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A SINGLE-DEGREE-OF-FREEDOM POLYNOMIAL SOLUTION TO THE OPTIMAL FEEDBACK/FEEDFORWARD STOCHASTIC TRACKING PROBLEM

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A solution to the optimal stochastic tracking problem in the presence of measurable and unmeasurable disturbances is obtained using polynomial techniques. The plant under consideration includes a coloured measurement noise signal and all disturbance and reference sub-systems may be unstable. The problem formulation involves a cost-function having dynamic weights. The analysis is for the single-degree-of-freedom controller structure. In addition, a feedforward compensator is incorporated in the overall optimisation procedure for the rejection of the measurable disturbance.

NOTATION

All systems considered are assumed to be linear, time-invariant and discrete-time. The systems are described in the time-domain by means of real polynomials in the delay operator d, and in the frequency-domain by means of real polynomials in the inverse of the z-transform complex number z. A polynomial $X(d)$ is *stable* (or *strictly Hurwitz*) iff it has no roots with magnitude less than or equal to unity. A polynomial $X(d)$ is *Hurwitz* iff it has no roots with magnitude less than unity. A polynomial $X(d)$ is *unstable* iff it has any roots with magnitude less than or equal to unity.

For simplicity the arguments of polynomials are often omitted so that $X(d)$ is denoted by X. The conjugate of a polynomial $X(d)$ is denoted by $X^*(d) \triangleq X(d^{-1})$, or simply X^*. The absolute coefficient of X is denoted by $\langle X \rangle$.

A transfer-function is *inverse stable* ('minimum phase') iff it has no zeros with magnitude less than or equal to unity.

The power spectrum of a signal $x(t)$ is denoted by ϕ_x.

1. INTRODUCTION

The polynomial equation approach to the linear stochastic optimal control problem was developed throughout the seventies by Kučera, whose pioneering work on the subject culminated in the publication of a book (Kučera [5]). The method was further refined by Kučera and Šebek [6].

Grimble [3] later included dynamic cost-function weights and coloured measurement noise in his analysis of the problem. Grimble's solution, however, was restricted to the case of asymptotically stable reference and disturbance sub-systems.

A recent development is the inclusion of a *feedforward* controller into the overall optimisation procedure for the rejection of *measurable* disturbances. The solution of this problem was given by Šebek et al [8] for the case of scalar cost-function weights and white measurement noise.

In this paper the complete general solution of the feedback/feedforward optimal stochastic tracking problem is obtained for a system which includes coloured measurement noise and where all disturbance and reference sub-systems may be unstable (the unstable sub-systems used to model signals such as steps, ramps or sinusoids are of greatest practical importance). In addition, the problem specification includes dynamic cost-function weights.

The analysis in this paper (as in Grimble [3]) is for the so-called *single-degree-of-freedom* (SDF) controller structure. In this structure the *observed* tracking error is processed by a single *cascade* compensator (in the two-degrees-of-freedom (2 DF) structure, on the other hand, the observed reference and observed output signals are processed independently). While the 2DF structure is known to lead to a lower optimal cost (Gawthrop [2]), it is not possible in some practical situations to realise a 2DF control structure since it is not always possible to measure the reference and output signals separately. For example, in many trajectory following problems it is only possible to measure the tracking error (i.e. the difference between the desired and actual trajectories) and in these cases a SDF control structure must be used.

To summarise, the problem considered is as follows:

(i) The cost-function includes dynamic weighting elements.

(ii) The system model includes a coloured output disturbance signal (measurement noise).

(iii) A feedforward compensator is incorporated in the overall design procedure for the rejection of measurable disturbances.

(iv) All disturbance and reference sub-systems may be unstable.

(v) A solution is obtained for the single-degree-of-freedom controller structure (including feedforward).

The paper is organised as follows: the system under consideration and the optimal control problem formulation are presented in Sections 2 and 3, respectively. In

Section 4 the problem solution is derived. Some important structural properties of the optimal control problem solution are outlined in Section 5. The paper is concluded in Section 6.

2. PLANT MODEL

The open-loop model for the single-input single-output *plant* under consideration is shown in Figure 1. The plant is governed by the equations:

(1) $$y(t) = p(t) + x(t) + d(t)$$

(2) $$= W_p\, u(t) + W_x\, l(t) + W_d\, \psi_d(t)$$

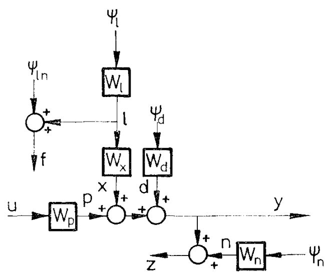

Fig. 1. Open-loop Plant Model.

