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NORMAL COVARIANCES

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The main goal of this paper is to characterize the class of normal covariances for random processes. This notion generalizes a known weakly stationary case and can be a suitable mathematical tool for describing real random processes which are not weakly stationary.

The notion of a normal covariance was introduced by the author in the paper [1]. Characterization of normal covariances for random sequences is given in [2]. The class of normal covariances was discovered by studying of locally stationary covariances which were introduced by Silverman in [3]. The name — normal covariances — is based on a close connection with the theory of normal operators in Hilbert spaces.

Definition 1. Let $R(\cdot, \cdot)$ be a covariance defined in the whole plane $\mathbb{R}_2$. The covariance $R(\cdot, \cdot)$ is called normal if it can be written in the form

$$R(s, t) = \iint_{-\infty}^{+\infty} e^{\lambda(s+t)} e^{i\mu(s-t)} \, dd F(\lambda, \mu), \quad s \in \mathbb{R}_1, \quad t \in \mathbb{R}_1$$

where $F(\cdot, \cdot)$ is a two-dimensional distribution function with finite variation.

The class of normal covariances is sufficiently large because every continuous weakly stationary covariance and every continuous symmetric covariance (for details see [1]) are normal. Their product is a normal covariance too. Let $\{Z(s)\}$, $s \in \mathbb{R}_1$, be a complex valued random process with everywhere vanishing expected value and with a normal covariance. Then by use of the Karhunen theorem such a process can be expressed in the form of a stochastic integral

$$Z(s) = \iint_{-\infty}^{+\infty} e^{s\lambda} e^{i\mu s} \, dd\xi(\lambda, \mu)$$

understood in the quadratic mean sense where $\xi(\cdot, \cdot)$ is a plane martingale satisfying $\mathsf{E}\{\xi(\lambda_1, \mu_1) \overline{\xi(\lambda_2, \mu_2)}\} = F(\min(\lambda_1, \lambda_2), \min(\mu_1, \mu_2))$.

Theorem 1. Every normal covariance function is continuous at the whole plane.

Proof. Let $R(\cdot, \cdot)$ be a normal covariance, let $(s, t) \in \mathbb{R}_2$ be quite arbitrary. Let h_1, h_2 be real numbers. We must estimate the difference $|R(s + h_1, t + h_2) - R(s, t)|$. Let $z = \lambda + i\mu$, then

$$|R(s + h_1, t + h_2) - R(s, t)| = \left|\iint_{-\infty}^{+\infty} (e^{(s+h_1)z} e^{(t+h_2)\bar{z}} - e^{sz} e^{t\bar{z}}) \, ddF(\lambda, \mu)\right| =$$

$$= \left|\iint_{-\infty}^{+\infty} e^{sz} e^{t\bar{z}}(e^{h_1 z} e^{h_2 \bar{z}} - 1) \, ddF(\lambda, \mu)\right| \leqq$$

$$\leqq \iint_{-\infty}^{+\infty} e^{(s+t)\lambda} |e^{\lambda(h_1+h_2)} e^{i\mu(h_1-h_2)} - 1| \, ddF(\lambda, \mu) = \iint_{-\infty}^{+\infty} e^{\lambda(s+t)} f(\lambda, \mu, h_1, h_2) \, ddF(\lambda, \mu),$$

where

$$f(\lambda, \mu, h_1, h_2) = |e^{\lambda(h_1+h_2)} \cos((h_1 - h_2)\mu) + i\, e^{\lambda(h_1+h_2)} \sin((h_1 - h_2)\mu) - 1|$$

and

$$\lim_{\substack{h_1 \to 0 \\ h_2 \to 0}} f(\lambda, \mu, h_1, h_2) = 0 \quad \text{for every} \quad (\lambda, \mu) \in \mathbb{R}_2 .$$

Further $f(\lambda, h_1, h_2) \leqq 2\, e^{\lambda(h_1+h_2)} + 1$. When $h_1 + h_2 \geqq 0$ and $\lambda \geqq 0$ then $e^{\lambda(h_1+h_2)} \leqq e^{2\lambda\varepsilon}$, if $\lambda \leqq 0$ then $e^{\lambda(h_1+h_2)} \leqq 1$ (it is possible to consider $|h_1| < \varepsilon$, $|h_2| < \varepsilon$ because of $h_1 \to 0$, $h_2 \to 0$). In case that $h_1 + h_2 < 0$ the situation is quite analogous. For every $\lambda \in \mathbb{R}_1$ and every h_1, h_2 with $|h_1| < \varepsilon$, $|h_2| < \varepsilon$

$$e^{\lambda(s+t)} e^{\lambda(h_1+h_2)} \leqq \max(1, e^{\lambda(s+t+2\varepsilon)})$$

that is an integrable majorant function as we assume the existence of integral $\iint_{-\infty}^{+\infty} e^{\lambda(s+t)} e^{i\mu(s-t)} \, ddF(\lambda, \mu)$ for every (s, t). Using the Lebesgue dominated theorem we immediately obtain that $F(\cdot, \cdot)$ is continuous at (s, t). □

Further properties of normal covariances are the following:

1. For every $(s, t) \in \mathbb{R}_2$

   $$|R(s, t)| \leqq \left(\int_{-\infty}^{+\infty} e^{2s\lambda} \, dF_1(\lambda)\right)^{1/2} \left(\int_{-\infty}^{+\infty} e^{2t\lambda} \, dF_1(\lambda)\right)^{1/2} = R^{1/2}(s, s) \,.\, R^{1/2}(t, t)$$

   where $F_1(\cdot)$ is the first marginal of $F(\cdot, \cdot)$, i.e.

   $$F_1(\lambda) = \int_{-\infty}^{+\infty} dF(\lambda, \mu) .$$

2. The function $R_1(s) = R(s, s) = \int_{-\infty}^{+\infty} e^{2s\lambda} \, dF_1(\lambda)$ is a nonnegative definite kernel with respect to sum, i.e.

   $$R_1(\tau_1 + \tau_2) = \int_{-\infty}^{+\infty} e^{2\lambda(\tau_1+\tau_2)} \, dF_1(\lambda)$$

   is a symmetric covariance in $(\tau_1, \tau_2) \in \mathbb{R}_2$.
3. Similarly, the function $R_2(t) = R(t, -t) = \int_{-\infty}^{+\infty} e^{2i\mu t} \, dF_2(\mu)$, where $F_2(\cdot)$ is the second marginal of $F(\cdot, \cdot)$, is a weakly stationary covariance.
4. Without loss of generality, we can put $R(0, 0) = 1$ that means the function $F(\cdot, \cdot)$ will be a probability distribution function in the plane.

