s) = u_{\max} \quad \text{for} \quad s \in [t_0 + (j - 1) h, t_0 + jh]$$

and

$$L_j = L_h^- , \quad \text{if} \quad u^*(s) = u_{\min} \quad \text{for} \quad s \in [t_0 + (j - 1) h, t_0 + jh] ,$$

The matrices $\exp((A + Bu_{\max}) h)$, $\exp((A + Bu_{\min}) h)$ and vectors

$$\int_0^h \exp((A + Bu_{\max})(h - s)) c \, ds \,, \quad \int_0^h \exp((A + Bu_{\min})(h - s)) c \, ds$$

can be either computed analytically or in more complicated cases can be approximated, e.g., for $\exp((A + Bu_{\max}) h)$:

$$\exp((A + Bu_{\max}) h) \approx I + (A + Bu_{\max}) h + (A + Bu_{\max})^2 h^2/2! \,.$$

Thus, the algorithm for determination of trajectories of time independent bilinear systems is as follows. We choose $h = (t_1 - t_0)/k$ according to the required accuracy, then compute L_h^+ and L_h^-. After these steps we construct for an arbitrary admissible control $u(s)$, $s \in [t_0, t_1]$, the function $u^*(s)$ according to Lemma 1 and then we can compute $x^*(t_i)$, $t_i = t_0 + ih$, $i = 1, \ldots, k$. In [3] the reader can find concrete examples of applications of this algorithm.

Now we intend to use this algorithm in order to modify the gradient method for finding the solution of the optimal control problem. We explore simple and well-known method of the gradient projection which is described in [6] in detail.

Let us consider time independent system (1) with the performance index

(40) $$J(x(t), u(t)) = g(x(t_1)) + \int_{t_0}^{t_1} (\langle a_0, x \rangle + \langle b_0, x \rangle u + c_0 u) \, dt \,,$$
$$a_0 \in R^n \,, \quad b_0 \in R^n \,, \quad c_0 \in R \,, \quad g(x) \in C^1(R^n)$$

which is to be minimized. Time interval is supposed to be fixed. Then, according to [6], the following minimization method can be used:

(41) $$u_{j+1} = P_{\mathscr{U}}(u_j - J(u_j) \alpha_j) \,,$$

where u_{j+1} is the „new" approximation, $P_{\mathscr{U}}$ is the projection operator on set of all admissible controls $\mathscr{U}$, $\alpha_j \in R$ is the stepsize, and $J'(\cdot)$ is the gradient of performance index (40) given by

(42) $$J'(u) = \langle b_0, x \rangle + c_0 - \langle (Bx + c), \psi(t, u) \rangle \,,$$

where

(43) $$\dot{\psi}(t, u) = (-A^* - B^* u(t)) \psi(t, u) + a_0 + b_0 u(t) \,,$$
$$\psi(t_1, u) = - \left. \frac{\partial g(x)}{\partial x} \right|_{x = x(t_1)} .$$

In the other words, in order to obtain the approximation u_{j+1} one has to solve the system (1) with initial condition (or left-end condition) $x(t_0) = x_0$ and for the control $u_j(t)$, $t \in [t_0, t_1]$, then to evaluate the right-end condition $\psi(t_1, u_j) = -\nabla_x g(x(t_1))$ and to solve the system (43) with this condition. After these steps it is possible to compute $J'(u_j)$, to choose a stepsize and then according to (41) obtain the approximation u_{j+1}. In the same time it is necessary to check if $J(u_{j+1}) < J(u_j)$. This not being the case the stepsize α_j must be adjusted, e.g., by using bisection procedure, until the condition $J(u_{j+1}) < J(u_j)$ is met.