The controlled output, $y(t)$, consists of the sum of three signals:

(i) The 'undisturbed' output $p(t) = W_p\, u(t)$, where $u(t)$ is the plant *control input*.
(ii) A disturbance signal $x(t) = W_x\, l(t)$, where $l(t)$ is a *measurable disturbance*.
(iii) A disturbance signal $d(t) = W_d\, \psi_d(t)$, where ψ_d is an *unmeasurable* stochastic signal.

The controlled output is corrupted by a *measurement noise* $n(t)$. The measured output, $z(t)$, is given by the equations:

(3) $$z(t) = y(t) + n(t)$$

(4) $$= y(t) + W_n\, \psi_n(t)$$

where $\psi_n(t)$ is an *unmeasurable* stochastic signal.

The measurable disturbance signal $l(t)$ is corrupted by a stochastic *measurement noise* $\psi_{ln}(t)$. The *disturbance measurement*, $f(t)$, is given by:

(5) $$f(t) = l(t) + \psi_{ln}(t)$$

The open-loop plant structure shown in Figure 1 is representative of many industrial control problems:

(i) The signal $l(t)$ typically represents a load disturbance which can be measured and used to provide feedforward control. The signal $\psi_{ln}(t)$ represents noise arising from the measurement of $l(t)$, so that the actual signal used for feedforward is $f(t)$.

(ii) The measured output available for feedback $(z(t))$ is usually different from the output to be controlled $(y(t))$ due to measurement noise which the controller should not attempt to regulate. Use of the filter W_n admits the modelling of many different forms of measurement noise. For example, in ship control systems $n(t)$ represents the high-frequency effect of waves to which the controller should not respond (see Grimble, [3]).

Command signal

In the optimal tracking control problem considered in the following the controlled output $y(t)$ will be required to follow as closely as possible a *reference* (or command) signal $r(t)$. The signal $r(t)$ may be represented as the output of a *generating sub-system* W_r which is driven by an external stochastic signal $\psi_r(t)$:

(6) $$r(t) = W_r \psi_r(t)$$

The reference signal $r(t)$ is corrupted by a stochastic *measurement noise* $\psi_{rn}(t)$. The *reference measurement*, $m(t)$, is given by:

(7) $$m(t) = r(t) + \psi_{rn}(t)$$

The *tracking error*, $e(t)$, is defined by:

(8) $$e(t) \triangleq r(t) - y(t)$$

Measurable disturbance

The measurable disturbance signal $l(t)$ may be represented as the output of a *generating sub-system* W_l driven by an external stochastic signal $\psi_l(t)$:

(9) $$l(t) = W_l \psi_l(t)$$

Polynomial form

It is always possible to express the various plant sub systems in terms of a least-common-denominator polynomial. Denoting the least-common-denominator of W_p, W_x, W_d and W_r by A these sub-systems may be expressed as:

(10) $$W_p = A^{-1}B$$

(11) $$W_d = A^{-1}C$$

(12) $$W_x = A^{-1}D$$

(13) $$W_r = A^{-1}E$$

where B, C, D and E are polynomials.

The sub-systems W_n and W_l are denoted by:

(14) $$W_n = A_n^{-1} C_n$$

(15) $$W_l = A_l^{-1} E_l$$

where A_n, C_n, A_l and E_l are polynomials.

Assumptions

(i) Each of the sub-systems is free of unstable hidden modes.
(ii) The signals ψ_d, ψ_n, ψ_l, ψ_r, ψ_{ln} and ψ_{rn} are sequences of mutually uncorrelated random variables having intensities σ_d, σ_n, σ_l, σ_r, σ_{ln} and σ_{rn}, respectively. All intensities are assumed to be non-zero.
(iii) Each of the sub-systems in equations (10)–(15) may have poles on the unit circle of the z-plane. Each sub-system is assumed, without loss of generality, to have no poles outside the unit circle of the z-plane.

3. PROBLEM DEFINITION

Control structure

In the single-degree-of-freedom control structure the *observed* tracking error is processed by a single *cascade* compensator. In addition, a *feedforward* compensator is employed to counter the effect of the measurable disturbance $l(t)$.

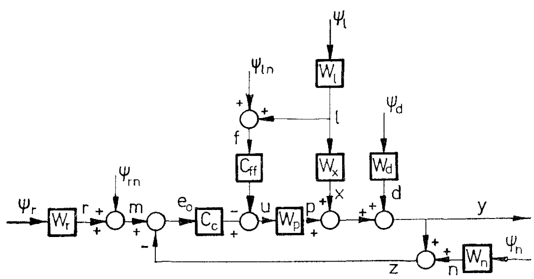

Fig. 2. SDF Control System With Feedforward.