The following theorem will characterize normal covariances as functions that are in some sense nonnegative definite.

Theorem 2. A covariance function $R(\cdot, \cdot)$ defined at the whole plane is normal if and only if

1) $R(0, 0) = 1$
2) $R(\cdot, \cdot)$ is continuous
3) there exists a function $S(\cdot, \cdot)$ such that for every $(s, t) \in \mathbb{R}_2$ $R(s, t) = S(s + t, s - t)$ and for every finite collection $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of complex numbers and every real numbers $u_1, \ldots, u_n, v_1, \ldots, v_n$

$$\sum_i \sum_j \alpha_i \bar{\alpha}_j S(u_i + u_j, v_i - v_j) \geqq 0 .$$

Proof. First, we shall construct a suitable Hilbert space. Let L be the linear set of all complex valued functions that are everywhere vanishing except a finite number of points in the plane, i.e. $f(\cdot, \cdot) \in L$ if and only if there exist $(u_i, v_i) \in \mathbb{R}_2$, $i = 1, 2, \ldots \ldots, n$ such that $f(u_i, v_i) \neq 0$ and $f(\cdot, \cdot) = 0$ otherwise. We can in L define an Hermite bilinear form $\langle f, g \rangle$, $f, g \in L$ by the relation

$$\langle f, g \rangle = \sum_{u,v} \sum_{x,y} f(u, v) \overline{g(x, y)} S(u + x, v - y) .$$

According to our assumption $\|f\|^2 \geqq 0$ and hence $\|\cdot\|$ is a seminorm in L. Let $h \in \mathbb{R}_1$ and let us define a shift-operator T_h in L in the following way

$$T_h f(u, v) = f(u - h, v - h) .$$

Let $N_0 \subset L$, $N_0 = \{f: \|f\| = 0\}$ and let us consider a factor space L/N_0. Then the bilinear form defined above is a scalar product and $\|\cdot\|$ is a norm. Let H be a completion of L/N_0 with respect the norm to $\|\cdot\|$. Then H is our underlying Hilbert space. As every T_h maps N_0 into N_0, there is a possibility to translate every operator T_h from L into H. The definition domain $\mathscr{D}(T_h)$ of every T_h will be the linear set L/N_0 in H, $\mathscr{D}(T_h)$ is thus everywhere dense in H. Let us consider for every operator T_h its adjoint operator T_h^* and let us prove that $\mathscr{D}(T_h^*) \supset L/N_0$. Let $f, g \in L/N_0$ then

$$\begin{aligned}\langle T_h f, g \rangle &= \sum_{u,v} \sum_{x,y} f(u - h, v - h) \overline{g(x, y)} S(u + x, v - y) = \\ &= \sum_{u,v} \sum_{x,y} f(u, v) \overline{g(x, y)} S(u + (x + h), v - (y - h)) = \\ &= \sum_{u,v} \sum_{x,y} f(u, v) \overline{g(x - h, y + h)} S(u + x, v - y) = \langle f, S_h g \rangle\end{aligned}$$

where $S_h g(x, y) = g(x - h, y + h)$. As this equality holds for every $f \in L/N_0$ that is everywhere dense in H the element $S_h g$ equals $T_h^* g$. We proved that $\mathscr{D}(T_h^*) \supset \supset L/N_0$. It means the every operator T_h can be closed, in other words, for every T_h there exists a closed operator $\overline{T}_h$ in H, $T_h \subset \overline{T}_h$. Now, we shall show that $T_h T_h^* = T_h^* T_h$ on L/N_0. When $f \in L/N_0$, then $T_h T_h^* f(u, v) = T_h f(u - h, v + h) = f(u - 2h, v) = T_h^* f(u - h, v - h) = T_h^* T_h f(u, v)$. This fact implies, further, that for every pair $f, g \in L/N_0$

$$\langle T_h f, T_h g \rangle = \langle T_h^* T_h f, g \rangle = \langle T_h T_h^* f, g \rangle = \langle T_h^* f, T_h^* g \rangle$$

because $T_h = T_h^{**}$ on L/N_0. At this moment we can construct a closed enlargement $\overline{T}_h$ of T_h. An element $f \in H$ belongs to $\mathscr{D}(\overline{T}_h)$ if there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset L/N_0$ such that $f_n \to f$ and $\{T_h f_n\}_{n=1}^{\infty}$ is convergent too. If $f_1 = \lim_{n\to\infty} T_h f_n$ then we put $f_1 = \overline{T}_h f$. There is no problem to prove that $\overline{T}_h$ is in this way defined unambiguously. Let $\{g_n\}_{n=1}^{\infty} \subset L/N_0$ be another sequence converging to f, but $T_h g_n \to g_1 \neq f_1$. Then for any $g \in L/N_0$

$$\langle g, f_1 - g_1\rangle = \langle g, \lim_{n\to\infty} T_h(f_n - g_n)\rangle = \lim_{n\to\infty} \langle g, T_h(f_n - g_n)\rangle =$$
$$= \lim_{n\to\infty} \langle T_h^* g, f_n - g_n\rangle = \langle T_h^* g, f - f\rangle = 0 .$$