Our modification of this method consists in constructing the control u_{j+1}^* to the control u_{j+1} by Lemma 1. Therefore we take u_{j+1}^* instead of u_{j+1} as an approximation of the optimal control. All these approximations are piecewise constant functions with values in $\{u_{\min}, u_{\max}\}$. Solution of systems (1) and (43) for these piecewise constant controls are obtained according to the algorithm which was described earlier. On the other hand, the corresponding trajectory $x_{j+1}^*(t)$, $t \in [t_0, t_1]$ approximates $x_{j+1}(t)$, $t \in [t_0, t_1]$, and $\psi_{j+1}^*(t)$ approximates $\psi_{j+1}(t)$ with first-order accuracy with respect to $h = (t_1 - t_0)/k$. When a certain $h = (t_1 - t_0)/k$ is chosen, we take e.g. $u_0 \equiv u_{\max}$ and start the computation. After certain number of steps we find that $u_{j+1}^* = u_j^*$, because the difference between u_{j+1} and u_j is so small that for given h we obtain, according to Lemma 1, the same u_{j+1}^* and u_j^*. Then the use of smaller h is necessary.

As a practical criterion of stopping this procedure we take $|J(u_{j+1}) - J(u_j)| < \varepsilon$, where ε is the required accuracy of approximation of optimal value of the performance index. As a result of this procedure we obtain a suboptimal (in the described sense) control which is a piecewise constant function with values in $\{u_{\min}, u_{\max}\}$.

This modificated gradient method was tested on several examples. First, time independent system (1) was considered:

$$n = 2\,, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad u_{\min} = -1$$

$$u_{\max} = 1\,, \quad t_0 = 0\,, \quad t_1 = \tfrac{1}{2}\pi\,, \quad x^0 = (2, 2)^{\mathrm{T}}\,.$$

Two types of performance indices were investigated

$$J_1(x(t), u(t)) = (x_1(t_1) - 2)^2 + (x_2(t_1))^2\,,$$

$$J_2(x(t), u(t)) = -\int_{t_0}^{t_1} x_2(t)\, u(t)\, \mathrm{d}t\,.$$

Theoretical optimal value of J_1 equals 2·0 and it can be obtained by an infinite number of controls, e.g., (i) constant optimal control $u(t) \equiv \tilde{u} = (2/\pi) \ln \frac{1}{2}$, (ii) bang-bang control $u(t) \equiv 1$, $t \in [0, \frac{1}{4}\pi + \frac{1}{2} \ln \frac{1}{2}]$; $u(t) \equiv -1$, $t \in [\frac{1}{4}\pi + \frac{1}{2} \ln \frac{1}{2}, \frac{1}{2}\pi]$, (iii) bang-bang control $u(t) \equiv -1$, $t \in [0, \frac{1}{4}\pi - \frac{1}{2} \ln \frac{1}{2}]$; $u(t) \equiv 1$, $t \in [\frac{1}{4}\pi - \frac{1}{2} \ln \frac{1}{2}, \frac{1}{2}\pi]$. Approximations u_j^* of optimal control converge in weak sense, i.e., in the norm $\max_{t \in [t_0, t_1]} \left| \int_{t_0}^{t} u(s)\, \mathrm{d}s \right|$, to the first indicated case (i). The number of switchings of control u_j^* increases to infinity. In fact, the sequence $\{u_j^*\}$ converges to a certain sliding rule (relaxed control). Let us note that constant theoretical optimal control is a singular one.

Theoretical optimal value of J_2 is equal to $-2·4423998$ and can be achieved by the unique optimal control of bang-bang type:

$$u(t) \equiv 1\,, \quad t \in [0, 0{\cdot}98226] \quad \text{and} \quad u(t) \equiv -1\,, \quad [0{\cdot}98226, \tfrac{1}{2}\pi]\,.$$

By the suggested method it was achieved $\tilde{J}_2 = -2·442389$ and the approximate optimal control was

$$\tilde{u}(t) \equiv 1\,, \quad t \in [0, 0{\cdot}98104]\,; \quad \tilde{u}(t) \equiv -1\,, \quad t \in [0, 98184, \tfrac{1}{2}\pi]$$

During not included negligible time interval $[0{\cdot}98104, 0{\cdot}98184]$ switching of $\tilde{u}(t)$ occurs. Total number of 7 iterations was needed:

2 iterations with $h = 0{\cdot}01$, 2 iterations with $h = 0{\cdot}001$ and 3 iterations with $h = 0{\cdot}0001$. All these computations including use of rather slow grafics Calcomp took about 5 minutes of CPU time on IBM 370/135. Let us note that one iteration with $h = 0{\cdot}0001$ took about hundred times more CPU time than one iteration with $h = 0{\cdot}01$. In this case it was also necessary to use DOUBLE PRECISION because of some integration procedures (especially for $h = 0{\cdot}0001$).