The closed-loop system for the single-degree-of-freedom controller including feedforward is shown in Figure 2. The observed error signal $e_0(t)$ is defined by:

(16) $$e_0(t) = m(t) - z(t)$$

The control law is given by:

(17) $$u(t) = C_c e_0(t) - C_{ff} f(t)$$

where the *cascade controller* C_c and the *feedforward controller* C_{ff} may be expressed as ratios of polynomials as:

$$C_c = C_{cd}^{-1} C_{cn} \tag{18}$$

$$C_{ff} = C_{ffd}^{-1} C_{ffn} \tag{19}$$

Cost-function

The desired optimal controller evolves from minimisation of the *cost-function*:

$$J = \mathsf{E}\{(H_q e)^2 (t) + (H_r u)^2 (t)\} \tag{20}$$

where H_q and H_r are dynamic (i.e. frequency-dependent) weighting elements which may be realised by rational transfer-functions.

Using Parseval's theorem the cost-function may be transformed to the frequency domain and expressed as:

$$J = \frac{1}{2\pi j} \oint_{|z|=1} \{Q_c \phi_e + R_c \phi_u\} \frac{dz}{z} \tag{21}$$

where ϕ_e and ϕ_u are the tracking error and control input spectral densities, respectively, and:

$$Q_c = H_q H_q^* , \quad R_c = H_r H_r^* \tag{22}$$

The weighting elements Q_c and R_c may be expressed as ratios of polynomials using:

$$Q_c \triangleq \frac{B_q^* B_q}{A_q^* A_q}, \quad R_c \triangleq \frac{B_r^* B_r}{A_r^* A_r} \tag{23}$$

Assumptions

(i) The weighting elements Q_c and R_c are *strictly positive* on $|z| = 1$.
(ii) A_q, B_q, A_r and B_r are strictly Hurwitz polynomials.

4. PROBLEM SOLUTION

Theorem 1. The optimal control problem has a solution if and only if:
(a) A and B have no unstable common factors.
(b) Any unstable factors of A_l are also factors of A and D.
(c) Any unstable factors of A_n are *not* also factors of A.

The strictly Hurwitz spectral factors D_c, D_f and D_{fd} are assumed to exist and are defined by:

$$D_c D_c^* = B A_r B_q B_q^* A_r^* B^* + A A_q B_r B_r^* A_q^* A^* \tag{24}$$

$$D_f D_f^* = (A_n C \sigma_d C^* A_n^* + A C_n \sigma_n C_n^* A^* + A_n E \sigma_r E^* A_n^* + A A_n \sigma_{rn} A_n^* A^* + A_n D \sigma_{ln} D^* A_n^*) \tag{25}$$

$$D_{fd} D_{fd}^* = A_l \sigma_{ln} A_l^* + E_l \sigma_l E_l^* \tag{26}$$

The cascade and feedforward parts of the control law (17) which minimises the cost-function (21) are as follows:

(i) *Optimal cascade controller*

(27) $$C_c = \frac{GA_r}{H}$$

where G, H (along with F) is the solution having the property:

$$(D_c^* D_f^* z^{-g1})^{-1} F \quad \text{strictly proper}$$

of the polynomial equations:

(28) $$D_c^* D_f^* z^{-g1} G + FAA_q A_n \quad = B^* A_r^* B_q^* B_q R_1$$

(29) $$D_c^* D_f^* z^{-g1} H - FBA_r A_q A_n = A^* R_2$$

where:

(30) $$R_1 = z^{-g1}(D_f D_f^* - C_n \sigma_n C_n^* AA^*)$$

(31) $$R_2 = z^{-g1}(D_f D_f^* A_q A_q^* B_r B_r^* + BB^* A_r A_r^* B_q B_q^* C_n \sigma_n C_n^*)$$

and $g1 > 0$ is the smallest integer which makes the equations (28)–(29) polynomial in z^{-1}.

(ii) *Optimal feedforward controller*

(32) $$C_{ff} = \frac{XA_r D_f - C_{cn} DD_{fd}}{D_{fd} AC_{cd}}$$

where X (along with Z and Y) is the solution having the property:

$$(D_c^* z^{-g2})^{-1} Z \quad \text{strictly proper}$$

of the polynomial equations:

(33) $$D_c^* z^{-g2} X + ZAA_q A_l = z^{-g2} B^* A_r^* B_q^* B_q DD_{fd}$$

(34) $$D_c^* z^{-g2} Y - ZBA_r A_l = z^{-g2} A^* A_q^* B_r^* B_r DD_{fd}$$

and $g2 > 0$ is the smallest integer which makes the equations (33)–(34) polynomial in z^{-1}.

The associated minimal cost is given by:

(35) $$J_{\min} = \frac{1}{2\pi j} \oint_{|z|=1} [\sum_{i=1}^{2} (T_i^- T_i^{-*}) + \phi_{01}] \frac{dz}{z}$$

where the terms T_i^-, $i = \{1, 2\}$ and ϕ_{01} are defined in the Appendix.