Thanks to the fact that L/N_0 is dense in H $g_1 = f_1$. As $\|T_h f\| = \|T_h^* f\|$ for every $f \in L/N_0$, we can prove that $\mathscr{D}(T_h^*) = \mathscr{D}(\overline{T}_h)$. Further, $T_h^* = (\overline{T}_h)^*$ and because of closeness of $\overline{T}_h$ $\overline{T}_h^{**} = \overline{T}_h$. It remains to prove that $\overline{T}_h^* \overline{T}_h = \overline{T}_h \overline{T}_h^*$ and after this we can state that $\overline{T}_h$ is a normal operator in H. First, we must prove that $\mathscr{D}(\overline{T}_h^* \overline{T}_h) = \mathscr{D}(\overline{T}_h \overline{T}_h^*)$. Let $f \in \mathscr{D}(\overline{T}_h^* \overline{T}_h)$, i.e. $\overline{T}_h f \in \mathscr{D}(\overline{T}_h^*)$ and simultaneously $f \in \mathscr{D}(\overline{T}_h)$. At the same moment $f \in \mathscr{D}(\overline{T}_h^*) = \mathscr{D}(T_h^*)$ and we can consider $T_h^* f$. Let $g \in L/N_0$ be quite arbitrary then $\langle T_h g, T_h^* f\rangle = \langle g, T_h^{**} T_h^* f\rangle = \langle g, \overline{T}_h T_h^* f\rangle$ that means that $T_h^* f \in \mathscr{D}(\overline{T}_h)$. We have proved that $\mathscr{D}(T_h^* \overline{T}_h) \subset \mathscr{D}(\overline{T}_h T_h^*)$. Quite analogously we can prove the opposite inclusion. We see that for every operator T_h there exists a normal enlargement $\overline{T}_h$, $T_h = \overline{T}_h$ on L/N_0 and $\{\overline{T}_h\}$, $h \in \mathbb{R}_1$, forms a group on the linear set L/N_0. For every $\overline{T}_h$ there exists a resolution of the identity in H $\{P_z^h\}$, $z \in \mathbb{C}$ such that

$$\overline{T}_h = \iint_{-\infty}^{+\infty} z \, \mathrm{d}P_z^h .$$

Let $\delta(\cdot, \cdot)$ be the element in L/N_0 defined as $\delta(0, 0) = 1$, $\delta(u, v) = 0$ otherwise. Let us calculate $\langle T_{h_1}\delta(\cdot, \cdot), T_{h_2}\delta(\cdot, \cdot)\rangle$. Thus

$$\langle T_{h_1}\delta(\cdot, \cdot), T_{h_2}\delta(\cdot, \cdot)\rangle =$$
$$= \sum_{u,v} \sum_{x,y} T_{h_1}\delta(u, v)\, \overline{T_{h_2}\delta(x, y)}\, S(u + x, v - y) =$$
$$= \sum_{u,v} \sum_{x,y} \delta(u - h_1, v - h_1)\, \delta(x - h_2, y - h_2)\, S(u + x, v - y) =$$
$$= \sum_{u,v} \sum_{x,y} \delta(u, v)\, \delta(x, y)\, S(h_1 + h_2, h_1 - h_2) = R(h_1, h_2) .$$

By use of the polar coordinates every operator $\overline{T}_h$ can be expressed as

$$\overline{T}_h = \int_{-\infty}^{+\infty} \int_{-\pi}^{+\pi} \mathrm{e}^{\lambda}\, \mathrm{e}^{\mathrm{i}\mu}\, \mathrm{d}E^h(\lambda, \mu)$$

where $\{E^h(\cdot, \cdot)\}$ is another resolution of the identity in H. At this moment we put $h = 1/n$ and the group property $T_{h_1}(T_{h_2}) = T_{h_2}(T_{h_1}) = T_{h_1+h_2}$ enables that for every integer $j \in \mathbb{Z}$

$$T_{1/n}^{j} = T_{j/n} .$$

Then we can write

$$R(j/n, k/n) = \int_{-\infty}^{+\infty} \int_{-\pi}^{+\pi} \mathrm{e}^{(j+k)\alpha}\, \mathrm{e}^{\mathrm{i}\beta(j-k)}\, \mathrm{d}\langle E_{(\alpha,\beta)}^{1/n}\delta(\cdot, \cdot), \delta(\cdot, \cdot)\rangle$$

using properties of the resolution of identity in H. We can continue and express

$$R(j/n, k/n) = \int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} e^{[(j+k)/n]\alpha}\, e^{i\beta[(j-k)/n]}\, d\langle E^{1/n}_{(\alpha/n,\beta/n)}\delta(\cdot,\cdot), \delta(\cdot,\cdot)\rangle .$$

Under the choice of suitable j, k such that $j/n \to s$, $k/n \to t$ if n tends to $+\infty$ the continuity of $R(\cdot,\cdot)$ gives that

$$R(s, t) = \lim_{n\to\infty}\int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} e^{[(j+k)/n]\alpha}\, e^{i\beta[(j-k)/n]}\, dd F^{1/n}_{(\alpha/n,\beta/n)}$$

if we denoted $\langle E^{1/n}_{(\alpha/n,\beta/n)}\delta(\cdot,\cdot), \delta(\cdot,\cdot)\rangle = F^{1/n}_{(\alpha/n,\beta/n)}$. Let

$$R_n(s, t) = \int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} e^{(s+t)\alpha}\, e^{i\beta(s-t)}\, dd F^{1/n}_{(\alpha/n,\beta/n)} ;$$

we see that $R_n(\cdot,\cdot)$ is a normal covariance and we shall prove that $R_n(s, t) \to R(s, t)$ as $n \to +\infty$. First, we estimate

$$\left|\int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} e^{\alpha(s+t)}\, e^{i\beta(s-t)}\, dd F^{1/n}_{(\alpha/n,\beta/n)} - \int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} e^{\alpha[(j+k)/n]}\, e^{i\beta[(j-k)/n]}\, dd F^{1/n}_{(\alpha/n,\beta/n)}\right| \leqq$$