Trajectories corresponding to the iterations of the method and to optimal control are shown in Fig. 1 (J_1) and Fig. 2 (J_2).

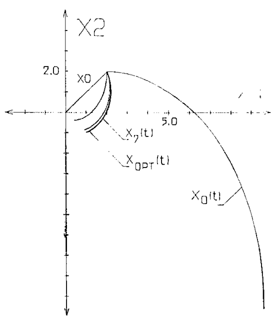

Fig. 1. Approximations of optimal trajectories for J_1.

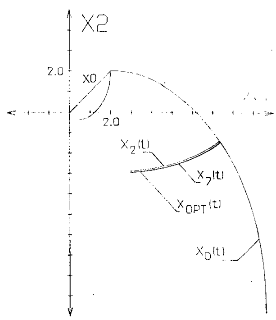

Fig. 2. Approximations of optimal trajectories for J_2.

Finally, the following bilinear time independent system (1) was considered:

$$n = 3\,, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 10 & 0 \\ 1 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$t_0 = 0\,, \quad t_1 = 1\,, \quad x^0 = (1, 0, 0)^T\,, \quad u_{max} = 1\,, \quad u_{min} = 0\,.$$

The performance index was taken as

$$J(x(t), u(t)) = x_1(1) + x_2(1) + x_3(1) - 1\,.$$

This system arises in problem of optimal design of multifunctional catalysts for chemical reactors (see [4], [5]). Optimal control for this system was computed and comparison with results of [4] was performed. In Fig. 3 theoretical optimal control as well as our approximation after 30 iterations (about 15 minutes of CPU time) are shown. Theoretical optimal value of the performance index was achieved with accuracy about 10^{-9}, but approximation of the optimal control rather differs from the theoretical optimal control. It is caused by fact that the value of performance index changes very little in the neighbourhood of the optimal control and therefore the convergence is slow (as well as in [4]). Moreover, the modified gradient method computes very well two parts of optimal control which are of bang-bang type, but it does not compute very well the part which is of singular type, as it can be seen from Fig. 3. Let us note that the approximation in Fig. 3 was computed for $h = 0{\cdot}001$.

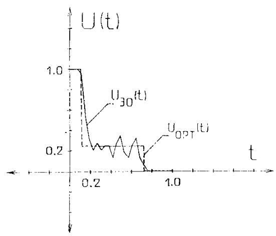

Fig. 3. Theoretical optimal control and its approximation after 30 iterations for J.

The depicted values of control were obtained using certain averaging process (taking always average of 40 neighbouring points). Let us also remark that 7 iterations with $h = 0{\cdot}01$ (performed in 15 seconds) suffice in order to achieve optimal value of the performance index with accuracy $2 \,.\, 10^{-4}$. Remaining 15 minutes are used for improving accuracy to 10^{-9}.

We can now summarize some experience connected with the use of the modified gradient method. It works quite well when we know a priori that the optimal control is of bang-bang type. In this case it is advantageous to use this method. When the optimal control is of a singular type, the method converges in certain sense to a sliding rule. Then it is necessary to make a concrete decision about any particular case and to use all a priori information in order to determine which kind of approximation is needed. (For example, if we know a priori that the singular part in Fig. 3 is constant, we may take this constant as an average value of all $u_{\min}$ and $u_{\max}$ through this part of time interval.) It is also advisable to use this method when it is suitable to have an approximation of the optimal control as a function of two values only (although with possible very high number of switchings).

5. CONCLUSIONS

Two different aims were followed in this paper. First, to show that the results of [3] concerning a problem of continuous dependence of trajectories of bilinear systems on control and their consequences can be extended to the case of time dependent bilinear systems. Second, to describe a modified gradient method for the solution of optimal control problems with time independent parameters on the fixed time interval. The modification was based on continuous dependence (in fact Lipschitzean dependence) of trajectories of bilinear systems on control. The method was constructed for time independent systems only for sake of simplicity, there are no theoretical obstacles to consider time dependent case, too.

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