Proof. The proof of Theorem 1 is given in the Appendix. □

Corollary 1. The polynomials G and H in equations (28) and (29) also satisfy the *implied cascade diophantine equation*:

(36) $$AH + BA_r G = D_f D_c$$

which also defines the closed-loop characteristic equation.

Proof. Multiplying equation (28) by BA_r, equation (29) by A and then adding results, using equation (24) and cancelling common factors, in equation (36). □

Corollary 2. The polynomials X and Y in equations (33) and (34) also satisfy the *implied feedforward diophantine equation*:

(37) $$AA_qY + BA_rX = D_cDD_{fd}$$

Proof. Multiplying equation (33) by BA_r, equation (34) by AA_q and adding results, after some algebraic manipulation, in equation (37). □

Corollary 3. The coloured measurement noise sub-system denominator polynomial A_n is a factor of the cascade controller numerator C_{cn}.

Proof. The diophantine equation (28) may be rewritten by substituting from equation (25) for $D_fD_f^*$ as:

$$D_c^*D_f^*z^{-g1}G + FAA_qA_n =$$
$$= B^*A_r^*B_q^*B_qz^{-g1}A_nA_n^*(C\sigma_dC^* + E\sigma_rE^* + A\sigma_{rn}A^* + D\sigma_{ln}D^*)$$

Since A_n divides both the right-hand side of this equation and the second term on the left side, it must also divide the term $D_c^*D_f^*z^{-g1}G$. Since the term D_cD_f is strictly Hurwitz by definition, the term $D_c^*D_f^*$ is non-Hurwitz. Since A_n can have no zeros outside the unit circle of the z-plane, A_n and $D_c^*D_f^*z^{-g1}$ can have no common factors. As a result, A_n must divide G and hence C_{cn}. □

Corollary 4. The expression C_{ffn}/A is *polynomial*.

Proof. From equation (32) C_{ffn}/A may be written:

$$\frac{C_{ffn}}{A} = \frac{XA_rD_f - C_{cn}DD_{fd}}{A}$$

Substituting from equations (36) and (37) the above transfer-function may be written, after some algebraic manipulation, as:

$$\frac{C_{ffn}}{A} = \frac{DD_{fd}C_{cd} - A_qYD_f}{B}$$

Multiplying equation (28) by $DD_{fd}z^{-g2}$, equation (33) by R_1 and comparing obtain, after some algebraic manipulation:

$$\frac{C_{ffn}}{A} = \left(D_c^*XC_n\sigma_nC_n^*A^* + FA_qA_nDD_{fd}z^{g1} - ZA_qA_lR_1z^{(g_1+g_2)}\right)A_r/D_c^*D_f^*\,.$$

Comparing the above three expressions for C_{ffn}/A the following conclusion can be drawn: since $D_f^*D_c^*$ is non-Hurwitz and since A and B cannot have any unstable common factors the expression for C_{ffn}/A is, in fact, *polynomial*. □

5. PROPERTIES AND STRUCTURE OF THE OPTIMAL SOLUTION

(i) The characteristic polynomial of the closed-loop system is stable (equation (36)) since the polynomials D_c and D_f are strictly Hurwitz, as is the spectral factor D_{fd}. In addition, Corollary 4 shows that the expression C_{ffn}/A is polynomial. Thus, bearing in mind that the controller must be realised as a single dynamical system having two inputs and one output, and applying the general theory relating to the stability of feedback systems given by Kučera [5] the closed-loop system is seen to be asymptotically stable.

(ii) The dynamic weighting elements in the cost-function allow frequency selective costing to be applied to the tracking error and control input signals. This feature is manifest in the fact that the control weighting denominator A_r is a factor of the numerators of each part of the controller and, when the output disturbance $n(t) = 0$, the error weighting denominator A_q is a factor of the denominators of each part of the controller. Thus, the magnitude of the loop-gain with respect to frequency is directly influenced by the choice of cost weights.

(iii) The denominator of the output disturbance sub-system (A_n) appears as a zero in the feedback loop. This fact is consistent with the well known transmission-blocking property of zeros (MacFarlane and Karcanias [7]) and has a natural interpretation since these disturbance modes should not, intuitively, be allowed to propagate through the system.

(iv) In line with the Internal Model Principle of Control (Francis and Wonham, [1]) the solvability conditions for the optimal control problem demand that any unstable reference and disturbance modes must also be modes of the plant input-output transfer-function.

(v) In the SDF controller structure the cascade part of the controller is independent of the feedforward part.

(vi) The feedforward part of the controller is causal and stable even when the plant is inverse unstable and when the delay associated with the plant is longer than the delay associated with the measurable disturbance sub-system (W_x). These plant conditions may cause serious difficulties in conventional feedforward controller designs.

