KYBERNETIKA — VOLUME 24 (1988), NUMBER 1

ON SYMMETRY AND REVERSIBLE SYMMETRY CONCERNING GENERALIZED DIRECTED DIVERGENCE

T. D. RAKHEJA, K. K. GULATI

The authors have proved a theorem on symmetry considering three probability distributions using reversible symmetry, a concept weaker than symmetry in the strict sense.

1. INTRODUCTION

Let

$$\Gamma_n = \{(p_1, p_2, \ldots, p_n): p_i \geqq 0, i = 1, 2, \ldots, n; \sum_{i=1}^{n} p_i = 1\}, \quad n = 2, 3, 4, \ldots$$

and

$$\Gamma_n^{(0)} = \{(p_1, p_2, \ldots, p_n): p_1 = 0, p_i \geqq 0, i = 2, 3, \ldots, n; \sum_{i=1}^{n} p_i = 1\}, \quad n = 2, 3, 4, \ldots$$

denote respectively the sets of all n-component complete discrete probability distributions with non-negative elements and with first component zero. Let G_n, $n = 2, 3, \ldots$ denote the set of all $3n$-tuples of the form $(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n)$ with $(p_1, p_2, \ldots, p_n) \in \Gamma_n$, $(q_1, q_2, \ldots, q_n) \in \Gamma_n$ and $(r_1, r_2, \ldots, r_n) \in \Gamma_n$ such that whenever r_i is zero, the corresponding q_i and p_i are also zero, $1 \leqq i \leqq n$.

A measure called the generalized directed divergence is defined as ([1], [4], [5], [7])

$$T_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n) = \sum_{i=1}^{n} p_i \log_2 (q_i/r_i) \tag{1}$$

Here the convention $0 \log_2 (0/x) = 0$, $x \geqq 0$ is used.

An important property of T_n is:

Postulate $\mathbf{I}_n$ (Symmetry). $T_n: G_n \to \mathbb{R}$ is symmetric under the simultaneous permutations of p_k, q_k and r_k, $k = 1, 2, \ldots, n$, that is,

$$T_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n) = T_n(p_{\pi(1)}, p_{\pi(2)}, \ldots p_{\pi(n)}; q_{\pi(1)}, q_{\pi(2)}, \ldots, q_{\pi(n)}; r_{\pi(1)}, r_{\pi(2)}, \ldots, r_{\pi(n)}) . \tag{2}$$

where π is an arbitrary permutation of $1, 2, \ldots, n$.

The object of this paper is to prove a theorem on symmetry using reversible symmetry, a concept weaker than that of symmetry in the strict sense. This theorem can be used in various characterization of the generalized directed divergence. For some related work concerning directed divergence, see [6].

2. REVERSIBLY SYMMETRIC FUNCTIONS

Definition. Let E be a non-empty set and $E^n = \dfrac{E \times E \times \ldots \times E}{n\text{-times}}$. A non-empty subset D_n of $E^n \times E^n \times E^n$ is said to be closed under reversible symmetry if

$$(x_1, x_2, \ldots, x_{n-1}, x_n; y_1, y_2, \ldots, y_{n-1}, y_n; z_1, z_2, \ldots, z_{n-1}, z_n) \in D_n \Rightarrow$$
$$\Rightarrow (x_n, x_{n-1}, \ldots, x_2, x_1; y_n, y_{n-1}, \ldots, y_2, y_1; z_n, z_{n-1}, \ldots, z_2, z_1) \in D_n$$

for all $(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n; z_1, z_2, \ldots, z_n) \in D_n$.

A function f_n: $D_n \to \mathbb{R}$ is said to be reversibly symmetric over the domain D_n if

$$f_n(x_1, x_2, \ldots, x_{n-1}, x_n; y_1, y_2, \ldots, y_{n-1}, y_n; z_1, z_2, \ldots, z_{n-1}, z_n) =$$
$$= f_n(x_n, x_{n-1}, \ldots, x_2, x_1; y_n, y_{n-1}, \ldots, y_2, y_1; z_n, z_{n-1}, \ldots, z_2, z_1)$$

for all $(x_1, x_2, \ldots, x_n; y_1, \ldots, y_n; z_1, z_2, \ldots, z_n) \in D_n$.

The above definition is motivated by reversible codes, see [3].