$$\leqq \left|\int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} e^{\alpha[(j+k)/n]}\left(e^{i\beta(s-t)} - e^{i\beta[(j-k)/n]}\right) dd F^{1/n}_{(\alpha/n,\beta/n)}\right| +$$

$$+ \left|\int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} e^{i\beta(s-t)}\left(e^{\alpha(s+t)} - e^{\alpha[(j+k)/n]}\right) dd F^{1/n}_{(\alpha/n,\beta/n)}\right| \leqq$$

$$\leqq \int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} e^{\alpha[(j+k)/n]}\left|e^{i\beta(s-t-(j-k)/n)} - 1\right| dd F^{1/n}_{(\alpha/n,\beta/n)} +$$

$$+ \left|\int_{-\infty}^{+\infty}\int_{-nk}^{+n\pi} e^{\alpha[(j+k)/n]}\left(e^{\alpha(s+t-(j+k)/n)} - 1\right) dd F^{1/n}_{(\alpha/n,\beta/n)}\right| .$$

We can choose $j, k \in \mathbb{Z}$ that $j/n \to s$, $k/n \to t$ when $n \to +\infty$ and $0 \leqq s - j/n < 1/n$, $0 \leqq t - k/n < 1/n$. Then, the first term can be estimated as

$$\int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} e^{\alpha[(j+k)/n]}\left|e^{i\theta_n\beta} - 1\right| dd F^{1/n}_{(\alpha/n,\beta/n)} \leqq \left(\int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} e^{2\alpha[(j+k)/n]}\, dd F^{1/n}_{(\alpha/n,\beta/n)}\right)^{1/2}$$

$$\left(\int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} \left|e^{i\theta_n\beta} - 1\right|^2 dd F^{1/n}_{(\alpha/n,\beta/n)}\right)^{1/2} =$$

$$= R^{1/2}((j+k)/n, (j+k)/n)\left(\int_{-n\pi}^{+n\pi} 2(1 - \cos(\theta_n\beta))\, d F^{1/n}_{2(\beta/n)}\right)^{1/2}$$

where $\theta_n = s - t - j/n + k/n$ and $F^{1/n}_{2(\beta/n)} = \int_{-\infty}^{+\infty} d F^{1/n}_{(\alpha/n,\beta/n)}$. Thanks to the continuity of $R(\cdot,\cdot)$, $R((j+k)/n, (j+k)/n) \to R(s+t, s+t)$ as $n \to +\infty$. Since $-1/n < \theta_n < 1/n$ $\cos(\theta_n n\beta) \geqq \cos(\beta)$ for every natural n and hence

$$\int_{-n\pi}^{+n\pi}(1 - \cos(\theta_n\beta))\, d F^{1/n}_{2(\beta/n)} = \int_{-\pi}^{+\pi}(1 - \cos(\theta_n n\beta))\, d F^{1/n}_{2(\beta)} \leqq$$

$$\leqq \int_{-\pi}^{+\pi}(1 - \cos(\beta))\, d F^{1/n}_{2(\beta)} = 1 - \operatorname{Re}\varphi_{1/n}(1) ,$$

where

$$\varphi_{1/n}(u) = \int_{-\pi}^{+\pi} e^{i\beta u}\, d F^{1/n}_{2(\beta)} = \int_{-\infty}^{+\infty}\int_{-\pi}^{+\pi} e^{i\beta u}\, dd F^{1/n}_{(\alpha,\beta)} = R_n(u/2n, -u/2n) .$$

It means we must prove that $\lim_{n\to\infty} R_n(1/2n, -1/2n) = 1$. Every operator $T_{1/n}$ can be expressed as $T_{1/n} = A_{1/n} . U_{1/n}$, where $A_{1/n}$ is a positive self-adjoint operator and $U_{1/n}$ is unitary. In our case

$$A_{1/n} = \int_{-\infty}^{+\infty}\int_{-\pi}^{+\pi} e^{\lambda}\, d E^{1/n}_{(\lambda,\mu)} , \quad U_{1/n} = \int_{-\infty}^{+\infty}\int_{-\pi}^{+\pi} e^{i\mu}\, d E^{1/n}_{(\lambda,\mu)} .$$

Then we can write that $U_{1/n}f(u, v) = f(u, v - 1/n)$ and $\langle U_{1/n}\delta(\cdot, \cdot), \delta(\cdot, \cdot)\rangle = S(0, 1/n) = R(1/2n, -1/2n)$. On the other hand,

$$\langle U_{1/n}\delta(\cdot,\cdot), \delta(\cdot,\cdot)\rangle = \int_{-\infty}^{+\infty}\int_{-\pi}^{+\pi} e^{i\mu}\, d\langle E_{(\lambda,\mu)}^{1/n}\delta(\cdot,\cdot), \delta(\cdot,\cdot)\rangle = \int_{-\pi}^{+\pi} e^{i\mu}\, dF_{2(\mu)}^{1/n} = \varphi_{1/n}(1).$$

As we suppose the continuity of $R(\cdot, \cdot)$ we obtain $\lim_{n\to\infty} \varphi_{1/n}(1) = 1$. The second term

$$\int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} e^{\alpha[(j+k)/n]}\left(e^{\alpha[s+t-(j+k)/n]} - 1\right) dd F_{(\alpha/n,\beta/n)}^{1/n}$$

can be estimated in the following way:

$$\left|\int_{-\infty}^{+\infty} e^{\alpha[(j+k)/n]}\left(e^{\alpha[s+t-(j+k)/n]} - 1\right) dF_{1(\alpha/n)}^{1/n}\right| \leqq$$

$$\leqq \left(\int_{-\infty}^{+\infty} e^{2\alpha[(j+k)/n]}\, dF_{1(\alpha/n)}^{1/n}\right)^{1/2}\left(\int_{-\infty}^{+\infty}(e^{\alpha\varrho_n} - 1)^2\, dF_{1(\alpha/n)}^{1/n}\right)^{1/2} =$$

$$= \left(R\left(\frac{j+k}{n}, \frac{j+k}{n}\right)\right)^{1/2}\left(\int_{-\infty}^{+\infty}(e^{\alpha\varrho_n} - 1)^2\, dF_{1(\alpha/n)}^{1/n}\right)^{1/2}, \quad 0 \leqq \varrho_n = s + t - \frac{j+k}{n} < \frac{2}{n}.$$