6. CONCLUSIONS

A polynomial solution to the optimal stochastic tracking problem has been obtained for a general system and a cost-function with dynamic weights. The system model includes a coloured measurement noise and both measurable and unmeasurable disturbances. A feedforward compensator is included in the design for rejection of the measurable disturbance.

The design procedure involves the solution of two couples of polynomial equations whose coefficients are obtained by spectral factorisation. Efficient numerical algorithms to perform the design can be found in Kučera [5].

The optimal controller consists of a cascade part which operates on the observed tracking error and a feedforward part which operates on the measured disturbance. The controller must be realised as a single dynamical system having two inputs and one output.

Further details of the theory presented in this paper can be found in Hunt [4].

APPENDIX: Proof of Theorem 1

The *closed loop transfer function M* and the *sensitivity function S* for the SDF control structure are defined as:

(A1)
$$M \triangleq \frac{C_c}{1 + W_p C_c}, \quad S \triangleq \frac{1}{1 + W_p C_c}$$

thus:

(A2)
$$M = C_c S, \quad S = 1 - W_p M$$

From the SDF system structure shown in Figure 2 the control input and tracking error signals may be written as:

(A3)
$$u = -M(d + n - m - W_x \psi_{ln}) - SC_{cf} f$$

(A4)
$$e = -(1 - W_p M)(d - m - W_x \psi_{ln}) + W_p M n - \psi_{rn} - (W_x - W_p S C_{cf}) f$$

where:

(A5)
$$C_{cf} \triangleq C_{ff} + C_c W_x$$

From equations (A3) and (A4) the control input and tracking error spectral densities may be written as:

(A6)
$$\phi_u = M(\phi_d + \phi_n + \phi_m + W_x \sigma_{ln} W_x^*) M^* + SC_{cf} \phi_f C_{cf}^* S^*$$

(A7)
$$\phi_e = (1 - W_p M)(\phi_d + \phi_m + W_x \sigma_{ln} W_x^*)(1 - W_p M)^* + W_p M \phi_n M^* W_p^* + \\ + \sigma_{rn} + (W_x - W_p S C_{cf}) \phi_f (W_x - W_p S C_{cf})^*$$

Denoting the integrand of the cost-function (21) as I, the integrand may be written:

(A8)
$$I = Q_c \phi_e + R_c \phi_u$$

Substituting the expressions for ϕ_u and ϕ_e given in equations (A6) and (A7) into equation (A8) the cost-function integrand may be written, after some algebraic manipulation, as:

(A9)
$$\begin{aligned} I = {} & (W_p Q_c W_p^* + R_c) SS^* (C_{cf} \phi_f C_{cf}^* + \\ & + C_c(\phi_d + \phi_n + \phi_m + W_x \sigma_{ln} W_x^*) C_c^*) + \\ & + Q_c(W_x \phi_f W_x^* + \phi_d + \phi_m + W_x \sigma_{ln} W_x^* + \sigma_{rn}) - \\ & - Q_c \phi_f (W_x C_{cf}^* S^* W_p^* + W_p S C_{cf} W_x^*) - \\ & - Q_c(\phi_d + \phi_m + W_x \sigma_{ln} W_x^*)(M^* W_p^* + W_p M) \end{aligned}$$

To further simplify the cost expression the *control* and *filter spectral factors* (Y_c and Y_f, respectively) are defined by:

(A10) $$Y_c Y_c^* \triangleq W_p Q_c W_p^* + R_c$$

(A11) $$Y_f Y_f^* \triangleq \phi_d + \phi_n + \phi_m + W_x \sigma_{ln} W_x^*$$

Similarly, the measurable disturbance spectral factor Y_{fd} is defined by:

(A12) $$Y_{fd} Y_{fd}^* \triangleq \phi_f$$

The following auxiliary spectra are defined by:

(A13) $$\phi_0 \triangleq Q_c(W_x \phi_f W_x^* + \phi_d + \phi_m + W_x \sigma_{ln} W_x^* + \sigma_{rn})$$

(A14) $$\phi_{h1} \triangleq Q_c \phi_f W_p^* W_x$$

(A15) $$\phi_{h2} \triangleq Q_c(\phi_d + \phi_m + W_x \sigma_{ln} W_x^*) W_p^*$$

Substituting from equations (A10)–(A15) into equation (A9), the cost-function integrand may be written as:

(A16) $$I = Y_c Y_c^* SS^*(C_{cf} Y_{fd} Y_{fd}^* C_{cf}^* + C_c Y_f Y_f^* C_c^*) + \\ + \phi_0 - \phi_{h1} C_{cf}^* S^* - \phi_{h1}^* S C_{cf} - \phi_{h2} M^* - \phi_{h2}^* M$$