3. SYSTEM OF POSTULATES

Postulate II$_m$ (Reversible Symmetry): T_m: $G_m \to \mathbb{R}$, $m \geqq 2$ is reversibly symmetric, that is,

$$(3) \qquad T_m(p_1, p_2, \ldots, p_{m-1}, p_m; q_1, q_2, \ldots, q_{m-1}, q_m; r_1, r_2, \ldots, r_{m-1}, r_m) =$$
$$T_m(p_m, p_{m-1}, \ldots, p_2, p_1; q_m, q_{m-1}, \ldots, q_2, q_1; r_m, r_{m-1}, \ldots, r_2, r_1)$$

for all $(p_1, p_2, \ldots, p_{m-1}, p_m; q_1, q_2, \ldots, q_{m-1}, q_m; r_1, r_2, \ldots, r_m) \in G_m$.

Postulate II$_m$ tells us that value of T_m remains unaltered if the order of probability estimates is reversed. It uses only two permutations of $1, 2, \ldots, m$, namely the identity permutation $1, 2, \ldots, m$ and the permutation $m, m - 1, \ldots, 3, 2, 1$.

Postulate I_m implies Postulate II$_m$. We give an example to show that the converse is not true.

Example I. Define F_n: $G_n \to \mathbb{R}$, $n = 3, 4, \ldots$ as

$$F_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n) =$$
$$\sum_{i=1}^{n-1} (p_i q_i r_i - p_{i+1} q_{i+1} r_{i+1})^2 .$$

Then for all integers $n \geqq 3$, F_n satisfies Postulate II_n but not I_n. Thus II_n is weaker than I_n in the strict sense.

For $n = 2$, I_2 and II_2 are equivalent.

Postulate III_n (Recursivity). For all probability distributions $(p_1, p_2, \ldots, p_n) \in \Gamma_n$ with $p_1 + p_2 > 0$, $(q_1, q_2, \ldots, q_n) \in \Gamma_n$, $(r_1, r_2, \ldots, r_n) \in \Gamma_n$ such that $(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n) \in G_n$,

$$(4) \qquad T_n(p_1, p_2, p_3, \ldots, p_n; q_1, q_2, q_3, \ldots, q_n; r_1, r_2, r_3, \ldots, r_n) =$$
$$T_{n-1}(p_1 + p_2, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n; r_1 + r_2, r_3, \ldots, r_n) +$$
$$(p_1 + p_2)\, T_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}; \frac{r_1}{r_1 + r_2}, \frac{r_2}{r_1 + r_2}\right),$$
$$p_1 + p_2 > 0\,.$$

Postulate IV_n. For all probability distributions $(0, 0, p_3, \ldots, p_n) \in \Gamma_n^{(0)}$, $(q_1, q_2, q_3, \ldots, q_n) \in \Gamma_n$, $(r_1, r_2, \ldots, r_n) \in \Gamma_n$ with $0 \leqq q_1 + q_2 < 1$, $0 \leqq r_1 + r_2 < 1$, such that $(0, 0, p_3, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n) \in G_n$,

$$(5) \qquad T_n(0, 0, p_3, \ldots, p_n; q_1, q_2, q_3, \ldots, q_n; r_1, r_2, r_3, \ldots, r_n) =$$
$$T_{n-1}(0, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n; r_1 + r_2, r_3, \ldots, r_n)\,.$$

Since $q_1 + q_2 = q_2 + q_1$ and $r_1 + r_2 = r_2 + r_1$, Postulate IV_n implies

$$(6) \qquad T_n(0, 0, p_3, \ldots, p_n; q_1, q_2, q_3, \ldots, q_n; r_1, r_2, r_3, \ldots, r_n) =$$
$$T_n(0, 0, p_3, \ldots, p_n; q_2, q_1, q_3, \ldots, q_n; r_2, r_1, r_3, \ldots, r_n)\,.$$

4. THEOREM ON SYMMETRY

The main result of this paper is the following theorem.

Theorem 1. Let $T_n: G_n \to \mathbb{R}$, $n = 2, 3, \ldots$ satisfy the Postulates II_m, for some fixed $m \geqq 4$, III_n $(n \geqq 3)$ and IV_n $(n \geqq 4)$ then $T_n: G_n \to \mathbb{R}$ is symmetric under the simultaneous permutation of p_i, q_i and r_i $(i = 1, 2, \ldots, n)$.