The last inequality implies that

$$(e^{2\alpha/n} - 1)^2 \geqq (1 - e^{\alpha\varrho_n})^2$$

for every α and hence

$$\int_{-\infty}^{+\infty}(e^{\alpha\varrho_n} - 1)^2\, dF_{1(\alpha/n)}^{1/n} \leqq \int_{-\infty}^{+\infty}(e^{2(\alpha/n)} - 1)^2\, dF_{(\alpha/n)}^{1/n} =$$

$$= \int_{-\infty}^{+\infty}(e^{4(\alpha/n)} - 2e^{2(\alpha/n)} + 1)\, dF_{1(\alpha/n)}^{1/n} = \int_{-\infty}^{+\infty}(e^{4\alpha} - 2e^{2\alpha} + 1)\, dF_{(\alpha)}^{1/n} =$$

$$= R(2/n, 2/n) - 2R(1/n, 1/n) + 1$$

and thanks to the continuity of $R(\cdot, \cdot)$ we can state that

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}(e^{\alpha\varrho_n} - 1)^2\, dF_{1(\alpha/n)}^{1/n} = 0.$$

We have proved that

$$|R_n(j/n, k/n) - R_n(s, t)| \to 0 \quad \text{as} \quad n \to \infty \quad \text{where} \quad j/n \to s, \quad k/n \to t.$$

As $R_n(j/n, k/n) = R(j/n, k/n)$ and $R(\cdot, \cdot)$ is continuous we obtained that

$$\lim_{n\to\infty} R_n(s, t) = R(s, t) \quad \text{for every} \quad (s, t) \in \mathbb{R}_2.$$

Further, we shall prove that the sequence of the first marginals $\{F_{1(\cdot)}^{1/n}\}_{n=1}^{\infty}$ is compact in the sense that there exists a subsequence converging to a probability distribution function. We know that for every $j \in \mathbb{Z}$

$$R_n(j, j) = S_n(2j, 0) = R(j, j) = \int_{-\infty}^{+\infty} e^{2j\alpha}\, dF_{1(\alpha/n)}^{1/n}$$

is not depending on $n \in \mathbb{N}$. Thus, for every $j > 0$

$$R(j, j) \geqq \int_{-\infty}^{\alpha_1} e^{2j\alpha}\, dF_{1(\alpha/n)}^{1/n} + (1 - F_{1(\alpha_1/n)}^{1/n})$$

that means

$$F_{1(\alpha_1/n)}^{1/n} \geqq 1 - R(j, j)\, e^{-2\alpha_1 j}.$$

When ε is chosen quite arbitrarily we can find $\alpha_1 = \alpha_1(\varepsilon)$ such that for every $n \in \mathbb{N}$

$$F^{1/n}_{1(\alpha_1/n)} > 1 - \varepsilon .$$

Similarly, one can prove that there exists a suitable $\alpha_0 = \alpha_0(\varepsilon)$ such that for every $n \in \mathbb{N}$

$$F^{1/n}_{1(\alpha_0/n)} < \varepsilon$$

because $R(j, j) = \int_{-\infty}^{+\infty} e^{2\alpha j} \, dF^{1/n}_{1(\alpha/n)}$ holds for negative $j \in \mathbb{Z}$ too. This fact shows that the sequence $\{F^{1/n}_{1(\cdot)}\}_{n=1}^{\infty}$ is compact, i.e. there exists a subsequence $\{F^{1/n_k}_{1(\cdot)}\}_{k=1}^{\infty}$ converging to $F_1(\cdot)$ at all points of continuity. Without loss of generality we can assume that $\{F^{1/n}_{1(\cdot)}\}$ is just convergent to $F_1(\cdot)$.

In the case of the sequence $\{F^{1/n}_{2(\cdot)}\}_{n=1}^{\infty}$ we shall consider the sequence of the corresponding characteristic functions $\{\varphi_{1/n(\cdot)}\}_{n=1}^{\infty}$ where

$$\varphi_{1/n}(t) = \int_{-n\pi}^{+n\pi} e^{it\beta} \, dF^{1/n}_{2(\beta/n)} .$$

We know that

$$\varphi_{1/n}(2k/n) = \int_{-n\pi}^{+n\pi} e^{2i(k/n)\beta} \, dF^{1/n}_{2(\beta/n)} = R(k/n, -k/n) = S_n(0, 2k/n) .$$

We can choose $k \in \mathbb{Z}$ in such a way that $k/n \to t$ as $n \to +\infty$ and $0 < t - k/n < 1/n$ for every $n \in \mathbb{N}$. First, we shall prove that

$$\varphi_{1/n}(k/n) - \varphi_{1/n}(t) \to 0 \quad \text{as} \quad n \to +\infty .$$