The integrand may now be split into terms which depend on each part of the controller, and terms which do not depend on the controller at all. Completing the squares in equation (A16) the integrand may be expressed as:

(A17) $$I = \left(Y_c S C_{cf} Y_{fd} - \frac{\phi_{h1}}{Y_c^* Y_{fd}^*}\right)\left(Y_c S C_{cf} Y_{fd} - \frac{\phi_{h1}}{Y_c^* Y_{fd}^*}\right)^* + \\ + \left(Y_c S C_c Y_f - \frac{\phi_{h2}}{Y_c^* Y_f^*}\right)\left(Y_c S C_c Y_f - \frac{\phi_{h2}}{Y_c^* Y_f^*}\right)^* + \phi_{01}$$

where:

(A18) $$\phi_{01} = \phi_0 - \frac{1}{Y_c Y_c^*}\left(\frac{\phi_{h1}\phi_{h1}^*}{Y_{fd} Y_{fd}^*} + \frac{\phi_{h2}\phi_{h2}^*}{Y_f Y_f^*}\right)$$

The term ϕ_{01} in equation (A17) does not depend on the controller and does not, therefore, enter into the following cost minimisation procedure. The first two terms in equation (A17) depend, respectively, on the feedforward and cascade parts of the controller.

Before proceeding it is necessary to express the spectral factors of equations (A10) to (A12) in polynomial form as follows:

(A19) $$Y_c Y_c^* \triangleq \frac{D_c D_c^*}{A_c A_c^*}$$

(A20) $$Y_f Y_f^* \triangleq \frac{D_f D_f^*}{A_f A_f^*}$$

(A21) $$Y_{fd}Y_{fd}^* \triangleq \frac{D_{fd}D_{fd}^*}{A_{fd}A_{fd}^*}$$

Using the common denominator form of the system model given in equations (10)–(15) and using the polynomial equation form of the cost-function weights given by equation (23), the spectral factors (A10)–(A12) may be written as:

(A22) $$Y_cY_c^* = \frac{BA_rB_qB_q^*A_r^*B^* + AA_qB_rB_r^*A_q^*A^*}{AA_qA_rA_r^*A_q^*A^*}$$

(A23) $$Y_fY_f^* = (A_nC\sigma_dC^*A_n^* + AC_n\sigma_nC_n^*A^* + A_nE\sigma_rE^*A_n^* + \\ + AA_n\sigma_{rn}A_n^*A^* + A_nD\sigma_{ln}D^*A_n^*)/AA_nA_n^*A^*$$

(A24) $$Y_{fd}Y_{fd}^* = \frac{A_l\sigma_{ln}A_l^* + E_l\sigma_lE_l^*}{A_lA_l^*}$$

Comparison of equations (A19)–(A21) with equations (A22)–(A24) then yields:

(A25) $$D_cD_c^* = BA_rB_qB_q^*A_r^*B^* + AA_qB_rB_r^*A_q^*A^*$$

(A26) $$D_fD_f^* = A_nC\sigma_dC^*A_n^* + AC_n\sigma_nC_n^*A^* + A_nE\sigma_rE^*A_n^* + \\ + AA_n\sigma_{rn}A_n^*A^* + A_nD\sigma_{ln}D^*A_n^*$$

(A27) $$D_{fd}D_{fd}^* = A_l\sigma_{ln}A_l^* + E_l\sigma_lE_l^*$$

and:

(A28) $$A_c = AA_qA_r$$

(A29) $$A_f = AA_n$$

(A30) $$A_{fd} = A_l$$

Each of the controller dependent terms in equation (A17) may now be simplified separately:

(i) C_c *dependent term*

From the plant model equations and spectral factor definitions obtain:

(A31) $$\frac{\phi_{h2}}{Y_c^*Y_f^*} = \frac{B^*A_r^*B_q^*B_q(D_fD_f^* - AC_n\sigma_nC_n^*A^*)}{AA_qA_nD_c^*D_f^*}$$

The diophantine equation (28) allows the strictly unstable part of equation (A31) to be separated as follows:

(A32) $$\frac{\phi_{h2}}{Y_c^*Y_f^*} = \frac{G}{AA_qA_n} + \frac{Fz^{g1}}{D_c^*D_f^*}$$

From the system equations and spectral factor definitions obtain:

(A33) $$Y_cSC_cY_f = \frac{D_cD_fC_{cn}}{AA_qA_nA_r(AC_{cd} + BC_{cn})}$$

From equations (A32) and (A33) obtain:

$$\text{(A34)} \qquad Y_c S C_c Y_f - \frac{\phi_{h2}}{Y_c^* Y_f^*} = \frac{D_c D_f C_{cn} - G A_r (A C_{cd} + B C_{cn})}{A A_q A_n A_r (A C_{cd} + B C_{cn})} - \frac{F z^{g1}}{D_c^* D_f^*}$$