To prove the above theorem, we need the following lemma:

Lemma 1. Postulates II_m for some fixed $m \geqq 4$, III_n $(n \geqq 3)$ and IV_n $(n \geqq 4)$ imply

$$(7) \qquad T_2(1, 0; 1, 0; 1, 0) = 0 = T_2(0, 1; 0, 1; 0, 1)$$

$$(8) \qquad T_{n+j}(p_1, p_2, \ldots, p_n, \underbrace{0, 0, \ldots, 0}_{j-\text{times}}; q_1, q_2, \ldots, q_n, \underbrace{0, 0, \ldots, 0}_{j-\text{times}}; r_1, r_2, \ldots, r_n,$$
$$\underbrace{0, 0, \ldots, 0}_{j-\text{times}}) = T_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n)\,,$$
$$p_1 + p_2 > 0\,, \quad j = 1, 2, \ldots; n = 2, 3, \ldots$$

(9) $$T_2(p_1, p_2; q_1, q_2; r_1, r_2) = T_2(p_2, p_1; q_2, q_1; r_2, r_1)$$

(10) $$T_3(p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3) = T_3(p_2, p_1, p_3; q_2, q_1, q_3; r_2, r_1, r_3)$$

(11) $$T_3(p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3) = T_3(p_3, p_2, p_1; q_3, q_2, q_1; r_3, r_2, r_1)$$

Proof. Fix $m \geqq 4$ arbitrarily. Then, by II_m, with $p \in [0, 1)$

(12) $$T_m(0, 0, \ldots, 0, 1 - p, p; 0, 0, \ldots, 0, 1 - p, p; 0, 0, \ldots, 0, 1 - p, p) = T_m(p, 1 - p, 0, \ldots, 0, 0; p, 1 - p, 0, \ldots, 0, 0; p, 1 - p, 0, \ldots, 0, 0)$$

Using IV_n $(4 \leqq n \leqq m)$ repeatedly, the L.H.S. of (12) reduces to $T_3(0, 1 - p, p; 0, 1 - p, p; 0, 1 - p, p)$. The R.H.S. of (12), after the repeated use of III_n $(n \geqq 3)$, reduces to $(m - 2)\ T_2(1, 0; 1, 0; 1, 0) + T_2(p, 1 - p; p, 1 - p; p, 1 - p)$. Thus, (12) reduces to

(13) $$T_3(0, 1 - p, p; 0, 1 - p, p; 0, 1 - p, p) = (m - 2)\ T_2(1, 0; 1, 0; 1, 0) + T_2(p, 1 - p; p, 1 - p; p, 1 - p)$$

Applying III_3 to the L.H.S. of (13), we obtain

(14) $$T_2(1 - p, p; 1 - p, p; 1 - p, p) + (1 - p)\ T_2(0, 1; 0, 1; 0, 1) = (m - 2)\ T_2(1, 0; 1, 0; 1, 0) + T_2(p, 1 - p; p, 1 - p; p, 1 - p)$$

Choosing $p = 0$ and $p = \frac{1}{2}$ respectively in (14), we get $(m - 3)\ T_2(1, 0; 1, 0; 1, 0) = 0$ and

$$T_2(1, 0; 1, 0; 1, 0) = \tfrac{1}{2} T_2(0, 1; 0, 1; 0,1)$$

from which (7) follows.

Equation (8) follows by the successive application of III_{n+b}, $b = j, j - 1, \ldots, 1$; $n = 2, 3, \ldots$ and (7).

To prove (9), we divide our discussion into four cases.

Case I. $p_1 = 0,\ p_2 = 1;\ q_1 = 0,\ q_2 = 1;\ r_1 = 0,\ r_2 = 1$.

Case II. $p_1 = 1,\ p_2 = 0;\ q_1 = 1,\ q_2 = 0;\ r_1 = 1,\ r_2 = 0$. In both these cases, (9) follows from (7).