As $\varphi_{1/n}(t) = \int_{-n\pi}^{+n\pi} e^{it\beta} \, dF^{1/n}_{2(\beta/n)}$ then

$$|\varphi_{1/n}(k/n) - \varphi_{1/n}(t)| = \left|\int_{-n\pi}^{+n\pi} (e^{it\beta} - e^{i(k/n)\beta}) \, dF^{1/n}_{2(\beta/n)}\right| =$$

$$= \left|\int_{-n\pi}^{+n\pi} e^{i(k/n)\beta} (e^{i(t-(k/n))\beta} - 1) \, dF^{1/n}_{2(\beta/n)}\right| \leqq \left(\int_{-n\pi}^{+n\pi} |e^{i(t-(k/n))} - 1|^2 \, dF^{1/n}_{2(\beta/n)}\right)^{1/2} =$$

$$= \sqrt{2}\left(\int_{-n\pi}^{+n\pi} (1 - \cos(t - k/n)\beta) \, dF^{1/n}_{2(\beta/n)}\right)^{1/2} \leqq$$

$$\leqq \sqrt{2}\left(\int_{-\pi}^{+\pi} (1 - \cos(\beta)) \, dF^{1/n}_{2(\beta)}\right)^{1/2} = \sqrt{(2)}\,(1 - \operatorname{Re} \varphi_{1/n}(1))^{1/2} \to 0 \quad \text{as} \quad n \to \infty$$

as was shown sooner. The assumptions of the theorem yield immediately that the function $R(t/2, -t/2) = S(0, t)$ is a characteristic function, i.e.

$$S(0, t) = \int_{-\infty}^{+\infty} e^{it\beta} \, dF_2(\beta) .$$

As

$\varphi_{1/n}(2k/n) = S(0, 2k/n) = \int_{-\infty}^{+\infty} e^{i(2k/n)\beta} \, dF_2(\beta)$ then $\varphi_{1/n}(2t) \to S(0, 2t)$ as $n \to \infty$; we have proved that

$$F^{1/n}_{2(\beta/n)} \to F_2(\beta)$$

at all points of continuity.

Now, we can estimate the measure of $K = [\alpha_0, \alpha_1) \times [\beta_0, \beta_1)$ in the plane $\iint_K dd F^{1/n}_{(\alpha/n,\, \beta/n)}$.

As

$$\iint_K dd F^{1/n}_{(\alpha/n,\beta/n)} = \iint_{-\infty}^{+\infty} \psi_{[\alpha_0,\alpha_1)}(\alpha)\, \psi_{[\beta_0,\beta_1)}(\beta) \, dd F^{1/n}_{(\alpha/n,\beta/n)} =$$

$$= \int_{\alpha_0}^{\alpha_1} \int_{-\infty}^{+\infty} dd F^{1/n}_{(\alpha/n,\beta/n)} - \int_{\alpha_0}^{\alpha_1} \int_{-\infty}^{\beta_0} dd F^{1/n}_{(\alpha/n,\beta/n)} - \int_{\alpha_0}^{\alpha_1} \int_{\beta_1}^{+\infty} dd F^{1/n}_{(\alpha/n,\beta/n)}$$

then

$$\iint_K \mathrm{dd}F^{1/n}_{(\alpha/n,\beta/n)} \geqq \left(F^{1/n}_{1(\alpha_1/n)} - F^{1/n}_{1(\alpha_0/n)}\right) - \int_{-\infty}^{+\infty}\int_{-\infty}^{\beta_0} \mathrm{dd}F^{1/n}_{(\alpha/n,\beta/n)} -$$
$$- \int_{-\infty}^{+\infty}\int_{\beta_1}^{\infty} \mathrm{dd}F^{1/n}_{(\alpha/n,\beta/n)} = \left(F^{1/n}_{1(\alpha_1/n)} - F^{1/n}_{1(\alpha_0/n)}\right) - \left(F^{1/n}_{2(\beta_0/n)} + 1 - F^{1/n}_{2(\beta_1/n)}\right) > 1 - 2\varepsilon .$$

because $F^{1/n}_{1(\cdot)} \to F_1(\cdot)$ and $F^{1/n}_{2(\cdot)} \to F_2(\cdot)$ as $n \to \infty$ and $F_1(\cdot)$, $F_2(\cdot)$ are probability distribution functions. This inequality proves that the sequence $\{F^{1/n}_{(\cdot/n,\cdot/n)}\}_{n=1}^{\infty}$ is compact. Hence, there exists a subsequence $\{F^{1/n_k}_{(\cdot/n_k,\cdot/n_k)}\}_{k=1}^{\infty}$ that is convergent to a probability distribution function $F(\cdot, \cdot)$. It remains to prove that

$$R(s, t) = \iint_{-\infty}^{+\infty} \mathrm{e}^{\lambda(s+t)}\, \mathrm{e}^{\mathrm{i}\mu(s-t)}\, \mathrm{dd}F(\lambda, \mu) .$$

We proved that

$$R(s, t) = \lim_{n\to\infty} \int_{-\infty}^{+\infty}\int_{-n\pi}^{+n\pi} \mathrm{e}^{\alpha(s+t)}\, \mathrm{e}^{\mathrm{i}\beta(s-t)}\, \mathrm{dd}F^{1/n}_{(\alpha/n,\beta/n)} .$$

At this moment we need possibility to change the order between integration and convergence. This change is possible under uniform integration of $|\mathrm{e}^{\alpha(s+t)}\, \mathrm{e}^{\mathrm{i}\beta(s-t)}| = \mathrm{e}^{\alpha(s+t)}$ with respect to the sequence $\{F^{1/n_k}_{(\cdot/n_k,\cdot/n_k)}\}_{k=1}^{\infty}$. We know that

$$R((s+t)/2, (s+t)/2) = S(s + t, 0) = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha(s+t)}\, \mathrm{d}F^{1/n}_{1(\alpha/n)}$$

is a continuous covariance function because the function $S(\cdot, 0)$ forms a nonnegative definite kernel with respect to the sum. Every function of these properties can be expressed as a bilateral Laplace transform

$$S(s + t, 0) = \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha(s+t)}\, \mathrm{d}G_1(\alpha)$$

where $G_1(\cdot)$ is a probability distribution function because $S(0, 0) = 1$. Sooner we proved that $F^{1/n}_{1(\cdot/n)} \to F_1(\cdot)$ as $n \to \infty$, and on the basis of the convergence of moments we state that $G_1(\cdot) = F_1(\cdot)$. We have proved