Substituting from the implied cascade diophantine equation (36), equation (A34) may be expressed as:

$$\text{(A35)} \qquad Y_c S C_c Y_f - \frac{\phi_{h2}}{Y_c^* Y_f^*} = \frac{C_{cn} H - G A_r C_{cd}}{A_q A_n A_r (A C_{cd} + B C_{cn})} - \frac{F z^{g1}}{D_c^* D_f^*}$$

Finally, equation (A35) may be expressed as:

$$\text{(A36)} \qquad Y_c S C_c Y_f - \frac{\phi_{h2}}{Y_c^* Y_f^*} = T_1^+ + T_1^-$$

where T_1^+ denotes the first term in equation (A35) and T_2^- denotes the second, strictly unstable, term.

(ii) C_{cf} *dependent term*

From the plant model equations and spectral factor definitions obtain:

$$\text{(A37)} \qquad \frac{\phi_{h1}}{Y_c^* Y_{fd}^*} = \frac{B^* A_r^* B_q^* B_q D D_{fd}}{A A_q A_l D_c^*}$$

The diophantine equation (33) allows the strictly unstable part of equation (A37) to be separated as follows:

$$\text{(A38)} \qquad \frac{\phi_{h1}}{Y_c^* Y_{fd}^*} = \frac{X}{A A_q A_l} + \frac{Z z^{g2}}{D_c^*}$$

From the system equations and spectral factor definitions obtain:

$$\text{(A39)} \qquad Y_c S C_{cf} Y_{fd} = \frac{D_c C_{cd} C_{cfn} D_{fd}}{C_{cfd} A_l A_q A_r (A C_{cd} + B C_{cn})}$$

From equations (A38) and (A39) obtain:

$$\text{(A40)} \qquad Y_c S C_{cf} Y_{fd} - \frac{\phi_{h1}}{Y_c^* Y_{fd}^*} = \frac{D_c C_{cd} C_{cfn} D_{fd} A - X A_r C_{cfd} (A C_{cd} + B C_{cn})}{C_{cfd} A_l A_q A_r A (A C_{cd} + B C_{cn})} - \frac{Z z^{g2}}{D_c^*}$$

Substituting from equations (27) and (36) this may be written:

$$\text{(A41)} \qquad Y_c S C_{cf} Y_{fd} - \frac{\phi_{h1}}{Y_c^* Y_{fd}^*} = \frac{C_{cd} C_{cfn} D_{fd} A - X A_r C_{cfd} D_f}{C_{cfd} A_l A_q A_r A D_f} - \frac{Z z^{g2}}{D_c^*}$$

Finally, equation (A41) may be expressed as:

$$\text{(A42)} \qquad Y_c S C_{cf} Y_{fd} - \frac{\phi_{h1}}{Y_c^* Y_{fd}^*} = T_2^+ + T_2^-$$

where T_2^+ denotes the first term in equation (A41) and T_2^- denotes the second, strictly unstable, term.

Minimisation

Substituting from equations (A36) and (A42) into equation (A17) the cost-function integrand may be written:

$$(A43) \qquad I = (T_1^+ + T_1^-)(T_1^+ + T_1^-)^* + (T_2^+ + T_2^-)(T_2^+ + T_2^-)^* + \phi_{01}$$

In equation (A43) the T_i^+ terms are stable and the T_i^- terms strictly unstable for $i = \{1, 2\}$. In the expansion of equation (A43) the terms $T_i^+ T_i^{-*}$ are therefore analytic in $|z| \geqq 1$. In addition, the terms $T_i^+ T_i^{-*}/z$ are also analytic in $|z| \geqq 1$ since the T_i^- terms contain the factors z^{g1} and z^{g2}, respectively.

Thus, using the identity:

$$(A44) \qquad \oint_c T^- T^{+*} \frac{dz}{z} = -\oint_c T^+ T^{-*} \frac{dz}{z}$$

and invoking Cauchy's Theorem, the contour integrals of the cross terms $T_i^+ T_i^{-*}$, $T_i^- T_i^{+*}$ in equation (A43) are zero. The cost function therefore simplifies to:

$$(A45) \qquad J = \frac{1}{2\pi j} \oint_{|z|=1} [\sum_{i=1}^{2} (T_i^+ T_i^{+*} + T_i^- T_i^{-*}) + \phi_{01}] \frac{dz}{z}$$