Case III. $0 \leqq p_1 < 1, 0 < p_2 \leqq 1; 0 < q_1 < 1, 0 < q_2 < 1; 0 < r_1 < 1, 0 < r_2 < 1$. Then

$$\begin{aligned} T_2(p_1, p_2; q_1, q_2; r_1, r_2) &\overset{(8)}{=} T_m(p_1, p_2, 0, \ldots, 0; q_1, q_2, 0, \ldots, 0; r_1, r_2, 0, \ldots, 0) \\ &\overset{(3)}{=} T_m(0, \ldots, 0, p_2, p_1; 0, \ldots, 0, q_2, q_1; 0, \ldots, 0, r_2, r_1) \\ &\overset{(5)}{=} T_3(0, p_2, p_1; 0, q_2, q_1; 0, r_2, r_1) \\ &\overset{(4)}{\underset{(7)}{=}} T_2(p_2, p_1; q_2, q_1; r_2, r_1)\,. \end{aligned}$$

Case IV. $p_1 = 1,\ p_2 = 0;\ 0 < q_1 < 1,\ 0 < q_2 < 1; 0 < r_1 < 1,\ 0 < r_2 < 1$. Now

(15) $$\begin{aligned} &T_m(0, 1, 0, \ldots, 0; 0, q_1, q_2, 0, \ldots, 0; 0, r_1, r_2, \ldots, 0) \\ &\overset{(3)}{=} T_m(0, \ldots, 0. 1, 0; 0, \ldots, 0, q_2, q_1; 0, \ldots, r_2, r_1, 0) \end{aligned}$$

The LHS of (15) by using III_n $(n \geqq 3)$ and (7) reduce to $T_{m-1}(1, 0, \ldots, 0; q_1, q_2, \ldots$ $\ldots, 0; r_1, r_2, \ldots, 0)$ which by the use of (8), reduces to

$T_2(1, 0; q_1, q_2; r_1, r_2)$. The RHS of (15), by using IV_n $(n \geqq 4)$, reduces to $T_3(0, 1, 0; q_2, q_1, 0; r_2, r_1, 0)$ which by using III_3 and (7), gives $T_2(0, 1; q_2, q_1; r_2, r_1)$. Thus (9) is proved.

To prove (10), we have the following cases:

Case I. $p_1 + p_2 = 0,\ p_3 = 1;\ 0 \leqq q_1 + q_2 < 1; 0 \leqq r_1 + r_2 < 1$. Then

$$\begin{aligned}
T_3(p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3) &= T_3(0, 0, 1; q_1, q_2, q_3; r_1, r_2, r_3)\\
&=^{(5)} T_m(0, \ldots, 0, 0, 1; 0, \ldots, q_1, q_2, q_3; 0, \ldots, r_1, r_2, r_3)\\
&=^{(3)} T_m(1, 0, 0, \ldots, 0; q_3, q_2, q_1, \ldots, 0; r_3, r_2, r_1, \ldots, 0)\\
&=^{(4)}_{(9)} T_m(0, 1, 0, \ldots, 0; q_2, q_3, q_1, \ldots, 0; r_2, r_3, r_1, \ldots, 0)\\
&=^{(3)} T_m(0, \ldots, 0, 1, 0; 0, \ldots, q_1, q_3, q_2; 0, \ldots, r_1, r_3, r_2)\\
&=^{(5)} T_3(0, 1, 0; q_1, q_3, q_2; r_1, r_3, r_2)\\
&=^{(4)}_{(9)} T_3(1, 0, 0; q_3, q_1, q_2; r_3, r_1, r_2)\\
&=^{(8)} T_m(1, 0, 0, 0, \ldots, 0; q_3, q_1, q_2, 0, \ldots, 0; r_3, r_1, r_2, 0, \ldots, 0)\\
&=^{(3)} T_m(0, \ldots, 0, 0, 0, 1; 0, \ldots, 0, q_2, q_1, q_3; 0, \ldots, 0, r_2, r_1, r_3)\\
&=^{(5)} T_3(0, 0, 1; q_2, q_1, q_3; r_2, r_1, r_3)\\
&= \quad T_3(p_2, p_1, p_3; q_2, q_1, q_3; r_2, r_1, r_3)
\end{aligned}$$

Case II. $0 < p_1 + p_2 \leqq 1; 0 < q_1 + q_2 \leqq 1; 0 < r_1 + r_2 \leqq 1$.

In this case, (10) follows from (4) and (9).