$$\lim_{k\to\infty} \iint_{-\infty}^{+\infty} |\mathrm{e}^{\alpha(s+t)}\, \mathrm{e}^{\mathrm{i}\beta(s-t)}|\, \mathrm{dd}F^{1/n}_{(\alpha/n_k,\beta/n_k)} = \iint_{-\infty}^{+\infty} |\mathrm{e}^{\alpha(s+t)}\, \mathrm{e}^{\mathrm{i}\beta(s-t)}|\, \mathrm{dd}F_{(\alpha,\beta)}$$

i.e. the function $e^{\alpha(s+t)}$ is uniformly integrable with respect to the sequence $\{F^{1/n_k}_{(\cdot/n_k,\cdot/n_k)}\}_{k=1}^{\infty}$. Further, the convergence $F^{1/n_k}_{(\cdot/n_k,\cdot/n_k)} \to F(\cdot, \cdot)$ as $k \to \infty$ implies the existence of $\iint_{-\infty}^{+\infty} \mathrm{e}^{\alpha(s+t)}\, \mathrm{e}^{\mathrm{i}\beta(s-t)}\, \mathrm{dd}\mathrm{F}(\alpha, \beta)$. Together, we can change order between integration and convergence and we can write

$$R(s, t) = \iint_{-\infty}^{+\infty} \mathrm{e}^{\alpha(s+t)}\, \mathrm{e}^{\mathrm{i}\beta(s-t)}\, \mathrm{dd}F(\alpha, \beta) .$$

On the contrary, let $R(\cdot, \cdot)$ be normal. Let

$$S(u, v) = \iint_{-\infty}^{+\infty} \mathrm{e}^{\alpha u}\, \mathrm{e}^{\mathrm{i}\beta v}\, \mathrm{dd}F(\alpha, \beta) .$$

Then for arbitrary $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and arbitrary $u_1, u_2, \ldots, u_n$, $v_1, v_2, \ldots, v_n \in \mathbb{R}_1$ surely

$$\sum_j \sum_k \alpha_j \bar{\alpha}_k S(u_j + u_k, v_j - v_k) \geqq 0$$

because

$$\sum_j \sum_k \alpha_j \bar{\alpha}_k\, \mathrm{e}^{\alpha(u_j+u_k)}\, \mathrm{e}^{\mathrm{i}\beta(v_j-v_k)} = \left| \sum_j \alpha_j\, \mathrm{e}^{\alpha u_j}\, \mathrm{e}^{\mathrm{i}\beta v_j} \right|^2 \geqq 0 .$$ □

The class of normal covariances can be also described by use of the corresponding reproducing kernel Hilbert space (RKHS).

Theorem 3. Let $R(\cdot, \cdot)$ be a continuous covariance function defined in the whole plane $\mathbb{R}_2$. $R(\cdot, \cdot)$ is normal if and only if

$$R(s + h, t) = \langle R(s, u); R(t, u + h)\rangle$$

holds for every real h, $\langle \cdot, \cdot \rangle$ is the scalar product in RKHS due to the covariance function $R(\cdot, \cdot)$.

Proof. Let $R(\cdot, \cdot)$ be normal. Then

$$R(s, t) = \iint_{-\infty}^{+\infty} e^{\alpha(s+t)} e^{i\beta(s-t)} \, dd F(\alpha, \beta), \quad (s, t) \in \mathbb{R}_2 .$$

Let h be any real number, thus

$$R(s + h, t) = \iint_{-\infty}^{+\infty} e^{\alpha(s+h+t)} e^{i\beta(s+h-t)} \, dd F(\alpha, \beta) =$$
$$= \iint_{-+}^{+\infty} e^{\alpha s} e^{i\beta s} e^{\alpha(t+k)} \overline{e^{i\beta(t-h)}} \, dd F(\alpha, \beta) .$$

Similarly,

$$R(t, u + h) = \iint_{-\infty}^{+\infty} e^{\alpha(t+u+h)} e^{i\beta(t-u-h)} \, dd F(\alpha, \beta) =$$
$$= \iint_{-\infty}^{+\infty} e^{\alpha(t+h)} e^{i\beta(t-h)} e^{\alpha u} \overline{e^{i\beta u}} \, dd F(\alpha, \beta)$$

and

$$R(s, u) = \iint_{-\infty}^{+\infty} e^{\alpha s} e^{i\beta s} e^{\alpha u} \overline{e^{i\beta u}} \, dd FF(\alpha, \beta) .$$

Now, by use of the "reproducing property" $m(s) = \langle m(\cdot); R(s, \cdot)\rangle$ holding for every $m(\cdot) \in$ RKHS we obtain immediately

$$R(s + h, t) = \langle R(s, u); R(t, u + h)\rangle .$$

On the contrary, let for every $h \in \mathbb{R}_1$ the covariance function $R(\cdot, \cdot)$ satisfy

$$R(s + h, t) = \langle R(s, u); R(t, u + h)\rangle$$

in the corresponding RKHS. Let us define the shift-operator T_h in the RKHS by the relation

$$T_h R(s, \cdot) = R(s + h, \cdot) .$$

The definition domain of every T_h is formed by all linear combinations $\sum_{i=1}^{n} \alpha_i R(s_i, \cdot)$, where α_i, $i = 1, \ldots, n$, are complex numbers. The construction of the RKHS gives that the definition domain $\mathscr{D}(T_h)$ of T_h is everywhere dense linear subset in the RKHS. Let T_h^* be the adjoint operator to T_h in the RKHS. Let us prove that $\mathscr{D}(T_h) \subset \mathscr{D}(T_h^*)$. Let $m(\cdot) \in \mathscr{D}(T_h)$. By definition of T_h $y(\cdot) \in$ RKHS belongs to $\mathscr{D}(T_h^*)$ if and only if for every $x(\cdot) \in \mathscr{D}(T_h)$

$$\langle T_h x(\cdot); y(\cdot)\rangle = \langle x(\cdot); T_h y(\cdot)\rangle .$$