Since the terms T_i^- and ϕ_{01} are independent of the controller the cost-function J is minimised by setting:

$$(A46) \qquad T_i^+ = 0, \quad i = \{1, 2\}$$

(i) *Cascade controller*

From equations (A35) and (A36), setting $T_1^+ = 0$ involves:

$$(A47) \qquad C_{cn} H - G A_r C_{cd} = 0 .$$

or:

$$(A48) \qquad C_c = \frac{G A_r}{H}$$

(ii) *Feedforward controller*

From equations (A41) and (A42), setting $T_2^+ = 0$ involves:

$$(A49) \qquad C_{cd} C_{cfn} D_{fd} A - X A_r C_{cfd} D_f = 0$$

or:

$$(A50) \qquad C_{cf} = \frac{X A_r D_f}{C_{cd} D_{fd} A}$$

Using the definition of C_{cf} in equation (A5), the feedforward controller becomes:

$$(A51) \qquad C_{ff} = \frac{X A_r D_f - C_{cn} D D_{fd}}{D_{fd} A C_{cd}}$$

Minimum Cost

Setting $T_i^+ = 0$, $i = \{1, 2\}$ in equation (A45), the minimum cost is found to be:

$$J_{\min} = \frac{1}{2\pi j} \oint_{|z|=1} \left[\sum_{i=1}^{2} (T_i^- T_i^{-*}) + \phi_{01} \right] \frac{dz}{z} \tag{A52}$$

Stability of T_i^+ terms

Implicit in the above proof is the requirement that the T_i^+ terms are asymptotically stable for $i = \{1, 2\}$. This is necessary for convergence of the cost. Stability of the T_i^+ terms may be demonstrated as follows:

(i) T_1^+ *term*

From equations (A35) and (A36) obtain:

$$T_1^+ = \frac{C_{cn}H - GA_rC_{cd}}{A_qA_nA_r(AC_{cd} + BC_{cn})} \tag{A53}$$

From equations (27) and (36) this equation may be re-written as:

$$T_1^+ = \frac{C_{cn}H - GA_rC_{cd}}{A_qA_nA_rD_fD_c} \tag{A54}$$

By definition A_q and A_r are strictly Hurwitz polynomials as are D_c and D_f. From Corollary 3 A_n divides both G and C_{cn}. T_1^+ is therefore asymptotically stable.

(ii) T_2^+ *term*

From equations (A41) and (A42) obtain:

$$T_2^+ = \frac{C_{cd}C_{cfn}D_{fd}A - XA_rC_{cfd}D_f}{C_{cfd}A_lA_qA_rAD_f} \tag{A55}$$

Substituting from the implied feedforward diophantine equation (37) and using equation (A50) the expression for T_2^+ may, after some algebraic manipulation, be written as:

$$T_2^+ = \frac{X(H - C_{cd})}{A_lA_qD_fD_c} \tag{A56}$$

By definition, A_q is strictly Hurwitz as are D_f and D_c. Condition (b) in Theorem 1 ensures that any unstable factors of A_l must divide A and D. Equations (33) and (34) show that any such unstable factors must also divide X (and Y). Thus, T_2^+ in equation (A56) is asymptotically stable.

Solvability conditions

It only remains to relate the conditions (a)–(c) in Theorem 1 to solvability of the optimal control problem. Solvability has already been demonstrated since the T_i^+ terms in the cost-function have been shown to be asymptotically stable. Problem solvability may also be demonstrated in a direct way by showing that the twelve transfer-functions between $e(t)$, $u(t)$ and the six external noise sources (ψ_d, ψ_n, ψ_l, ψ_r, ψ_{ln} and ψ_{rn}) are asymptotically stable. This is straightforward using the plant model equations and the conditions (a)–(c) in Theorem 1.

Finally, the conditions (a)–(c) may be given a straightforward physical interpretation as follows:

(i) *Condition (a)*

Common factors in A and B represent those disturbance and reference modes which are not present in the plant transfer-function W_p. These modes are clearly required to be stable to ensure a stable control signal (and hence a finite cost).

(ii) *Condition (b)*

The plant W_p must be able to reproduce any unstable modes in the path between the disturbance ψ_l, and the output (see Figure 2). Denote the unstable poles of W_l by A_{lu} and the unstable poles of W_x by A_{xu}. Thus, the product $A_{lu}A_{xu}$ must also be poles of the plant (this also means that A_{lu} must firstly be a factor of A). The polynomial D (equation (12)) includes those poles of the plant which are *not* also poles of W_x. From above, $A_{lu}A_{xu}$ must appear as poles of the plant. A_{lu} must therefore also be a factor of D.

(iii) *Condition (c)*

From Corollary 3, A_n is a factor of the cascade controller numerator C_{cn}. Any unstable factors of A_n which are also poles of the plant W_p would therefore lead to an unstable pole/zero cancellation and stability of the closed-loop system would be destroyed. Thus, any unstable factors of A_n must not appear in A. This problem is made clear by equation (25) since any common factors of A and A_n lying on the unit circle would also appear in the spectral factor D_f.

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