To prove (11) we have the following cases:

Case I. $p_1 + p_2 = 0, p_3 = 1; 0 \leqq q_1 + q_2 < 1; 0 \leqq r_1 + r_2 < 1$.
Then

$$\begin{aligned}
T_3(p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3) &= T_3(0, 0, 1; q_1, q_2, q_3; r_1, r_2, r_3)\\
&=^{(5)} T_m(0, \ldots, 0, 0, 1; 0, \ldots, q_1, q_2, q_3; 0, \ldots, r_1, r_2, r_3)\\
&=^{(3)} T_m(1, 0, 0, \ldots, 0; q_3, q_2, q_1, 0, \ldots, 0; r_3, r_2, r_1, \ldots, 0)\\
&=^{(8)} T_3(1, 0, 0; q_3, q_2, q_1; r_3, r_2, r_1)\\
&= \quad T_3(p_3, p_2, p_1; q_3, q_2, q_1; r_3, r_2, r_1).
\end{aligned}$$

Case II. $0 < p_1 + p_2 \leqq 1; 0 < q_1 + q_2 \leqq 1; 0 < r_1 + r_2 \leqq 1$.
Then

$$\begin{aligned}
&T_3(p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3)\\
&\quad =^{(8)} T_m(p_1, p_2, p_3, 0, \ldots, 0; q_1, q_2, q_3, 0, \ldots, 0; r_1, r_2, r_3, 0, \ldots, 0)\\
&\quad =^{(3)} T_m(0, \ldots, 0, p_3, p_2, p_1; 0, \ldots, 0, q_3, q_2, q_1; 0, \ldots, 0, r_3, r_2, r_1)\\
&\quad =^{(5)} T_3(p_3, p_2, p_1; q_3, q_2, q_1; r_3, r_2, r_1) \text{ if } p_3 = 0\\
&\quad =^{(4)}_{(7)} T_3(p_3, p_2, p_1; q_3, q_2, q_1; r_3, r_2, r_1) \quad \text{if} \quad p_3 > 0.
\end{aligned}$$

Thus Lemma is proved.

Proof of the Main Theorem.

For $n = 2$, the theorem follows from (9). For $n = 3$, it follows from (10) and (11). We prove the theorem for all $n \geqq 4$ by induction on n. We assume that T_n is symmetric under the simultaneous permutation of p_i, q_i and r_i $(i = 1, 2, \ldots, j)$, $j = n \geqq 3$ and then prove that T_{n+1} is symmetric. For this, it is enough to prove the following:

(16) $$T_{n+1}(p_1, p_2, \ldots, p_{n+1}; q_1, q_2, \ldots, q_{n+1}; r_1, r_2, \ldots, r_{n+1})$$
$$= T_{n+1}(p_2, p_1, \ldots, p_{n+1}; q_2, q_1, \ldots, q_{n+1}; r_2, r_1, \ldots, r_{n+1})$$

(17) $$T_{n+1}(p_1, p_2, p_3, \ldots, p_{n+1}; q_1, q_2, q_3, \ldots, q_{n+1}; r_1, r_2, r_3, \ldots, r_{n+1})$$
$$= T_{n+1}(p_1, p_2, p_{k(3)}, \ldots, p_{k(n+1)}; q_1, q_2, q_{k(3)}, \ldots, q_{k(n+1)}; r_1, r_2, r_{k(3)}, \ldots, r_{k(n+1)})$$

where k is an arbitrary permutation of $3, 4, \ldots, (n + 1)$ and

(18) $$T_{n+1}(p_1, p_2, p_3, p_4, \ldots, p_{n+1}; q_1, q_2, q_3, q_4, \ldots q_{n+1}; r_1, r_2, r_3, r_4, \ldots, r_{n+1})$$
$$= T_{n+1}(p_1, p_3, p_2, p_4, \ldots, p_{n+1}; q_1, q_3, q_2, q_4, \ldots, q_{n+1}; r_1, r_3, r_2, r_4, \ldots, r_{n+1})$$

To prove (16), we have the following cases:

Case I. $p_1 + p_2 = 0$. In this case, (16) follows from (6).

Case II. $0 < p_1 + p_2 \leqq 1$. In this case, (16) follows from III_n and (9).

To prove (17), we have the following cases:

Case I. $p_1 + p_2 = 0$. In this case, (17) follows from (5) and the induction hypothesis.

Case II. $0 < p_1 + p_2 \leqq 1$. In this case, (17) follows from III_n $(n \geqq 3)$ and the induction hypothesis.

To prove (18), we have the following cases:

Case I. $p_1 + p_2 = 0$; $0 \leqq q_1 + q_2 < 1$; $0 \leqq r_1 + r_2 < 1$.