When $x(\cdot) = \sum_i \alpha_i R(s_i, \cdot)$, $m(\cdot) = \sum_j \beta_j R(t_j, \cdot)$, then $T_h x(\cdot) = \sum_i \alpha_i R(s_i + h, \cdot)$ and

$$\langle T_h x(\cdot); m(\cdot)\rangle = \sum_i \sum_j \alpha_i \bar{\beta}_j \langle R(s_i + h, u); R(t_j, u)\rangle =$$
$$= \sum_i \sum_j \alpha_i \bar{\beta}_j R(s_i + h, t_j) = \sum_i \sum_j \alpha_i \bar{\beta}_j \langle R(s_i, u); R(t_j, u + h)\rangle = \langle x(\cdot); m^*(\cdot)\rangle$$

where $m^*(\cdot) = \sum_j \beta_j R(t_j, (\cdot) + h) = T_h^* m(\cdot)$. In other words, the adjoint operator T_h^* is represented by the shift operator in the argument in the RKHS, i.e.

$$T_h^* R(s, \cdot) = R(s, (\cdot) + h).$$

It implies that T_h^* is everywhere densily defined and hence the operator T_h possesses a closed enlargement $\overline{T}_h$ in the RKHS. There is no problem to show that for every $m_1(\cdot), m_2(\cdot) \in \mathscr{D}(T_h)$

$$\langle T_h m_1(\cdot); T_h m_2(\cdot)\rangle = \langle T_h^* m_1(\cdot); T_h^* m_2(\cdot)\rangle$$

and since this moment we can follow the proof of Theorem 2. We shall prove that every operator T_h in the RKHS is normal and by means of their spectral resolutions one can show in the same way as was used in the proof of Theorem 2 that the covariance function $R(\cdot, \cdot)$ is normal. □

Remark. In this part we shall consider two covariances. The first one

$$R_1(s, t) = e^{c(s+t)^2/D}\, e^{-a(s-t)^2/D}\, e^{-i\,b(s^2-t^2)/D}, \quad D = 4ac - b^2 > 0,$$

is normal, the other one

$$R_2(s, t) = e^{-\gamma(s^2+t^2)}, \quad \gamma > 0,$$

is not normal. The covariance $R_1(s, t)$ is normal because

$$R_1(s, t) = \iint_{-\infty}^{+\infty} e^{\alpha(s+t)}\, e^{i\beta(s-t)}\, e^{-(a\alpha^2 + b\alpha\beta + c\beta^2)}\, d\alpha\, d\beta$$

In case when $b = 0$ we obtain a locally stationary covariance because then $R_1(s, t) = S_1[(s + t)/2]\, S_2(s - t)$ where $S_1(u) = e^{(4c/D)u^2} > 0$, $S_2(v) = e^{-av^2/D}$ is a characteristic function. Further, this case is interesting also because the correlation function corresponding to $R_1(s, t)$ for $b = 0$

$$\varrho(s, t) = \frac{R_1(s, t)}{R_1^{1/2}(s, s)\, R_1^{1/2}(t, t)} = e^{-[(a+c)/D](s-t)^2}$$

is depending on $s - t$ only.

On the other hand, the covariance $R_2(s, t)$ is locally stationary also because

$$R_2(s, t) = e^{-2\gamma[(s+t)/2]^2}\, e^{\gamma(s-t)^2/2},$$

although the first term $e^{-2\gamma[(s+t)/2]^2}$ is not covariance. When we put, in the case of $R_1(\cdot, \cdot)$ $a = c = \sqrt{2}/2$ we have

$$R_1(s, t) = e^{2st} = e^{(s+t)^2/2}\, e^{-(s-t)^2/2},$$

and $\gamma = 1$ for the case of $R_2(\cdot, \cdot)$, then

$$R_2(s, t) = e^{-(s^2+t^2)} = e^{-[(s+t)/2]^2}\, e^{-(s-t)^2/2}.$$

The function $e^{[(s+t)/2]^2}$ is a covariance but $e^{-[(s+t)/2]^2}$ is not covariance. If we consider the shift-operator T_h in the RKHS due to the covariance e^{2st} then

$$R_1(s + h, t) = e^{2(s+k)t} = e^{2st} \cdot e^{2kt} = R_1(s, t) \cdot R_1(h, t).$$

This fact yields that for every $m(\cdot) \in \mathscr{D}(T_h)$

$$T_h(m\cdot) = m(\cdot)\, R_1(h, \cdot)\,,$$

that shows $\langle T_h m(\cdot); x(\cdot)\rangle$ is a continuous linear functional on $\mathscr{D}(T_h)$ and hence $\mathscr{D}(T_h^*) \supset \mathscr{D}(T_h)$. On the other hand, a similar resolution in the case of $R_2(\cdot, \cdot)$ is not possible because

$$R_2(s + h, t) = \mathrm{e}^{-(s+h)^2}\, \mathrm{e}^{-t^2} = \mathrm{e}^{-(s^2+t^2)}\, \mathrm{e}^{-2hs} \,.\, \mathrm{e}^{-h^2} =$$

$$= R_2(s, t)\, R_2\left(\frac{h\sqrt{2}}{2}, \frac{h\sqrt{2}}{2}\right) \mathrm{e}^{-2hs}\,.$$

Thus

$$T_h m(\cdot) = R_2\left(\frac{h\sqrt{2}}{2}, \frac{h\sqrt{2}}{2}\right) \sum_{i=1}^{n} \alpha_i R_2(s_i, \cdot)\, \mathrm{e}^{-2hs_i}$$

when $m(\cdot) = \sum_{i=1}^{n} \alpha_i R(s_i, \cdot)$ and the assumption $m_n(\cdot) \to 0$ in $\mathscr{D}(T_h)$ need not imply, in general, that $\langle T_h m_n(\cdot);\ x(\cdot)\rangle \to 0$. This fact causes that the adjoint operator T_h^* is not well defined $((\mathscr{D}(T_h^*) \not\supset \mathscr{D}(T_h))$ and T_h cannot be normal in the RKHS.

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