Then

$$T_{n+1}(p_1, p_2, p_3, p_4, \ldots, p_{n+1}; q_1, q_2, q_3, q_4, \ldots, q_{n+1}; r_1, r_2, r_3, r_4, \ldots, r_{n+1})$$
$$= T_{n+1}(0, 0, p_3, p_4, \ldots, p_{n+1}; q_1, q_2, q_3, q_4, \ldots, q_{n+1}; r_1, r_2, r_3, r_4, \ldots, r_{n+1})$$
$$=^{(5)} T_{n+2}(0, 0, 0, p_3, p_4, \ldots, p_{n+1}; 0, q_1, q_2, q_3, q_4, \ldots, q_{n+1}; 0, r_1, r_2, r_3, r_4, \ldots, r_{n+1})$$
$$=^{(17)} T_{n+2}(0, 0, p_3, 0, p_4, \ldots, p_{n+1}; 0, q_1, q_3, q_2, q_4, \ldots, q_{n+1}; 0, r_1, r_3, r_2, r_4, \ldots, r_{n+1})$$
$$=^{(5)} T_{n+1}(0, p_3, 0, p_4, \ldots, p_{n+1}; q_1, q_3, q_2, q_4, \ldots, q_{n+1}; r_1, r_3, r_2, r_4, \ldots, r_{n+1})$$
$$= T_{n+1}(p_1, p_3, p_2, p_4, \ldots, p_{n+1}; q_1, q_3, q_2, q_4, \ldots, q_{n+1}; r_1, r_3, r_2, r_4, \ldots, r_{n+1})$$

Case II. $0 < p_1 + p_2 \leqq 1; 0 < q_1 + q_2 \leqq 1; 0 < r_1 + r_2, \leqq 1$.

In this case, (18) $(n \geqq 4)$ follows from III_n $(n \geqq 3)$ and the symmetry of T_3 by proceeding in the same way as on page 60 in [2].

This completes the proof of the theorem.

COMMENTS

A code is defined to be reversible if its code-word set is invariant under a reversal of the digits in each code word. An important subclass of the BCH codes consists entirely of reversible codes.

Suppose that information has been encoded into a block code and the code word placed in a storage medium. It may be advantageous to read out the stored data beginning from either end of the stored block.

Suppose, however, that the code can be decoded digit-by-digit by feeding the block into a sequential circuit. If the code is reversible, then the same decoding circuit can be used regardless of which end of the block is processed first. But it is possible that much greater potential utility lies in exploiting the additional symmetry provided by reversibility to simplify the decoding procedure for a reversible code.

Just as a reversible code remains invariant under a reversal of the digits in each code word; in an analogous way, the average amount of information $H_n(p_1, p_2, \ldots, p_n)$ associated with the probability distribution also remains unchanged if the elements of $(p_1, p_2, \ldots, p_n)$ are reversed so that $H_n(p_n, \ldots, p_2, p_1)$ is the average amount of information associated with $(p_n, p_{n-1}, \ldots, p_2, p_1)$ i.e.

$$H_n(p_1, p_2, \ldots, p_n) = H_n(p_n, \ldots, p_2, p_1)$$

This property of H_n is known as the reversible symmetry of the Shannon entropy H_n. This sort of analogy can be extended to other measures of information like directed divergence and generalized directed divergence also. In this paper, we have exhibited such an analogy between the reversible codes and the reversible symmetry pertaining to generalized directed divergence.

ACKNOWLEDGEMENTS

The authors express their gratitudes to the referee for his valuable suggestions.

(Received April 18, 1986.)

REFERENCES

[1] J. Aczel and P. Nath: Axiomatic characterization of some measures of divergence in information. Z. Wahrsch. Verw. Gebiete *21* (1972), 215—224.

[2] J. Aczel and Z. Daroczy: On measures of information and their characterization. Academic Press, New York 1975.

[3] J. L. Massey: Reversible Codes. Inform. and Control 7 (1964), 369—384.

[4] Pl. Kannappan and P. N. Rathie: An axiomatic characterization of generalized directed divergence. Kybernetika *9* (1973), 330—337.

[5] P. Nath: On measures of error in information. J. Math. Sci. *3* (1968), 1—16.

[6] P. Nath and R. Singh: Reversible symmetry and the directed divergence in information theory. SCIMA *V-13* (1984), 1, 42—53.

[7] H. Theil: Economics and Information Theory. North Holland Publishing Co., Amsterdam 1967.

Thakar Dass Rakheja, Kanwar Kumar Gulati, Faculty of Mathematics, University of Delhi, Delhi-110007. India.